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A Global Glance on Categories in Logic

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Abstract. We explore the possibility and some potential payoffs of using the theory of accessible categories in the study of categories of logics. We illustrate this by two case studies focusing on the category of finitary structural logics and its subcategory of algebraizable logics.

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1. Introduction

1.1. Categories of logics

This work responds to an increasing tendency to consider logics by their relations to other logics. Accordingly, the (potential) use of categories in logic we are considering here is not to give semantics for formal languages or perform proof-theoretical considerations. Rather we are considering the use of categories for what was their original purpose, to study the "sociology of mathematical objects", and thus we are considering categories *of* logics, i.e., categories whose objects are logical systems and whose morphisms are translations.

This is a relatively recent point of view which has largely come into consideration through the topic of combination of logics: The goal of combining two logics L_1 and L_2 has been described as to obtain "the smallest logic system for the combined language which is a conservative extension of both L_1 and L_2". In [24] it was proposed that this rather informal statement could be given precise meaning by considering the combination of logics as a colimit construction in an appropriate category of logics. Following this idea the notions of modulated fibring, metafibring and combination of institutions and π-institutions have been presented as colimit constructions in different categories.

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It has also been observed that a presentation of a logic as a colimit of others can be seen as a splitting of this logic into other, simpler, logics, possibly helping to understand the more complex logic. Possible translation semantics and remote algebraization (see, e.g., [11]) are concrete developments of this point of view. We observe that the situation of a logic L that is completely determined by the translation of other logics L_i into L can be seen as a "covering" of L by the L_i. In Section 4 we turn this intuition into a mathematical statement by choosing a category of logics and giving a rigorous definition of covering.

Thus it seems reasonable to adopt a global perspective on logic and consider not only constructions with particular logics but the whole category of (some kind of) logics. Defining such a category is not a completely straightforward task. It means giving a partial answer to the *identity problem*, amply discussed in the book *Logica Universalis: Towards a general theory of logic* (see [7]), i.e., the question of when two given presentations of logic systems can be considered to describe the same logic: Two such presentations should be expected to describe isomorphic objects in their ambient category. This gives only a partial answer, since the identity problem also includes the task of comparing logic systems which are given in different styles of presentation, e.g. a Hilbert calculus and a sequent calculus. It seems difficult to unite two such differently presented systems in one category.

However, the identity problem is not our main concern here and we shall only briefly readdress it towards the end of the article. What is our main concern, and this gives the second intended meaning to the title, is to investigate global properties of categories of logics, like those of being complete, connected, accessible or locally presentable (the latter two notions will be explained in a moment). We believe that in choosing a category of logics it should ultimately be of advantage to take into account such global categorial properties and not only the ad hoc requirements of the constructions one wishes to perform. In particular we believe that the theory of accessible categories has to offer theorems and intuitions that can be of use in tackling technical as well as conceptual questions arising in Universal Logic. To confirm and illustrate this we will in the following present two case studies taken from our previous publications [4, 5], of which this article is an amalgamation and expansion. The main point of these case studies, in each of which we prove a certain category of logics to be accessible, is twofold: first, to show that and how the notion of accessibility *can* apply to categories of logics in a natural way (this is not obvious, see the last section) and, second that some benefits can be gained thereof. We are, however, aware that the categories we present are not a good ambient for proper logical studies since they give unsatisfactory answers to the identity problem. It remains a project for the future to give a category of logics with good global properties *and* an appropriate notion of isomorphism of logics; we outline a possible solution for this task in Section 5.

1.2. Locally presentable and accessible categories

With the advent of category theory came the task of characterizing categories "of an algebraic character" by the means of categorial language. One answer that has been given is through the theory of monads and its variations; see [14] for a survey. Another one came from sketch theory; the varieties from Universal Algebra are exactly the categories of **Set**-models of finite product sketches — this is close in spirit to the usual characterization of varieties as categories of **Set**-models of equational theories.

A more general class of "algebraic" categories are the locally finitely presentable categories. The key observation leading to the definition of these is that in the familiar algebraic categories every object is a directed colimit of finitely presentable objects, i.e., objects specifiable by a finite number of generators and relations (a directed colimit is the colimit of a directed poset, i.e., a non empty poset such that for every pair of elements there is one greater than each of the two). In these familiar cases the property of an object A being finitely presentable has an equivalent categorial description: A is finitely presentable iff the functor $Hom(A, -)$ preserves directed colimits (or equivalently, filtered colimits)[1]. A category is called locally finitely presentable if it is cocomplete and has a *set* of finitely presentable objects such that every object is a directed (or filtered) colimit of objects from this set. Examples of such categories are all varieties of finitary many-sorted algebras as well as the categories of sets and posets and categories of **Set**-valued functors on a small category.

A further generalization is the notion of locally λ-presentable category, where λ is a regular cardinal: A poset is called λ-directed if every set of elements of cardinality strictly lesser than λ has an upper bound, an object A is called λ-presentable if $Hom(A, -)$ preserves colimits of (diagrams over) such posets and a category is λ-presentable iff it is cocomplete and there is a set of λ-presentable objects such that each object is a λ-directed colimit of these objects. Finally, a category is called locally presentable if it is locally λ-presentable for some regular cardinal λ. Model-theoretically the locally presentable categories have been described as

1. Categories of **Set**-models of limit sketches;
2. Categories of **Set**-models of essentially algebraic theories, i.e., equational theories of partial operations in which the domain of each operation is defined by equations in the preceding operations;
3. Categories of **Set**-models of so-called limit theories, which are certain infinitary first order theories. A locally λ-presentable category is the category of models of an $L_{\lambda\lambda}$theory. In particular, for $\lambda = \omega$, this means that locally finitely presentable theories are categories of models of finitary first order theories.

Examples of locally presentable (but not finitely presentable) categories include the category of Banach spaces and linear contractions as morphisms, that of convergence spaces and any complete lattice considered as a category.

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[1] The technical definition of filtered category is not needed in this article. The curious reader who wants know this definition and the sense of the "equivalence" between directed colimit and filtered colimit must see the results in the pages 13–16 of [1].

The last notion we will use here is that of λ-accessible category, which are defined like the locally λ-presentable categories except for requiring only λ-directed colimits to exist instead of arbitrary ones. An accessible category is, again, a category which is λ-accessible for some λ. These categories are exactly the categories of **Set**-models of arbitrary sketches or, alternatively, the categories of **Set**-models of so-called basic sentences of the infinitary first order logic $L_{\infty\infty}$ (which allows disjunctions and quantification over arbitrary (small) sets of formulas/variables). Examples not included in the previous classes are the categories of fields and Hilbert spaces.

For an introduction to these notions and their theories see [1] and [21]. There are also other related notions like locally multipresentable categories, weakly locally presentable categories and D-accessible categories (where D is some class of small categories) to some of which the related types of sketches and first order theories have also been identified.

The theory of accessible and locally presentable categories provides some powerful tools and also some conceptual clarifications. As an example of the first, we mention the theorem of [18] which says that an accessible category is locally presentable iff it is complete iff it is cocomplete. In Section 3 we prove a certain category of logics to be accessible and, since we know it not to have an initial object (hence not to be cocomplete), we know that it is not complete either. By investigating whether this category is of any of the types of categories mentioned in the previous paragraph we could further try to discover which type of limit is missing. For the second recall that the above types of categories were meant to capture the essential properties of categories of algebraic objects. Now there is a general duality phenomenon in mathematics between algebraic and geometrical/topological objects as witnessed by Stone duality, Gelfand duality, the duality between algebraic sets and k-algebras and many others. In [18] this informal observation is turned into a real mathematical statement by a theorem saying that the dual of a locally presentable category can not also be locally presentable except when it is a (category coming from a) poset. This seems to indicate that the notion of local presentability has succeeded in capturing some of the features of algebraic categories. In view of the ever returning question of "how algebraic" is logic, it thus seems to be interesting to investigate whether categories of logics are locally presentable.

Having seen these attempts to characterize categories of algebras, one is naturally led to think about whether the parallel between universal logic and universal algebra could be continued here. It has been argued, as mentioned in [7], that logical structures should be seen as one of the fundamental species of mathematical structures in the sense of Bourbaki. Assuming this to be the case one can wonder if it is possible to recognize the logical character of a category by categorial properties. If so, it should be in some different vein than the limit closure and generator properties defining locally presentable categories; after all we show below that a certain category of logics is locally presentable. The theory of fibred categories seems to fit very well for considering categories of logics as is convincingly shown in [19]. Indeed it seems reasonable that a category of logics should be fibred over a base category of signatures and finding further decisive properties (maybe like the base category being a category of free algebras) of the involved categories could give a sharp categorial picture.

2. The categories of signatures and logics

In the following subsection we will define a category of signatures which is equivalent to $\mathbf{Set}^{\mathbb{N}}$ and the main results will be that this category is locally finitely presentable (a special case of [1], ex. 1.12, p. 18) and that the finitely presentable objects are precisely those signatures which have finitely many connectives (a special case of [1], ex. 1.2(2), p. 9). The reader who is content with these explanations can skip to section 2.2 without many problems, for the others we will give a detailed exposition.

2.1. The category $\mathcal{S}$

The category $\mathcal{S}$ is the category of signatures and morphisms of signatures. In what follows, let $X = \{x_0, x_1, \ldots, x_n, \ldots\}$ be an enumerable set (written in a fixed order) as in [10].

2.1.1. What is $\mathcal{S}$? The objects of $\mathcal{S}$ are signatures. A signature Σ is a sequence of sets $\Sigma = (\Sigma_n)_{n\in\omega}$ such that $\Sigma_i \cap \Sigma_j = \emptyset$ for all $i < j < \omega$. We write $|\Sigma| = \bigsqcup_{n\in\omega} \Sigma_n = \bigcup_{n\in\omega} \Sigma_n \times \{n\}$ and we denote by $F(\Sigma)$ the set of all (propositional) formulas built with signature Σ over the variables in X. The notion of complexity $l(\varphi)$ of the formula φ is the usual:

- $l(\varphi) = 1$ if $\varphi \in X \cup \Sigma_0$;
- $l(\varphi) = 1 + l(\psi_0) + \cdots l(\psi_{n-1})$ if $\varphi = c(\psi_0, \ldots, \psi_{n-1})$, where $c \in \Sigma_n$ and $n > 0$.

If Σ, Σ' are signatures then a morphism $f : \Sigma \longrightarrow \Sigma'$ is a sequence of functions $f = (f_n)_{n\in\omega}$, where $f_n : \Sigma_n \longrightarrow \Sigma'_n$. For each morphism $f : \Sigma \longrightarrow \Sigma'$ there is only one function $\widehat{f} : F(\Sigma) \longrightarrow F(\Sigma')$, called the *extension of* f, such that:

- $\widehat{f}(x) = x$ if $x \in X$;
- $\widehat{f}(c) = f_0(c)$ if $c \in \Sigma_0$;
- $\widehat{f}(c(\psi_0, \ldots, \psi_{n-1}) = f_n(c)(\widehat{f}(\psi_0), \ldots, \widehat{f}(\psi_{n-1}))$ if $c \in \Sigma_n$, $n > 0$.

Then $\widehat{f}(\varphi(\psi_0, \ldots, \psi_{n-1}) = \widehat{f}(\varphi)(\widehat{f}(\psi_0), \ldots, \widehat{f}(\psi_{n-1}))$, by induction on $l(\varphi)$.

Composition in $\mathcal{S}$ is componentwise. The extension of the formula algebra of a composition is the extensions' composition. Identities in $\mathcal{S}$ are the sequences of identities on each level n, for $n \in \omega$. The extension of an identity is the identity function on the formula algebra.

2.1.2. Some facts about $\mathcal{S}$.

Remark 2.1. *About stratification:* For each signature Σ and for each $n \in \omega$ we consider the set of Σ-formulas: $F(\Sigma)[n] = \{\varphi \in F(\Sigma)$: the set of variables that occur in φ is *precisely* $\{x_0, \dots, x_{n-1}\}\}$. Let $f : \Sigma \longrightarrow \Sigma'$ be a signature morphism and $\widehat{f} : F(\Sigma) \longrightarrow F(\Sigma')$ be the induced formula algebra function; we can see directly by induction on the complexity of Σ-formulas that $\widehat{f}$ "preserves stratification": if $\varphi \in F(\Sigma)[n]$ then $\widehat{f}(\varphi) \in F(\Sigma')[n]$.

Fact 1. *About substitution*:

- For any *substitution* function $\sigma : X \longrightarrow F(\Sigma)$, there is only one *extension* $\widetilde{\sigma} : F(\Sigma) \longrightarrow F(\Sigma)$ such that $\widetilde{\sigma}$ is an "homomorphism": $\widetilde{\sigma}(x) = \sigma(x)$, for all $x \in X$ and $\widetilde{\sigma}(c_n(\psi_0, \dots, \psi_{n-1}) = c_n(\widetilde{\sigma}(\psi_0)), \dots, \widetilde{\sigma}(\psi_{n-1}))$, for all $c_n \in \Sigma_n$, $n \in \omega$; it follows that for any $\theta(x_0, \dots, x_{n-1}) \in F(\Sigma)\widetilde{\sigma}(\theta(x_0, \dots, x_{n-1})) = \theta(\sigma(x_0), \dots, \sigma(x_{n-1}))$. The *identity substitution* induces the identity homomorphism on the formula algebra; the *composition substitution* of the substitutions $\sigma', \sigma : X \longrightarrow F(\Sigma)$ is the substitution $\sigma'' : X \longrightarrow F(\Sigma)$, $\sigma'' = \sigma' \star \sigma \doteq \widetilde{\sigma'} \circ \sigma$ and $\widetilde{\sigma''} = \widetilde{\sigma' \star \sigma} \doteq \widetilde{\sigma'} \circ \widetilde{\sigma}$.
- Let $f : \Sigma \longrightarrow \Sigma'$ be a $\mathcal{S}$-morphism. Then for any substitution $\sigma : X \longrightarrow F(\Sigma)$ there is another substitution $\sigma' : X \longrightarrow F(\Sigma')$ such that $\widetilde{\sigma'} \circ \widehat{f} = \widehat{f} \circ \widetilde{\sigma}$.

$$\begin{tikzcd}
F(\Sigma) \arrow[r, "\widehat{f}"] \arrow[d, "\widetilde{\sigma}"'] \arrow[dr, phantom, "\circlearrowleft"] & F(\Sigma') \arrow[d, "\widetilde{\sigma'}"] \\
F(\Sigma) \arrow[r, "\widehat{f}"'] & F(\Sigma')
\end{tikzcd}$$

Proposition 2.2. *$\mathcal{S}$ is a complete and cocomplete category.*

Proof. Observe that $\mathcal{S}$ is equivalent to the functor category $\mathbf{Set}^{\mathbb{N}}$, where $\mathbb{N}$ is the discrete category with object class $\mathbb{N}$, then $\mathcal{S}$ has all small limits and colimits and they are componentwise.

Here we write the constructions but omit the (standard) verfications:

Limits. Let $\mathcal{I}$ be a small category and $D : \mathcal{I} \longrightarrow \mathcal{S}$, $(\Sigma^i \xrightarrow{f^h} \Sigma^j)_{(h:i \to j) \in \mathcal{I}}$ a diagram. Then $(\Sigma, (\pi^i)_{i \in Obj(\mathcal{I})})$ is the limit of this diagram if we take:

- $\Sigma_n = \{c = (c_i)_{i \in Obj(\mathcal{I})} \in \prod_{i \in obj\mathcal{I}} \Sigma^i_n$: for all $\mathcal{I}$-arrow $(i \xrightarrow{h} j)$, $f^h_n(c_i) = c_j\}$;
- $\pi^i_n : \Sigma_n \longrightarrow \Sigma^i_n$ such that if $c = (c_i)_{i \in Obj(\mathcal{I})} \in \Sigma_n$ then $\pi^i(c) = c_i$, $n \in \omega$ and $i \in I$.

Now we describe the most important kind of colimits in this work.

Filtered colimits. Let $(I, \leq)$ be a directed ordered set and $D : (I, \leq) \longrightarrow \mathcal{S}$, $(\Sigma^i \xrightarrow{f^{ij}} \Sigma^j)_{(i \leq j) \in I}$ a diagram. Then $(\Sigma, (\gamma^i)_{i \in I})$ is the colimit of this diagram if we take:

- $\Sigma_n = (\bigsqcup_{i \in I} (\Sigma^i)_n)/\sim_n$ where, if $c_i \in \Sigma^i_n, c_j \in \Sigma^j_n (c_i, i) \sim_n (c_j, j)$ iff there is a $k \geq i, j$ such that $(f^{ik})_n(c_i) = (f^{jk})_n(c_j)$, $n \in \omega$: it follows from the directness assumption that $\sim_n$ is an equivalence relation on $\bigsqcup_{i \in I} (\Sigma^i)_n$;
- $\gamma^i_n : \Sigma^i_n \longrightarrow \Sigma_n$ such that if $c_i \in \Sigma^i_n$ then $\gamma^i_n(c_i) = [(c_i, i)]$, $n \in \omega$ and $i \in I$.

Colimits. Let $\mathcal{I}$ be a small category and $D : \mathcal{I} \longrightarrow \mathcal{S}$, $(\Sigma^i \xrightarrow{f^h} \Sigma^j)_{(h:i \to j) \in \mathcal{I}}$ a diagram. Then $(\Sigma, (\gamma^i)_{i \in Obj(\mathcal{I})})$ is the colimit of this diagram if we take:

- $\Sigma_n = (\bigsqcup_{i \in Obj(\mathcal{I})} (\Sigma^i)_n)/\sim_n$, $n \in \omega$ where $\sim_n$ is the smallest equivalence relation on $\bigsqcup_{i \in I} (\Sigma^i)_n$ such that for all $\mathcal{I}$-arrow $(i \xrightarrow{h} j)$ if $c_i \in \Sigma^i_n$ then $(c_i, i) \sim_n (f^h_n(c_i), j)$;
- $\gamma^i_n : \Sigma^i_n \longrightarrow \Sigma_n$ such that if $c_i \in \Sigma^i_n$ then $\gamma^i_n(c_i) = [(c_i, i)]$, $n \in \omega$ and $i \in Obj(\mathcal{I})$. □

Fact 2. *About monomorphisms and epimorphisms in* $\mathcal{S}$: Let $f : \Sigma \longrightarrow \Sigma'$ be a signature morphism. Then

(i) f is a $\mathcal{S}$-monomorphism iff, for all $n \in \omega$, $f_n : \Sigma_n \longrightarrow \Sigma'_n$ is injective; f is a $\mathcal{S}$-epimorphism iff, for all $n \in \omega$, $f_n : \Sigma_n \longrightarrow \Sigma'_n$ is surjective;

(ii) if f is a $\mathcal{S}$-monomorphism then $\widehat{f} : F(\Sigma) \longrightarrow F(\Sigma')$ is injective; if f is a $\mathcal{S}$-epimorphism then $\widehat{f} : F(\Sigma) \longrightarrow F(\Sigma')$ is surjective.

2.1.3. $\mathcal{S}$ is a locally presentable category. We have just seen that the category $\mathcal{S}$ is complete and cocomplete. Furthermore, it has other nice categorial property: it is a *finitely accessible category*. Therefore $\mathcal{S}$ is a finitely locally presentable category (a complete and cocomplete finitely accessible category).

Fact 3. *Aditional facts on filtered colimits in* $\mathcal{S}$: Let $D : (I, \leq) \longrightarrow \mathcal{S}$, $(\Sigma^i \xrightarrow{f^{ij}} \Sigma^j)_{(i \leq j) \in I}$ be a directed diagram and let $(\Sigma', (\alpha^i)_{i \in I})$ be a commutative cocone over the diagram D:

(i) $(\Sigma', (\alpha^i)_{i \in I})$ is "the" universal colimit cocone of diagram D iff:
  - $\Sigma'_n = \bigcup_{i \in I} \alpha^i_n[\Sigma^i_n]$, $n \in \omega$;
  - if $c_i \in \Sigma^i_n, c_j \in \Sigma^j_n$ are such that $\alpha^i_n(c_i) = \alpha^j_n(c_j)$, then there is a $k \geq i, j$ such that $f^{ik}_n(c_i) = f^{jk}_n(c_j)$, $n \in \omega$.

(ii) If $(\Sigma', (\alpha^i)_{i \in I})$ is "the" universal colimit cocone of diagram D, as noticed above, for all $n \in \omega$, $\Sigma'_n = \bigcup_{i \in I} \alpha^i_n[\Sigma^i_n]$. It follows easily from the directness condition, by induction on complexity, that any formula in the colimit signature can be "obtained at given defined time", that is, $F(\Sigma') = \bigcup_{i \in I} \widehat{\alpha^i}[F(\Sigma^i)]$ and, analogously, any finite set of formulas in the colimit signature can be "obtained at a given defined time";

(iii) if, for all $(i \leq j) \in I$ and all $n \in \omega$, $f^{ij}_n : \Sigma^i_n \longrightarrow \Sigma^j_n$ is injective, then if $(\Sigma', (\alpha^i)_{i\in I})$ is "the" universal colimit cocone of diagram D, then for all $n \in \omega$, $\alpha^i_n : \Sigma^i_n \longrightarrow \Sigma'_n$ is injective.

Proof. Here we only prove item (iii). Let $j \in I$ and $n \in \omega$. Let $c^i, d^i \in \Sigma^i_n$ such that $\alpha^i_n(c^i) = \alpha^i_n(d^i)$. Therefore, by item (i) above, there is $k \geq i$ such that $f^{ik}_n(c^i) = f^{ik}_n(d^i)$ and, as $f^{ik}_n : \Sigma^i_n \longrightarrow \Sigma^k_n$ is injective, we have $c^i = d^i$. ☐

Proposition 2.3. *Any signature is a directed colimit of finite type signatures.*

Proof. Consider I as the set of all Σ' such that, for all $n \in \omega$, $\Sigma'_n \subseteq \Sigma_n$ and $|\Sigma'| \subseteq_{fin} |\Sigma|$. Take in I the pointwise order relation $\Sigma' \leq \Sigma''$ iff for all $n \in \omega$, $\Sigma'_n \subseteq \Sigma_n$. Then:

- $(I, \leq)$ is a directed ordered set;
- The obvious diagram $D : (I, \leq) \longrightarrow \mathcal{S} : (\Sigma' \leq \Sigma'') \mapsto (\Sigma' \hookrightarrow \Sigma'')$ is such that $(\Sigma, (\Sigma' \overset{i_{\Sigma'}}{\hookrightarrow} \Sigma)_{\Sigma' \in I})$ is a commutative D-cocone;
- By the characterization in Fact 3.(i), $(\Sigma, (\Sigma' \overset{i_{\Sigma'}}{\hookrightarrow} \Sigma)_{\Sigma' \in I})$ is a colimit cocone over D, so Σ is a directed colimit of finite type (sub)signatures. ☐

Proposition 2.4. *A signature is finitely presentable if and only if it is of finite type.*

Proof.
$(\Leftarrow)$. Let Σ' be a signature of finite type, that is, $|\Sigma'|$ is finite, and consider $D : (I, \leq) \longrightarrow \mathcal{S}$, $(\Sigma^i \xrightarrow{f^{ij}} \Sigma^j)_{(i \leq j) \in I}$ a directed diagram of signatures. Then the canonical arrow $k : colim_{i\in I}\mathcal{S}(\Sigma', \Sigma_i) \longrightarrow \mathcal{S}(\Sigma', colim_{i\in I}\Sigma_i)$ is an isomorphism.

$$\left[\left(\Sigma' \xrightarrow{h^i} \Sigma_i\right), i)\right] \mapsto \left(\Sigma' \xrightarrow{h^i} \Sigma_i \xrightarrow{\gamma_i} colim_{i\in I}\Sigma_i\right).$$

In order to prove that k is surjective, we have to prove that, for each signature morphism $h : \Sigma' \longrightarrow colim_{i\in I}\Sigma_i$, there is an $i \in I$ and a signature morphism $h^i : \Sigma' \longrightarrow \Sigma_i$ such that $(\Sigma' \xrightarrow{h} colim_{i\in I}\Sigma_i) = (\Sigma' \xrightarrow{h_i} \Sigma_i \xrightarrow{\gamma_i} colim_{i\in I}\Sigma_i)$. As $|\Sigma'|$ is a finite set, there is only a finite set $\{n_0, \ldots, n_{t-1}\} \subseteq \mathbb{N}$ such that, for all $r < t$, Σ'_{n_r} is a finite non-empty set. Since that for each n_r, Σ_{n_r} is finite and $(I, \leq)$ is a directed ordered set, there is an $i_r \in I$ such that $h_{n_r}[\Sigma'{}_{n_r}] \subseteq \gamma^{i_r}_{n_r}[\Sigma^{i_r}_{n_r}]$. As there is only a finite set $\{n_0, \ldots, n_{t-1}\} \subseteq \mathbb{N}$ such that for all $r < t$, $\Sigma'_{n_r} \neq \emptyset$ and $(I, \leq)$ is a directed ordered set, take an $i \geq i_0, \ldots, i_{t-1}$. As $\gamma^{i_r} = \gamma^i \circ f^{i_r i}$, it follows that, for all $n \in \omega$, $h_n[\Sigma'{}_n] \subseteq \gamma^i_n[\Sigma^i_n]$. Just take, for each $n \in \omega$, $h^i_n : \Sigma'_n \longrightarrow \Sigma^i_n$ such that, for each $c'_n \in \Sigma'_n$, $h^i_n(c'_n) \in \Sigma^i_n$ is such that $h_n(c'_n) = [(h^i_n(c'_n), i)]$ and so $h = \gamma^i \circ h^i$.

In order to prove that k is injective, we have to prove that, for each signature morphism $h : \Sigma' \longrightarrow colim_{i\in I}\Sigma_i$ such that there are $i_0, i_1 \in I$ and signature morphisms $h^{i_0} : \Sigma' \longrightarrow \Sigma_{i_0}$, $h^{i_1} : \Sigma' \longrightarrow \Sigma_{i_1}$ such that $\gamma^{i_0} \circ h^{i_0} = h = \gamma^{i_1} \circ h^{i_1}$, then there is a $j \geq i_0, i_1$ and a $h^j : \Sigma' \longrightarrow \Sigma_j$ such that $f^{i_0 j} \circ h^{i_0} = h^j = f^{i_1 j} \circ h^{i_1}$. Since that for each $i \in \{i_0, i_1\}$ and $n \in \omega$, $h^i_n : \Sigma'_n \longrightarrow \Sigma^i_n$ is such that for each $c'_n \in \Sigma'_n$, $h^i_n(c'_n) \in \Sigma^i_n$ is such that $[(h^{i_0}_n(c'_n), i_0)] = h_n(c'_n) = [(h^{i_1}_n(c'_n), i_1)]$,

because $h = \gamma^i \circ h^i$, then there is a $j(c'_n) \geq i_0, i_1$ such that $f_n^{i_0 j(c'_n)}(h_n^{i_0}(c'_n)) = f_n^{i_1 j(c'_n)}(h_n^{i_1}(c'_n)) \in \Sigma_n^{j(c'_n)}$. As $|\Sigma'|$ is a finite set then $\{j(c'_n) : c'_n \in \Sigma'_n\}$ is a finite set and, as $(I, \leq)$ is a directed ordered set, there is a $j_n \geq j(c'_n)$ for each $c'_n \in \Sigma'_n$. Again, as $|\Sigma'|$ is a finite set, there is only a finite set $\{n_0, \ldots, n_{t-1}\} \subseteq \mathbb{N}$ such that, for all $r < t$, $\Sigma'_{n_r} \neq \emptyset$. As $(I, \leq)$ is a directed ordered set, take a $j \geq j_{n_0}, \ldots, j_{n_t}$. Then, as D is a diagram, for each $n \in \omega$, $f_n^{i_0 j} \circ h_n^{i_0} = f_n^{i_1 j} \circ h_n^{i_1}$ take $h^j = f^{ij} \circ h^i$, $i \in \{i_0, i_1\}$.
($\Rightarrow$). Let Σ be a finitely presentable logic. Then, by Proposition 2.3, there is a directed diagram of finite type logics $D : (I, \leq) \longrightarrow \mathcal{S}$, $(\Sigma^i \xrightarrow{f^{ij}} \Sigma^j)_{(i \leq j) \in I}$ such that there is an isomorphism $h : l \xrightarrow{\cong} colim_{i \in I} \Sigma^i$. Then, as the canonical morphism is invertible $k : colim_{i \in I} \mathcal{S}(\Sigma, \Sigma^i) \xrightarrow{\cong} \mathcal{S}(\Sigma, colim_{i \in I} \Sigma^i)$, there is a factorization of $h : (\Sigma \xrightarrow{h} colim_{i \in I} \Sigma^i) = (\Sigma \xrightarrow{h^i} \Sigma^i \xrightarrow{\gamma^i} colim_{i \in I} \Sigma^i)$. Then, as h is an isomorphism, $h^i : \Sigma \longrightarrow \Sigma^i$ is an $\mathcal{S}$-section. In particular there is a sequence of injections $(h_n^i : \Sigma_n \rightarrowtail \Sigma_n^i)_{n \in \omega}$ so, as $|\Sigma^i|$ is finite, then $|\Sigma|$ is finite. □

Theorem 2.5. *The category $\mathcal{S}$ is a finitely locally presentable category, i.e., $\mathcal{S}$ is an accessible category that is cocomplete and complete.*

Proof. Direct consequence of Propositions 2.3, 2.4 and 2.2. □

Corollary 2.6. (i) *The Yoneda functor $Y : \mathcal{S}_{fp} \longrightarrow Set^{(\mathcal{S}_{fp})^{op}}$ has an extension to a functor $Y' : \mathcal{S} \longrightarrow Set^{(\mathcal{S}_{fp})^{op}}$, $\Sigma \mapsto Y'(\Sigma) = \mathcal{L}(\iota(.), \Sigma)$ that is full and faithful;*

(ii) *Let $Flat(\mathcal{S}_{fp}, Set)$ be the full subcategory of $Set^{(\mathcal{S}_{fp})^{op}}$ whose objects are the functors that are filtered colimits of representable functors. Then $Flat(\mathcal{S}_{fp}, Set)$ is the "essential image" of Y' and so his restriction functor $E : \mathcal{S} \longrightarrow Flat(\mathcal{S}_{fp}, Set)$ is an equivalence of categories;*

(iii) *$Flat(\mathcal{S}_{fp}, Set)$ coincides with the category of Set-valued functors that preserve finite limits;*

(iv) *Y' has a left adjoint.*

Proof. For (i) and (ii) see [9], Theorem 5.3.5 (p. 265) or [1], Theorem 2.26 (p. 83) or [22], Observation 1.6 (p. 46). For (iii) and (iv) see [1], Theorem 1.46 (p. 38). □

2.2. The category $\mathcal{L}$

The category $\mathcal{L}$ is the category of propositional logics and translations as morphisms. This is a category "built above" the category $\mathcal{S}$, that is, there is an obvious forgetful functor $U : \mathcal{L} \longrightarrow \mathcal{S}$.

2.2.1. What is $\mathcal{L}$? The objects of $\mathcal{L}$ are logics. A logic is an ordered pair $l = (\Sigma, \vdash)$ where Σ is an object of $\mathcal{S}$ and $\vdash$ codifies the "consequence operator" on $F(\Sigma)$ — $\vdash$ is a binary relation, a subset of $Parts(F(\Sigma)) \times F(\Sigma)$, such that $Cons(\Gamma) = \{\varphi \in F(\Sigma) : \Gamma \vdash \varphi\}$, for all $\Gamma \subseteq F(\Sigma)$, gives a structural finitary closure operator on $F(\Sigma)$:

- inflationary: $\Gamma \subseteq Cons(\Gamma)$;

- increasing: $\Gamma_0 \subseteq \Gamma_1 \Rightarrow Cons(\Gamma_0) \subseteq Cons(\Gamma_1)$;
- idempotent: $Cons(Cons(\Gamma)) \subseteq Cons(\Gamma)$;
- finitary: $Cons(\Gamma) = \bigcup\{Cons(\Gamma') : \Gamma' \subseteq_{fin} \Gamma\}$;
- structural: $\widetilde{\sigma}(Cons(\Gamma)) \subseteq Cons(\widetilde{\sigma}(\Gamma))$, for each substitution $\sigma : X \to F(\Sigma)$.

In [20], Łoś and Suszko give a characterization of usual provability notion in Hilbert calculi by the consequence operators with the above properties.

We say that a logic $l = (\Sigma, \vdash)$ is of *finite type* when $|\Sigma|$ is a finite set and $\vdash$ is determined, in the sense of [20][2], by a finite set of axioms and inference rules.

If $l = (\Sigma, \vdash), l' = (\Sigma', \vdash')$ are logics then a *translation morphism* $f : l \longrightarrow l'$ is a signature morphism $f : \Sigma \longrightarrow \Sigma'$ that "preserves the consequence relation", that is, for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, if $\Gamma \vdash \psi$ then $\widehat{f}[\Gamma] \vdash' \widehat{f}(\psi)$. We say that a morphism $f : l \longrightarrow l'$ is a *conservative translation morphism* if for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, $\Gamma \vdash \psi$ if and only if $\widehat{f}[\Gamma] \vdash' \widehat{f}(\psi)$. Composition and identities are similar to $\mathcal{S}$.

2.2.2. Some facts about $\mathcal{L}$.

Definition 2.7. There is a natural definition of order between consequence relations on each signature Σ: for each pair $\vdash, \vdash'$ of consequence relations over Σ we have the equivalence between the items below:

- For each $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, $\Gamma \vdash \psi \Rightarrow \Gamma \vdash' \psi$;
- The identity signature morphism over Σ, $id_\Sigma : \Sigma \longrightarrow \Sigma$, is a translation morphism $id_\Sigma : (\Sigma, \vdash) \longrightarrow (\Sigma, \vdash')$.

We write $\vdash \, \leq \, \vdash'$ when the conditions above are satisfied.

Fact 4. The set of consequence relations on a signature Σ, denoted by $\mathcal{L}_\Sigma$, is a complete lattice. It is in fact an algebraic lattice where the compact elements are the "finitely generated logics", the logics over Σ given by a finite set of axioms and a finite set of (finitary) inference rules.

Proof. Here we just give a sketch of proof. In the following we have other similar (but more general) propositions where we supply full proofs.

Let Σ be a signature.

Infs. Consider I a set and $D = \{l^i = (\Sigma, \vdash_i)\}_{i \in I}$ a family of logics over the signature Σ. Now, for each $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, define that $\Gamma \vdash \psi \Leftrightarrow$ there is $\Gamma' \subseteq_{fin} \Gamma$ such that $(\forall i \in I)(\Gamma' \vdash_i \varphi)$, then $(\Sigma, \vdash)$ is a logic and $l = (\Sigma, \vdash)$ is the infimum of the family D in $\mathcal{L}$. In fact, this follows from them items below:

- $l \in Obj(\mathcal{L})$;
- $id_\Sigma \in \mathcal{L}(l, l^j)$ for all $j \in I$;
- if $l' = (\Sigma, \vdash')$ is a logic over the signature Σ such that $id_\Sigma \in \mathcal{L}(l', l^j)$ for all $j \in I$, then $id_\Sigma \in \mathcal{L}(l', l)$.

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[2] A formula is demonstrable from a given set of hypothesis iff there is a finite sequence of formulas such that the last one is the thesis and each formula is an hypothesis or an instance of axiom or is obtained from the previous formulas in the sequence by an instantiation of a (finitary) inference rule.

Directed sups. Consider I a set and $D = \{l^i = (\Sigma, \vdash_i)\}_{i \in I}$ an upward directed family of logics over the signature Σ, that is, for each $i, j \in I$ there is a $k \in I$ such that $id_\Sigma \in \mathcal{L}(l^i, l^k)$, $id_\Sigma \in \mathcal{L}(l^j, l^k)$. Now, for each $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, define that $\Gamma \vdash \psi \Leftrightarrow$ there is $\Gamma' \subseteq_{fin} \Gamma$ and there is an $i \in I$ such that $\Gamma' \vdash_i \varphi$, then $(\Sigma, \vdash)$ is a logic and $l = (\Sigma, \vdash)$ is the supremum of the family D in $\mathcal{L}$. In fact, this follows from the items below:

- $l \in Obj(\mathcal{L})$;
- $id_\Sigma \in \mathcal{L}(l^j, l)$ for all $j \in I$;
- if $l' = (\Sigma, \vdash')$ is a logic over the signature Σ such that $id_\Sigma \in \mathcal{L}(l^j, l')$ for all $j \in I$, then $id_\Sigma \in \mathcal{L}(l, l')$.

Sups. As usual, the supremum of a family of logics can be obtained taking the infimum of the set of upper bounds of that family of logics. A more objective characterization of suprema can be given but we postpone that because this can be easily described by more general results below (see Proposition 2.11).

Compact consequence relations. A consequence relation $\vdash'$ over Σ is compact if for each set I, each $D = \{l^i = (\Sigma, \vdash_i)\}_{i \in I}$ a upward directed family of logics over the signature Σ, if $\vdash' \leq \bigvee_{i \in I} \vdash_i$ then there is an $i \in I$ such that $\vdash' \leq \vdash_i$. It follows easily that this condition is equivalent to the "stronger" condition: for each set J, each $D = \{l^j = (\Sigma, \vdash_j)\}_{j \in J}$ a family of logics over the signature Σ, if $\vdash' \leq \bigvee_{j \in J} \vdash_j$ then there is a finite subset $J' \subseteq J$ such that $\vdash' \leq \bigvee_{j \in J'} \vdash_j$.[3] A consequence relation on Σ is compact if and only if it is a finitely generated consequence relation on Σ. Any consequence relation on Σ is the directed supremum of its compact (sub)consequence relations on Σ. $\square$

Remark 2.8. As the set of consequence operators (or consequence relations) on a signature Σ is a complete lattice, there exists the logic generated by any function $W : P(F(\Sigma)) \longrightarrow P(F(\Sigma))$: it is enough to take the infimum of the family of all consequence relations on Σ that are upper bounds of the "proto-consequence relation" associated with the "proto-consequence operator" W.

Definition 2.9. Direct image and inverse image: let $f : \Sigma \longrightarrow \Sigma'$ be a $\mathcal{S}$-morphism:

- **Inverse image:** if $l' = (\Sigma', \vdash') \in Obj(\mathcal{L})$ then for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$ define $\Gamma \vdash_{f^\star(\vdash')} \psi$ iff $\widehat{f}[\Gamma] \vdash' \widehat{f}(\psi)$;
- **Direct image:** if $l = (\Sigma, \vdash) \in Obj(\mathcal{L})$ then for all $\Gamma' \cup \{\psi'\} \subseteq F(\Sigma')$ define $\Gamma' \vdash_{f_\star(\vdash)} \psi'$ iff there is a finite sequence of Σ'-formulas $(\phi'_0, \dots, \phi'_t)$ such that:
    - $\phi'_t = \psi'$;
    - for all $p \leq t$ at least one of the alternatives below occurs:
        - "ϕ'_p is a hypothesis": $\phi'_p \in \Gamma'$;
        - "ϕ'_p is an instance of an l-axiom": there is a $\theta_p \in F(\Sigma)$ such that $\vdash \theta_p$ and there is a substitution $\sigma' : X \longrightarrow F(\Sigma')$ such that $\widetilde{\sigma}'(\widehat{f}(\theta_p)) = \phi'_p$;

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[3] Just observe that any sup of a family coincides with a sup of a directed family: for each set J take $I = P_{fin}(J)$ then, for each $J' \subseteq_{fin} J$, define $\vdash_{J'} = \bigvee_{j \in J'} \vdash_j \dots$.

$\ast$ “ϕ'_p is a direct consequence of an instance of l-inference rule applied over previous members in the sequence”: there is a $\Delta_p \cup \{\theta_p\} \subseteq_{fin} F(\Sigma)$ such that $\Delta_p \vdash \theta_p$ and there is a substitution $\sigma' : X \longrightarrow F(\Sigma')$ such that $\widetilde{\sigma}'(\widehat{f}(\theta_p)) = \phi'_j$ and $\widetilde{\sigma}'[\widehat{f}[\Delta_p]] \subseteq \{\phi'_0, \ldots, \phi'_{j-1}\}$.

Fact 5. *About direct image and inverse image:* Let $f : \Sigma \longrightarrow \Sigma'$ be a $\mathcal{S}$-morphism and let $l = (\Sigma, \vdash), l' = (\Sigma', \vdash')$ be logics $l, l' \in Obj(\mathcal{L})$. Then (i)* and (i)$_\star$ hold, and (ii)*, (ii)$_{tm}$ and (ii)$_\star$ are equivalent:

(i)* if $l' = (\Sigma', \vdash') \in Obj(\mathcal{L})$ then $f^\star(l') = (\Sigma, \vdash_{f^\star(\vdash')}) \in Obj(\mathcal{L})$;
(i)$_\star$ if $l = (\Sigma, \vdash) \in Obj(\mathcal{L})$ then $f_\star(l) = (\Sigma', \vdash_{f_\star(\vdash)}) \in Obj(\mathcal{L})$.
(ii)* $\vdash \leq \ f^\star(\vdash')$;
(ii)$_{tm}$ $f : (\Sigma, \vdash) \longrightarrow (\Sigma', \vdash')$ is a translation morphism;
(ii)$_\star$ $f_\star(\vdash) \leq' \vdash'$.

Proof. (Sketch)

(i) The proof of (i)* is omitted; the proof of (i)$_\star$ is analogous to the item Colimits.(a) in Proposition 2.11;

(ii) The equivalence (ii)* $\Leftrightarrow$ (ii)$_{tm}$ follows directly from the definitions; the implication (ii)$_\star$ $\Rightarrow$ (ii)$_{tm}$ is analogous to the item Colimits.(b) in Proposition 2.11; the implication (ii)$_{tm}$ $\Rightarrow$ (ii)$_\star$ is analogous to the item Colimits.(c) in Proposition 2.11. □

Remark 2.10. It follows easily from the facts above that the forgetful functor $U : \mathcal{L} \longrightarrow \mathcal{S} : ((\Sigma, \vdash) \xrightarrow{f} (\Sigma', \vdash')) \mapsto (\Sigma \xrightarrow{f} \Sigma')$ has left and right adjoint functors: the left adjoint $T : \mathcal{S} \longrightarrow \mathcal{L}$ and the right adjoint $V : \mathcal{S} \longrightarrow \mathcal{L}$ take a signature Σ to, respectively, $T(\Sigma) = (\Sigma, \vdash_{min})$ (the first element of $\mathcal{L}_\Sigma$) and $V(\Sigma) = (\Sigma, \vdash_{max})$ (the last element of $\mathcal{L}_\Sigma$).

Proposition 2.11. *The category $\mathcal{L}$ is complete and cocomplete and the forgetful functor $U : \mathcal{L} \longrightarrow \mathcal{S}$ creates all small limits and colimits.*

Proof. Before begining the proof (that is long and tedious!) let us claim that this result is not surprising and has an “easy proof” in terms of the concepts of infima/suprema and inverse/direct image: the limit cones in $\mathcal{L}$ are the underlying limit cones in $\mathcal{S}$ when we take the “limit consequence relation” on the limit signature as the *infimum* of the set of the *inverse images* of consequence relations through each $\mathcal{S}$-morphism in the $\mathcal{S}$-limit cone; analogously the colimit cocones in $\mathcal{L}$ are the underlying colimit cocones in $\mathcal{S}$ when we take the “colimit consequence relation” on the colimit signature as the *supremum* of the set of the *direct images* of consequence relations through each $\mathcal{S}$-morphism in the $\mathcal{S}$-colimit cocone. However we choose present the full proofs by two reasons: first we have not present any explicit calculation for the concepts of infima/suprema and inverse/direct image (see, respectively Fact 4 and Fact 5), second, and most importantly, we do need the explicit construction of the *direct colimit logic* in the proofs of Section 3.

We split the proof in three sections: limits[4], filtered colimits and colimits.

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[4] In [10] there is a similar proof for products, but with another notion of signature morphism.

Limits. Let $\mathcal{I}$ be a small category and $D : \mathcal{I} \longrightarrow \mathcal{L}$, $((\Sigma^i, \vdash_i) \xrightarrow{f^h} (\Sigma^j, \vdash_j))_{(h:i\to j)\in\mathcal{I}}$ a diagram, and take $(\Sigma, (\pi^i)_{i\in Obj(\mathcal{I})})$ the limit of the underlying diagram ($\mathcal{I} \xrightarrow{D} \mathcal{S} \xrightarrow{U} \mathcal{L}$). For all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, define that $\Gamma \vdash \psi \Leftrightarrow$ there is $\Gamma^- \subseteq_{fin} \Gamma$ such that for all $i \in Obj(\mathcal{I})$ $\widehat{\pi}^i[\Gamma^-] \vdash_i \widehat{\pi}^i(\psi)$[5], then $l = (\Sigma, \vdash)$ is a logic and $(l, (\pi^i)_{i\in Obj(\mathcal{I})})$ is the limit of D in $\mathcal{L}$. In fact, this follows from (a), (b) and (c) below:

(a) $l \in Obj(\mathcal{L})$;
(b) $\pi^j \in \mathcal{L}(l, l^j)$, for all $j \in I$;
(c) if $(l', (\alpha^i)_{i\in Obj(\mathcal{I})})$ is a commutative cone over the diagram D then the unique signature morphism $\alpha : \Sigma' \longrightarrow \Sigma$ such that $\alpha^i = \pi^i \circ \alpha$, $i \in Obj(\mathcal{I})$ preserves the consequence relation.

Now we prove (a), (b) and (c).

(a) it follows directly from the definition of $\vdash$ that it gives a *finitary* and *increasing* consequence operator. It is also *inflationary* because if $\psi \in \Gamma$, take any $\Gamma^- \subseteq_{fin} \Gamma$ such that $\psi \in \Gamma^-$ then, for all $i \in Obj(\mathcal{I})$, $\widehat{\pi}^i(\psi) \in \widehat{\pi}^i[\Gamma^-]$. Since $\vdash_i$ is inflationary, then $\widehat{\pi}^i[\Gamma^-] \vdash_i \widehat{\pi}^i(\psi)$. We have $\Gamma \vdash \psi$.

Idempotent: Let $\psi \in F(\Sigma)$ such that $\overline{\Gamma} \vdash \psi$, where $\overline{\Gamma} = \{\theta \in F(\Sigma) : \Gamma \vdash \theta\}$. Let us prove that $\Gamma \vdash \psi$. Since $\vdash$ is finitary, let $\Delta \subseteq_{fin} \overline{\Gamma}$ be such that $\Delta \vdash \psi$. Then, for each $\theta \in \Delta$, let $\Gamma^\theta \subseteq_{fin} \Gamma$ be such that $\Gamma^\theta \vdash \theta$. It follows that $\Gamma^- = \bigcup_{\theta\in\Delta}\Gamma^\theta$ satisfies $\Gamma^- \subseteq_{fin} \Gamma$ and, as $\vdash$ is increasing, for each $\theta \in \Delta$, $\Gamma^- \vdash \theta$. Then, as $\vdash_i$ gives an inflationary operator, for each $i \in Obj(\mathcal{I})$, it follows from the definition of $\Delta \vdash \psi$ that for each $i \in Obj(\mathcal{I})$, $\widehat{\pi}^i[\Delta] \vdash_i \widehat{\pi}^i(\psi)$. Analogously, as $\Gamma^- \vdash \theta$ for each $\theta \in \Delta$, then for each $i \in Obj(\mathcal{I})$, $\widehat{\pi}^i[\Gamma^-] \vdash_i \widehat{\pi}^i(\theta)$, for each $\theta \in \Delta$. Now, as $\vdash_i$ gives an idempotent operator, then $\widehat{\pi}^i[\Gamma^-] \vdash_i \widehat{\pi}^i(\psi)$, for each $i \in Obj(\mathcal{I})$. Therefore, as $\Gamma^- \subseteq_{fin} \Gamma$, we have $\Gamma \vdash \psi$.

Structural: Let $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$ be such that $\Gamma \vdash \psi$. We have to prove that for any substitution $\sigma : X \longrightarrow F(\Sigma)$ we have $\widetilde{\sigma}[\Gamma] \vdash \widetilde{\sigma}(\psi)$. Let $\Gamma^- \subseteq_{fin} \Gamma$ be such that, for all $i \in Obj(\mathcal{I})$, $\widehat{\pi}^i[\Gamma^-] \vdash_i \widehat{\pi}^i(\psi)$. Since $\widetilde{\sigma}[\Gamma^-] \subseteq_{fin} \widetilde{\sigma}[\Gamma]$, we have $\widetilde{\sigma}[\Gamma] \vdash \widetilde{\sigma}(\psi)$ if we prove that $\widehat{\pi}^i[\widetilde{\sigma}[\Gamma^-]] \vdash_i \widehat{\pi}^i(\widetilde{\sigma}(\psi))$, for each $i \in Obj(\mathcal{I})$. Now, from Fact 1.(ii), for all $i \in Obj(\mathcal{I})$ there is a substitution $\sigma^i : X \longrightarrow F(\Sigma^i)$ such that $\widehat{\pi}^i \circ \widetilde{\sigma} = \widetilde{\sigma}^i \circ \widehat{\pi}^i$. Then, for each $i \in Obj(\mathcal{I})$, since $\widehat{\pi}^i[\Gamma^-] \vdash_i \widehat{\pi}^i(\psi)$ and $\vdash_i$ gives a structural operator, we have that $\widetilde{\sigma}^i[\widehat{\pi}^i[\Gamma^-]] \vdash_i \widetilde{\sigma}^i(\widehat{\pi}^i(\psi))$. So we have $\widehat{\pi}^i[\widetilde{\sigma}[\Gamma^-]] \vdash_i \widehat{\pi}^i(\widetilde{\sigma}(\psi))$, for each $i \in Obj(\mathcal{I})$. Therefore $\widetilde{\sigma}[\Gamma] \vdash \widetilde{\sigma}(\psi)$.

(b) Let $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$ be such that $\Gamma \vdash \psi$. Then select a $\Gamma^- \subseteq_{fin} \Gamma$ such that for all $i \in Obj(\mathcal{I})$, $\widehat{\pi}^i[\Gamma^-] \vdash_i \widehat{\pi}^i(\psi)$. Since $\widehat{\pi}^i[\Gamma^-] \subseteq \widehat{\pi}^i[\Gamma]$ and $\vdash_i$ is inflationary, for each $i \in Obj(\mathcal{I})$, we have $\widehat{\pi}^i[\Gamma] \vdash_i \widehat{\pi}^i(\psi)$. So the signature morphism $\pi^i : \Sigma \longrightarrow \Sigma^i$ gives also a *translation morphism* $\pi^i : (\Sigma, \vdash) \longrightarrow (\Sigma^i, \vdash_i)$, for each $i \in Obj(\mathcal{I})$.

(c) Let $\Gamma' \cup \{\psi'\} \subseteq F(\Sigma')$ be such that $\Gamma' \vdash' \psi'$. We have to prove that $\widehat{\alpha}[\Gamma'] \vdash \widehat{\alpha}(\psi')$. Since $\vdash'$ is finitary, select $\Gamma'^- \subseteq_{fin} \Gamma'$ such that $\Gamma'^- \vdash' \psi'$. Then, as $\alpha^i : l' \longrightarrow l^i$ is a translation morphism, $\widehat{\alpha}^i[\Gamma'^-] \vdash_i \widehat{\alpha}^i(\psi')$, for each $i \in Obj(\mathcal{I})$. So,

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[5] This definition also works for the terminal logic $l = (\Sigma, \vdash)$ where Σ is the terminal signature ($card(\Sigma_n) = 1, \forall n \in \omega$), and for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, $\Gamma \vdash \psi$.

as $\alpha^i = \pi^i \circ \alpha$, we have $\widehat{\pi}^i[\widehat{\alpha}[\Gamma'^{-}]] \vdash_i \widehat{\pi}^i(\widehat{\alpha}(\psi'))$, for each $i \in Obj(\mathcal{I})$. Now, since $\widehat{\alpha}[\Gamma'^{-}] \subseteq_{fin} \widehat{\alpha}[\Gamma']$, we have from the definition that $\widehat{\alpha}[\Gamma'] \vdash \widehat{\alpha}(\psi')$. So the signature morphism $\alpha : \Sigma' \longrightarrow \Sigma$ gives also a translation morphism $\alpha : (\Sigma', \vdash') \longrightarrow (\Sigma, \vdash)$.

Filtered colimits. Let $(I, \leq)$ be a directed ordered set and $D : (I, \leq) \longrightarrow \mathcal{L}$, $((\Sigma^i, \vdash_i) \xrightarrow{f^{ij}} (\Sigma^j, \vdash_j))_{(i \leq j) \in I}$ be a diagram. Take $(\Sigma, (\gamma^i)_{i \in I})$ the colimit of the underlying diagram $(\mathcal{I} \xrightarrow{D} \mathcal{S} \xrightarrow{U} \mathcal{L})$. Now, for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, define that $\Gamma \vdash \psi \Leftrightarrow$ there is $\Gamma^{-} \subseteq_{fin} \Gamma$ and there is an $i \in I$ such that $\Gamma^{-} \cup \{\psi\} \subseteq \widehat{\gamma}^i[F(\Sigma^i)]$ and there is $\Gamma^{-^{i}} \cup \{\psi^i\} \subseteq_{fin} F(\Sigma^i)$ such that $\widehat{\gamma}^i[\Gamma^{-^{i}}] = \Gamma^{-}$, $\widehat{\gamma}^i(\psi^i) = \psi$ and $\Gamma^{-^{i}} \vdash_i \psi^i$. Then $l = (\Sigma, \vdash)$ is a logic and $(l, (\gamma^i)_{i \in I})$ is the colimit of D in $\mathcal{L}$. In fact, this follows from (a), (b) and (c) below:

(a) $l \in Obj(\mathcal{L})$;
(b) $\gamma^j \in \mathcal{L}(l^j, l)$, for all $j \in I$;
(c) if $(l', (\alpha^i)_{i \in I})$ is a commutative cocone over the diagram D then the unique signature morphism $\alpha : l \longrightarrow l'$ such that $\alpha^i = \alpha \circ \gamma^i$, $i \in I$ preserves the consequence relations.

Now we prove (a), (b) and (c).

(a) It follows directly from the definition of $\vdash$ that it gives a *finitary* and *increasing* consequence operator. It is also *inflationary* because if $\psi \in \Gamma$, take $\Gamma^{-} = \{\psi\}$ and any $i \in I$ such that $\psi \in \widehat{\gamma}^i[\Sigma^i]$, take $\psi^i \in F(\Sigma^i)$ such that $\widehat{\gamma}^i(\psi^i) = \psi$ and $\Gamma^{-^{i}} = \{\psi^i\}$ then, as $\vdash_i$ gives a inflationary operator, $\Gamma^{-^{i}} \vdash_i \psi^i$.

Idempotent: Let $\psi \in F(\Sigma)$ be such that $\overline{\Gamma} \vdash \psi$ where $\overline{\Gamma} = \{\theta \in F(\Sigma) : \Gamma \vdash \theta\}$. Let us prove that $\Gamma \vdash \psi$. Since $\vdash$ is finitary, let $\Delta \subseteq_{fin} \overline{\Gamma}$ be such that $\Delta \vdash \psi$. Then, for each $\theta \in \Delta$, let $\Gamma^\theta \subseteq_{fin} \Gamma$ be such that $\Gamma^\theta \vdash \theta$. Then $\Gamma^{-} = \bigcup_{\theta \in \Delta} \Gamma^\theta$ is such that $\Gamma^{-} \subseteq_{fin} \Gamma$ and, since $\vdash$ is increasing, for each $\theta \in \Delta$, $\Gamma^{-} \vdash \theta$. We can choose $i \in I$ and $\Delta^i \cup \{\psi^i\} \subseteq_{fin} F(\Sigma^i)$ with $\widehat{\gamma}^i[\Delta^i] = \Delta$, $\widehat{\gamma}^i(\psi^i) = \psi$ and $\Delta^i \vdash_i \psi^i$ because, by definition, $\Delta \vdash \psi$ iff there is a subset $\Delta^{-} \subseteq_{fin} \Delta$ and a $j \in I$ $\Delta^{-^{j}} \cup \{\psi^j\} \subseteq_{fin} F(\Sigma^j)$ with $\widehat{\gamma}^j[\Delta^{-^{j}}] = \Delta$, $\widehat{\gamma}^j(\psi^j) = \psi$ and $\Delta^{-^{j}} \vdash_j \psi^j$ and, since Δ is a finite set and $(I, \leq)$ is a directed ordered set, there is an $i \geq j$ and a $\Delta^i \subseteq_{fin} F(\Sigma^i)$ such that $\widehat{f}^{ji}[\Delta^{-^{j}}] \subseteq \Delta^i$ and $\widehat{\gamma}^i[\Delta^i] = \Delta$. Now we have $\widehat{f}^{ji}[\Delta^{-^{j}}] \vdash_i \widehat{f^{ji}(\psi^j)}$ and since $(\Sigma, (\gamma^i)_{i \in I})$ is a commutative cocone over the diagram D, taking $\psi^i = \widehat{f^{ji}}(\psi^j)$ we have, since $\vdash_i$ is increasing, $\Delta^i \cup \{\psi^i\} \subseteq_{fin} F(\Sigma^i)$ with $\widehat{\gamma}^i[\Delta^i] = \Delta$, $\widehat{\gamma}^i(\psi^i) = \psi$ and $\Delta^i \vdash_i \psi^i$. Analogously we can choose, for each $\theta \in \Delta$, an $i_\theta \in I$ such that there is $\Gamma^{-^{i_\theta}} \cup \{\theta^{i_\theta}\} \subseteq_{fin} F(\Sigma^{i_\theta})$ with $\widehat{\gamma}^{i_\theta}[\Gamma^{-^{i_\theta}}] = \Gamma^{-}$, $\widehat{\gamma}^{i_\theta}(\theta^{i_\theta}) = \theta$ and $\Gamma^{-^{i_\theta}} \vdash_{i_\theta} \theta^{i_\theta}$. Then, since $(I, \leq)$ is a directed ordered set, take $j \geq i, i_\theta$ for all $\theta \in \Delta$. Now, since $(\Sigma, (\gamma^i)_{i \in I})$ is a commutative cocone over the diagram $D : (I, \leq) \longrightarrow \mathcal{L}$, then, with $\Gamma^{-^{j}} = \bigcup_{\theta \in \Delta} \widehat{f}^{i_\theta j}[\Gamma^{-^{i_\theta}}] \subseteq_{fin} F(\Sigma^j)$ and $\theta^j = \widehat{f}^{i_\theta j}(\theta^{i_\theta})$, we have, since $\vdash_j$ is increasing, $\Gamma^{-^{j}} \vdash_j \theta^j$ for all $\theta \in \Delta$. We can also suppose j such that $\Delta^j = \{\theta^j : \theta \in \Delta\}$ satisfies $\Delta^j \cup \{\psi^j\} \subseteq_{fin} F(\Sigma^j)$ with $\widehat{\gamma}^j[\Delta^j] = \Delta$, $\widehat{\gamma}^j(\psi^j) = \psi$ and $\Delta^j \vdash_j \psi^j$. Finally, since $\vdash_j$ is *idempotent*, we have $\Gamma^{-^{j}} \vdash_j \psi^j$ and we have $\widehat{\gamma}^j[\Gamma^{-^{j}}] = \bigcup_{\theta \in \Delta} \widehat{\gamma}^j[\widehat{f}^{i_\theta j}[\Gamma^{-^{i_\theta}}]] = \bigcup_{\theta \in \Delta} \widehat{\gamma}^{i_\theta}[\Gamma^{-^{i_\theta}}] = \bigcup_{\theta \in \Delta} \Gamma^{-} = \Gamma^{-} \subseteq_{fin} \Gamma$ and $\widehat{\gamma}^j(\psi^j) = \widehat{\gamma}^j(\widehat{f}^{ij}(\psi^i)) = \widehat{\gamma}^i(\psi^i) = \psi$.

Structural: Let $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$ be such that $\Gamma \vdash \psi$. We have to prove that for any substitution $\sigma : X \longrightarrow F(\Sigma)$ we have $\widetilde{\sigma}[\Gamma] \vdash \widetilde{\sigma}(\psi)$.

Let $\Gamma^{-} \subseteq_{fin} \Gamma$, an $i \in I$ such that $\Gamma^{-} \cup \{\psi\} \subseteq \widehat{\gamma}^{i}[F(\Sigma^{i})]$ and let $\Gamma^{-^{i}} \cup \{\psi^{i}\} \subseteq_{fin} F(\Sigma^{i})$ be such that $\widehat{\gamma}^{i}[\Gamma^{-^{i}}] = \Gamma^{-}$, $\widehat{\gamma}^{i}(\psi^{i}) = \psi$ and $\Gamma^{-^{i}} \vdash_{i} \psi^{i}$. Now, since $\Gamma^{-} \cup \{\psi\}$ is a finite set of Σ-formulas then also $\Gamma^{-} \cup \{\psi\} \cup \widetilde{\sigma}[\Gamma^{-}] \cup \{\widetilde{\sigma}(\psi)\} \subseteq_{fin} F(\Sigma)$. So there is an $n \in \omega$ such that the X-variables that occur in $\Gamma^{-} \cup \{\psi\} \cup \widetilde{\sigma}[\Gamma^{-}] \cup \{\widetilde{\sigma}(\psi)\} \subseteq_{fin} F(\Sigma)$ are in the finite set $\{x_0, \ldots, x_{n-1}\} \subseteq_{fin} X$. Then, since $(I, \leq)$ is a directed ordered set, there is a $k \geq i$ such that $\sigma[\{x_0, \ldots, x_{n-1}\}] \subseteq \gamma^{k}[\Sigma^{k}]$. Now take a substitution $\sigma^{k} : X \longrightarrow F(\Sigma^{k})$ such that for any $m < n$, $\sigma^{k}(x_m) \in (\widehat{\gamma}^{k})^{-1}[\sigma(x_m)]$. Now, with $\Gamma^{-^{k}} = \widehat{f}^{ik}[\Gamma^{-^{i}}]$ and $\psi^{k} = \widehat{f}^{ik}(\psi^{i})$, we have $\Gamma^{-^{k}} \cup \{\psi^{k}\} \subseteq_{fin} F(\Sigma^{k})$ such that $\widehat{\gamma}^{k}[\Gamma^{-^{k}}] = \Gamma^{-}$, $\widehat{\gamma}^{k}(\psi^{k}) = \psi$ and $\Gamma^{-^{k}} \vdash_{k} \psi^{k}$. So, since $\vdash_{k}$ is *structural*, it follows that $\widetilde{\sigma}^{k}[\Gamma^{-^{k}}] \vdash_{k} \widetilde{\sigma}^{k}(\psi^{k})$. Since $\widetilde{\sigma}^{k}[\Gamma^{-^{k}}] \cup \{\widetilde{\sigma}^{k}(\psi^{k})\} \subseteq_{fin} F(\Sigma^{k})$ it is enough to show that for all Σ-formula θ in the finite set $\Gamma^{-} \cup \{\psi\} \subseteq_{fin} F(\Sigma)$ the Σ^{k}-formula $\theta^{k} \in \Gamma^{-^{k}} \cup \{\psi^{k}\} \subseteq_{fin} F(\Sigma^{k})$ such that $\widehat{\gamma}^{k}(\theta^{k}) = \theta$ also satisfies $\widetilde{\sigma}(\theta) = \widehat{\gamma}^{k}(\widetilde{\sigma}^{k}(\theta^{k}))$, in order to prove that $\widetilde{\sigma}[\Gamma] \vdash \widetilde{\sigma}(\psi)$. As we have seen above, $\theta = \theta(x_0, \ldots, x_{n-1})$. This equation together with Fact 1.(i) give $\widetilde{\sigma}(\theta(x_0, \ldots, x_{n-1})) = \theta(\sigma(x_0), \ldots, \sigma(x_{n-1})) = \widehat{\gamma}^{k}(\theta^{k})(\sigma(x_0), \ldots, \sigma(x_{n-1})) = \widehat{\gamma}^{k}(\theta^{k})(\widehat{\gamma}^{k}(\sigma^{k}(x_0)), \ldots, \widehat{\gamma}^{k}(\sigma^{k}(x_{n-1}))) = \widehat{\gamma}^{k}(\theta^{k}(\sigma^{k}(x_0), \ldots, \sigma^{k}(x_{n-1}))) = \widehat{\gamma}^{k}(\widetilde{\sigma}^{k}(\theta^{k}(x_0, \ldots, x_{n-1})))$.

(b) Let $j \in I$ and $\Gamma^{j} \cup \{\psi^{j}\} \subseteq F(\Sigma^{j})$ be such that $\Gamma^{j} \vdash_{j} \psi^{j}$. Now since $\vdash_{j}$ is finitary, select $\Gamma^{-^{j}} \subseteq_{fin} \Gamma^{j}$ such that $\Gamma^{-^{j}} \vdash_{j} \psi^{j}$. For any $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$ such that $\Gamma = \widehat{\gamma}^{j}[\Gamma^{j}]$, $\psi = \widehat{\gamma}^{j}(\psi^{j})$. We have that $\Gamma \vdash \psi$. In fact, there is a $\Gamma^{-} \subseteq_{fin} \Gamma$ (take $\Gamma^{-} = \widehat{\gamma}^{j}[\Gamma^{-^{j}}]$) and there is an $i \in I$ (take $i = j$) such that $\Gamma^{-} \cup \{\psi\} \subseteq \widehat{\gamma}^{i}[F(\Sigma^{i})]$ and there is $\Gamma^{-^{i}} \cup \{\psi^{i}\} \subseteq_{fin} F(\Sigma^{i})$ such that $\widehat{\gamma}^{i}[\Gamma^{-^{i}}] = \Gamma^{-}$, $\widehat{\gamma}^{i}(\psi^{i}) = \psi$ and $\Gamma^{-^{i}} \vdash_{i} \psi^{i}$.

(c) Let $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$ be such that $\Gamma \vdash \psi$. Then, by definition, there is $\Gamma^{-} \subseteq_{fin} \Gamma$ and there is an $i \in I$ such that $\Gamma^{-} \cup \{\psi\} \subseteq \widehat{\gamma}^{i}[F(\Sigma^{i})]$ and there is $\Gamma^{-^{i}} \cup \{\psi^{i}\} \subseteq_{fin} F(\Sigma^{i})$ such that $\widehat{\gamma}^{i}[\Gamma^{-^{i}}] = \Gamma^{-}$, $\widehat{\gamma}^{i}(\psi^{i}) = \psi$ and $\Gamma^{-^{i}} \vdash_{i} \psi^{i}$. Since $\alpha^{i} : l^{i} \longrightarrow l'$ is a *translation morphism*, we have $\widehat{\alpha}^{i}[\Gamma^{-^{i}}] \vdash' \widehat{\alpha}^{i}(\psi^{i})$ and since $\alpha^{i} = \alpha \circ \gamma^{i}$, $\widehat{\alpha}[\widehat{\gamma}^{i}[\Gamma^{-^{i}}]] \vdash' \widehat{\alpha}(\widehat{\gamma}^{i}(\psi^{i}))$ we have $\widehat{\alpha}[\Gamma^{-}] \vdash' \widehat{\alpha}(\psi)$ and $\widehat{\alpha}[\Gamma] \vdash' \widehat{\alpha}(\psi)$ ($\vdash'$ is increasing).

Colimits. Let $\mathcal{I}$ be a small category and $D : \mathcal{I} \longrightarrow \mathcal{L}$, $((\Sigma^{i}, \vdash_{i}) \xrightarrow{f^{h}} (\Sigma^{j}, \vdash_{j}))_{(h:i \to j) \in \mathcal{I}}$ be a diagram, and take $(\Sigma, (\gamma^{i})_{i \in Obj(\mathcal{I})})$ the colimit of the underlying diagram $(\mathcal{I} \xrightarrow{D} \mathcal{S} \xrightarrow{U} \mathcal{L})$. Now, for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, define that $\Gamma \vdash \psi \Leftrightarrow$ there is a finite sequence of Σ-formulas $(\phi_0, \ldots, \phi_t)$, where $\phi_t = \psi$ and for all $p \leq t$ one of these alternative occurs:

- "ϕ_p is an hypothesis": $\phi_p \in \Gamma$;
- "ϕ_p is an axiom": there are $i \in Obj(\mathcal{I})$, $\theta^{i} \in F(\Sigma^{i})$, $\sigma : X \longrightarrow F(\Sigma)$ such that $\vdash_{i} \theta^{i}$ and $\phi_p = \widetilde{\sigma}(\widehat{\gamma}^{i}(\theta^{i}))$;

- "ϕ_p is a consequence of a inference rule": there are $i \in Obj(\mathcal{I})$, $\Delta^i \cup \{\theta^i\} \subseteq_{fin} F(\Sigma^i)$, $\sigma : X \longrightarrow F(\Sigma)$ such that $\Delta^i \vdash_i \theta^i$ and $\widetilde{\sigma}[\widehat{\gamma}^i[\Delta^i]] \subseteq \{\phi_0, \ldots, \phi_{p-1}\}$, $\phi_p = \widetilde{\sigma}(\widehat{\gamma}^i(\theta^i))$;

Then $l = (\Sigma, \vdash)$ is a logic and $(l, (\gamma^i)_{i \in Obj(\mathcal{I})})$ is the colimit of D in $\mathcal{L}$. In fact, this follows from (a), (b) and (c) below:

(a) $l \in Obj(\mathcal{L})$;
(b) $\gamma^j \in \mathcal{L}(l^j, l)$, for all $j \in Obj(\mathcal{I})$;
(c) if $(l', (\alpha^i)_{i \in I})$ is a commutative cocone over the diagram D then the unique *signature morphism* $\alpha : l \longrightarrow l'$ such that $\alpha^i = \alpha \circ \gamma^i$, $i \in Obj(\mathcal{I})$ preserves the consequence relation.

Now we prove (a), (b) and (c).

(a) It follows directly from the definition of $\vdash$ that it gives a *finitary*, *increasing* and *inflationary* consequence operator.

Idempotent: Let $\psi \in F(\Sigma)$ be such that $\overline{\Gamma} \vdash \psi$, where $\overline{\Gamma} = \{\theta \in F(\Sigma) : \Gamma \vdash \theta\}$. Let us prove that $\Gamma \vdash \psi$. Choose a "proof of ψ from hypothesis in $\overline{\Gamma}$", that is, a finite sequence of Σ-formulas $(\phi_0, \ldots, \phi_t)$ where $\phi_t = \psi$ and for all $p \leq t$, "ϕ_p is a $\overline{\Gamma}$-hypothesis" or "ϕ_p is an instance of axiom" or "ϕ_p is a consequence of previous formulas in the sequence by an instance of a inference rule". Now, for each $p \leq t$ such that "ϕ_p is an $\overline{\Gamma}$-hypothesis", that is, $\Gamma \vdash \phi_p$, select a "proof of ϕ_p from hypothesis in Γ": a finite sequence of Σ-formulas $(\varphi^p_0, \ldots, \varphi^p_{t_p})$ where $\varphi^p_{t_p} = \phi_p$ and for all $m \leq t_p$, "φ^p_m is a Γ-hypothesis" or "φ^p_m is an instance of axiom" or "φ^p_m is a consequence of previous formulas in the sequence by an instance of a inference rule". Now merge the original sequence $(\phi_0, \ldots, \phi_t)$ with the sequences $(\varphi^p_0, \ldots, \varphi^p_{t_p})$, for each $p \leq t$ such that ϕ_p is a $\overline{\Gamma}$-hypothesis in the obvious way: replace that ϕ_p by the sequence $(\varphi^p_0, \ldots, \varphi^p_{t_p})$. Then since $\varphi^p_{t_p} = \phi_p$, we get a finite sequence of Σ-formulas such that the last one is ψ and each formula is "a Γ-hypothesis" or is "an instance of axiom" or "a consequence of previous formulas in the sequence by an instance of a inference rule". So this resulting finite sequence of Σ-formulas is a "proof of ψ from hypothesis in Γ".

Structural: Let $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$ be such that $\Gamma \vdash \psi$. We have to prove that for any substitution $\sigma' : X \longrightarrow F(\Sigma)$ we have $\widetilde{\sigma'}[\Gamma] \vdash \widetilde{\sigma'}(\psi)$. Choose a "proof of ψ from hypothesis in Γ", that is, a finite sequence of Σ-formulas $(\phi_0, \ldots, \phi_t)$ where $\phi_t = \psi$ and for all $p \leq t$, "ϕ_p is a Γ-hypothesis" or "ϕ_p is an instance of axiom" or "ϕ_p is a consequence of previous formulas in the sequence by an instance of a inference rule". We will see that $(\widetilde{\sigma'}(\phi_0), \ldots, \widetilde{\sigma'}(\phi_t))$ is a "proof of $\widetilde{\sigma'}(\psi)$ from hypothesis in $\widetilde{\sigma'}[\Gamma]$": Since $\widetilde{\sigma'}(\phi_t) = \widetilde{\sigma'}(\psi)$, then we have to show that for all $p \leq t$, "$\widetilde{\sigma'}(\phi_p)$ is an $\widetilde{\sigma'}[\Gamma]$-hypothesis" or "$\widetilde{\sigma'}(\phi_p)$ is an instance of axiom" or "$\widetilde{\sigma'}(\phi_p)$ is a consequence of previous formulas in the sequence by an instance of a inference rule":

- If "ϕ_p is a Γ-hypothesis" then $\phi_p \in \Gamma$ so $\widetilde{\sigma'}(\phi_p) \in \widetilde{\sigma'}[\Gamma]$, that is, "$\widetilde{\sigma'}(\phi_p)$ is a $\widetilde{\sigma'}[\Gamma]$-hypothesis";

- If "ϕ_p is an instance of axiom" then select $i \in Obj(\mathcal{I})$, $\theta^i \in F(\Sigma^i)$, $\sigma : X \longrightarrow F(\Sigma)$ such that $\vdash_i \theta^i$ and $\phi_p = \widetilde{\sigma}(\widehat{\gamma}^i(\theta^i))$, then take the substitution $\sigma'' : X \longrightarrow F(\Sigma)$ such that $(X \xrightarrow{\sigma''} F(\Sigma)) = (X \xrightarrow{\sigma} F(\Sigma) \xrightarrow{\widetilde{\sigma}'} F(\Sigma))$. Then, as $\sigma'' = \widetilde{\sigma}' \circ \sigma$, we have, by uniqueness of extensions, $\widetilde{\sigma}'' = \widetilde{\sigma}' \circ \widetilde{\sigma}$. It follows that we have $i \in Obj(\mathcal{I})$, $\theta^i \in F(\Sigma^i)$, $\sigma'' : X \longrightarrow F(\Sigma)$ such that $\vdash_i \theta^i$ and $\widetilde{\sigma'}(\phi_p) = \widetilde{\sigma'}(\widetilde{\sigma}(\widehat{\gamma}^i(\theta^i))) = \widetilde{\sigma''}(\widehat{\gamma}^i(\theta^i))$. So "$\widetilde{\sigma'}(\phi_p)$ is an instance of axiom";
- If "ϕ_p is a consequence of previous formulas in the sequence by an instance of a inference rule" then select $i \in Obj(\mathcal{I})$, $\Delta^i \cup \{\theta^i\} \subseteq_{fin} F(\Sigma^i)$, $\sigma : X \longrightarrow F(\Sigma)$ such that $\Delta^i \vdash_i \theta^i$ and $\widetilde{\sigma}[\widehat{\gamma}^i[\Delta^i]] \subseteq \{\phi_0, \ldots, \phi_{p-1}\}$, $\phi_p = \widetilde{\sigma}(\widehat{\gamma}^i(\theta^i))$. Then, as above, take the substitution $\sigma'' : X \longrightarrow F(\Sigma)$ such that $\sigma'' = \widetilde{\sigma}' \circ \sigma$, then we have $i \in Obj(\mathcal{I})$, $\Delta^i \cup \{\theta^i\} \subseteq_{fin} F(\Sigma^i)$, $\sigma'' : X \longrightarrow F(\Sigma)$ such that $\Delta^i \vdash_i \theta^i$ and $\widetilde{\sigma}''[\widehat{\gamma}^i[\Delta^i]] = \widetilde{\sigma}'[\widetilde{\sigma}[\widehat{\gamma}^i[\Delta^i]]] \subseteq \widetilde{\sigma}'[\{\phi_0, \ldots, \phi_{p-1}\}] = \{\widetilde{\sigma}'(\phi_0), \ldots, \widetilde{\sigma}'(\phi_{p-1})\}$ and $(\widetilde{\sigma}''(\widehat{\gamma}^i(\theta^i)) = \widetilde{\sigma}'(\widetilde{\sigma}(\widehat{\gamma}^i(\theta^i))) = \widetilde{\sigma}'(\phi_p)$. So "$\widetilde{\sigma'}(\phi_p)$ is a consequence of previous formulas in the sequence by an instance of a inference rule".

(b) Let $j \in Obj(\mathcal{I})$ and $\Gamma^j \cup \{\psi^j\} \subseteq F(\Sigma^j)$ be such that $\Gamma^j \vdash_j \psi^j$. Since $\vdash_j$ is finitary, select $\Gamma^{-j} \subseteq_{fin} \Gamma^j$ such that $\Gamma^{-j} \vdash_j \psi^j$. Now take $\{\phi_0, \ldots, \phi_{t-1}\}$ a enumeration of the finite set $\widehat{\gamma}^j[\Gamma^{-j}]$ and consider the finite sequence of Σ-formulas $(\phi_0, \ldots, \phi_{t-1}, \phi_t)$ such that $\phi_t = \widehat{\gamma}^j(\psi^j)$. Then $\widehat{\gamma}^j[\Gamma^j] \vdash \widehat{\gamma}^j(\psi^j)$. In fact, the finite sequence of Σ-formulas $(\phi_0, \ldots, \phi_t)$ is a "proof of $\widehat{\gamma}^j(\psi^j)$ from hypothesis in $\widehat{\gamma}^j[\Gamma^j]$": for each $p < t$, ϕ_p is a "$\widehat{\gamma}^j[\Gamma^j]$-hypothesis", because $\phi_p \in \widehat{\gamma}^j[\Gamma^{-j}] \subseteq \widehat{\gamma}^j[\Gamma^j]$ and, for $p = t$, $\phi_p = \widehat{\gamma}^j(\psi^j)$ is a "consequence of previous formulas in the sequence", because we have a $j \in Obj(\mathcal{I})$ and $\Delta^j \cup \{\theta^j\} \subseteq_{fin} F(\Sigma^j)$ such that $\Delta^j \vdash_j \theta^j$ (take $\Delta^j = \Gamma^{-j}$ and $\theta^j = \psi^j$) and we have a substitution $\sigma : X \longrightarrow F(\Sigma)$ (take $\sigma(x_n) = x_n$, for all $n \in \omega$. Clearly $\widetilde{\sigma} = id : F(\Sigma) \longrightarrow F(\Sigma)$) such that $\widetilde{\sigma}(\widehat{\gamma}^j(\theta^j)) = \phi_t = \widehat{\gamma}^j(\psi^j)$ and $\widetilde{\sigma}[\widehat{\gamma}^j[\Delta^j]] = \widehat{\gamma}^j[\Gamma^{-j}] \subseteq \{\phi_0, \ldots, \phi_{t-1}\}$.

(c) Let $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$ be such that $\Gamma \vdash \psi$. We show that $\widehat{\alpha}[\Gamma] \vdash' \widehat{\alpha}(\psi)$, by "induction on the rank of a formula demonstrable from hypothesis in Γ": if $\Lambda \vdash \varphi$, $rk(\varphi)$ is the least $t \in \omega$ such there is a sequence of Σ-formulas $(\phi_0, \ldots, \phi_t)$ that is a "proof of φ from hypothesis in Λ".

Let $t = rk(\psi)$:

- If $t = 0$ then $\psi = \phi_t$ is a hypothesis or an axiom:
  - If ψ is a hypothesis then $\psi \in \Gamma$, so $\widehat{\alpha}(\psi) \in \widehat{\alpha}[\Gamma]$ and, as $\vdash'$ is inflationary, $\widehat{\alpha}[\Gamma] \vdash' \widehat{\alpha}(\psi)$;
  - If ψ is an axiom then select $i \in Obj(\mathcal{I})$, $\theta^i \in F(\Sigma^i)$, $\sigma : X \longrightarrow F(\Sigma)$ such that $\vdash_i \theta^i$ and $\psi = \widetilde{\sigma}(\widehat{\gamma}^i(\theta^i))$. As α^i is a translation morphism, we have $\vdash' \widehat{\alpha}^i(\theta^i)$ and as $\alpha^i = \alpha \circ \gamma^i$, $\vdash' \widehat{\alpha}(\widehat{\gamma}^i(\theta^i))$. Now, by Fact 1.(ii), take a substitution $\sigma' : X \longrightarrow F(\Sigma')$ such that $\widetilde{\sigma}' \circ \widehat{\alpha} = \widehat{\alpha} \circ \widetilde{\sigma}$. As $\vdash'$ is structural we get $\vdash' \widetilde{\sigma}'(\widehat{\alpha}(\widehat{\gamma}^i(\theta^i)))$ and then, as $\widetilde{\sigma}'(\widehat{\alpha}(\widehat{\gamma}^i(\theta^i))) = \widehat{\alpha}(\widetilde{\sigma}(\widehat{\gamma}^i(\theta^i))) = \widehat{\alpha}(\psi)$, we have $\vdash' \widehat{\alpha}(\psi)$ so, as $\vdash'$ is increasing, $\widehat{\alpha}[\Gamma] \vdash' \widehat{\alpha}(\psi)$.
- If $t > 0$ then $\psi = \phi_t$ is a consequence of a inference rule: there are $i \in Obj(\mathcal{I})$, $\Delta^i \cup \{\theta^i\} \subseteq_{fin} F(\Sigma^i)$, $\sigma : X \longrightarrow F(\Sigma)$ such that $\Delta^i \vdash_i \theta^i$, $\psi = \phi_t = \widetilde{\sigma}(\widehat{\gamma}^i(\theta^i))$ and $\widetilde{\sigma}[\widehat{\gamma}^i[\Delta^i]] \subseteq \{\phi_0, \ldots, \phi_{t-1}\}$. As for each $m < t$ the subsequence

$(\phi_0, \ldots, \phi_m)$ is a "proof of ϕ_m from hypothesis in Γ" then, for each $m < t$, the formula ϕ_m is such that $rk(\phi_m) \leq m < t$ so, by the *induction hypothesis*, $\widehat{\alpha}[\Gamma] \vdash' \widehat{\alpha}(\phi_m)$, for each $m < t$. As α^i is a translation morphism, we have $\widehat{\alpha}^i[\Delta^i] \vdash' \widehat{\alpha}^i(\theta^i)$ and as $\alpha^i = \alpha \circ \gamma^i$, $\widehat{\alpha}[\widehat{\gamma}^i[\Delta^i]] \vdash' \widehat{\alpha}(\widehat{\gamma}^i(\theta^i))$. Now, by Fact 1.(ii), take a substitution $\sigma' : X \longrightarrow F(\Sigma')$ such that $\widetilde{\sigma'} \circ \widehat{\alpha} = \widehat{\alpha} \circ \widetilde{\sigma}$. As $\vdash'$ is structural we get $\widetilde{\sigma}'[\widehat{\alpha}[\widehat{\gamma}^i[\Delta^i]]] \vdash' \widetilde{\sigma}'(\widehat{\alpha}(\widehat{\gamma}^i(\theta^i)))$. As $\widetilde{\sigma}'(\widehat{\alpha}(\widehat{\gamma}^i(\theta^i))) = \widehat{\alpha}(\widetilde{\sigma}(\widehat{\gamma}^i(\theta^i))) = \widehat{\alpha}(\psi)$ and $\widetilde{\sigma}'[\widehat{\alpha}[\widehat{\gamma}^i[\Delta^i]]] = \widehat{\alpha}[\widetilde{\sigma}[\widehat{\gamma}^i[\Delta^i]]]$ we have $\widehat{\alpha}[\widetilde{\sigma}[\widehat{\gamma}^i[\Delta^i]]] \vdash' \widehat{\alpha}(\psi)$ so, as $\vdash'$ is inflationary, $\widehat{\alpha}[\{\phi_0, \ldots, \phi_{t-1}\}] \vdash' \widehat{\alpha}(\psi)$. Finally, as $\widehat{\alpha}[\Gamma] \vdash' \widehat{\alpha}(\phi_m)$ for each $m < t$, and $\vdash'$ is idempotent, we get $\widehat{\alpha}[\Gamma] \vdash' \widehat{\alpha}(\psi)$. □

Proposition 2.12. *Monomorphisms and epimorphisms in $\mathcal{L}$: let $U : \mathcal{L} \longrightarrow \mathcal{S}$ be the forgetful functor. Then a morphism f in $\mathcal{L}$ is monic (epic) iff $U(f)$ is monic (epic) in $\mathcal{S}$.*

Proof. The right to left implication is easy. Let $f : l \longrightarrow l'$ be a morphism in $\mathcal{L}$ such that $U(f) : U(l) \longrightarrow U(l')$ is not $\mathcal{S}$-monic. Then there exists $\Sigma'' \in Obj(\mathcal{S})$ and $g, h : \Sigma'' \longrightarrow U(l)$ such that $g \neq h$ and $U(f) \circ g = U(f) \circ h$. These g, h are $\mathcal{L}$-morphisms from the least logic over Σ (i.e., the logic whose closure operator is the identity) to l which satisfy $g \neq h$ and $f \circ g = f \circ h$, showing that f is not $\mathcal{L}$-monic. For the "epic" part proceed similarly, taking two "counterexample" arrows in $\mathcal{S}$, $g, h : U(l') \longrightarrow \Sigma''$ such that $g \neq h$ and $g \circ U(f) = h \circ U(f)$. These g, h become $\mathcal{L}$-morphisms equipping their codomain with the greatest consequence relation there (where "everything can be deduced from anything and/or nothing") or with the "logic generated" by the direct image logic of this $\mathcal{S}$-morphism ($\vdash'' = g_\star(\vdash) \vee h_\star(\vdash)$). □

2.2.3. $\mathcal{L}$ is a locally presentable category.

Fact 6. *Additional facts on filtered colimits in $\mathcal{L}$*: Let $D : (I, \leq) \longrightarrow \mathcal{L}$, $(l^i \xrightarrow{f^{ij}} l^j)_{(i \leq j) \in I}$ be a directed diagram and let $(l', (\alpha^i)_{i \in I})$ be a commutative cocone over the diagram D:

(i) $(l', (\alpha^i)_{i \in I})$ is "the" universal colimit cocone of diagram D iff:
  - $\Sigma'_n = \bigcup_{i \in I} \alpha^i_n[\Sigma^i_n]$, $n \in \omega$;
  - If $c_i \in \Sigma^i_n, c_j \in \Sigma^j_n$ are such that $\alpha^i_n(c_i) = \alpha^j_n(c_j)$ then there is $k \geq i, j$ such that $(f^{ik})_n(c_i) = (f^{jk})_n(c_j)$, $n \in \omega$;
  - For all $\Gamma' \cup \{\psi'\} \subseteq F(\Sigma')$ such that $\Gamma' \vdash' \psi' \Leftrightarrow$, there is $\Gamma'^{-} \subseteq_{fin} \Gamma'$ and there is $i \in I$ such that $\Gamma'^{-} \cup \{\psi'\} \subseteq \widehat{\alpha}^i[F(\Sigma^i)]$ and there is $\Gamma^{-i} \cup \{\psi^i\} \subseteq_{fin} F(\Sigma^i)$ such that $\widehat{\alpha}^i[\Gamma^{-i}] = \Gamma'^{-}$, $\widehat{\alpha}^i(\psi^i) = \psi'$ and $\Gamma^{-i} \vdash_i \psi^i$.

(ii) If, for all $(i \leq j) \in I$, $f^{ij} : l^i \longrightarrow l^j$ is a conservative monomorphism, then if $(l', (\alpha^i)_{i \in I})$ is "the" universal colimit cocone of diagram D then $\alpha^i : l^i \longrightarrow l'$ is a conservative monomorphism.

Proof. We prove only the part of item (ii) concerning conservativeness: Let $j \in I$ and $\Gamma^j \cup \{\psi^j\} \subseteq F(\Sigma^j)$ such that $\widehat{\alpha}^j[\Gamma^j] \vdash' \widehat{\alpha}^j(\psi^j)$. Now take $\Gamma' \cup \{\psi'\} \subseteq F(\Sigma')$

such that $\Gamma' = \widehat{\alpha}^j[\Gamma^j]$, $\psi' = \widehat{\alpha}^j(\psi^j)$. As $\Gamma' \vdash' \psi'$ then, by item (i) above, there is $\Gamma'^- \subseteq_{fin} \Gamma'$ and there is $i \in I$ such that $\Gamma'^- \cup \{\psi'\} \subseteq \widehat{\alpha}^i[F(\Sigma^i)]$ and there is $\Gamma^{-^i} \cup \{\psi^i\} \subseteq_{fin} F(\Sigma^i)$ such that $\widehat{\alpha}^i[\Gamma^{-^i}] = \Gamma'^-$, $\widehat{\alpha}^i(\psi^i) = \psi'$ and $\Gamma^{-^i} \vdash_i \psi^i$. As $(I, \leq)$ is a directed ordered set, there is $k \geq i, j$, then as $\alpha^i = \alpha^k \circ f^{ik}$, $\Gamma'^- \cup \{\psi'\} \subseteq \widehat{\alpha}^k[F(\Sigma^k)]$ and there is $\Gamma^{-^k} \cup \{\psi^k\} \subseteq_{fin} F(\Sigma^k)$ (take $\Gamma^{-^k} = \widehat{f}^{ik}[\Gamma^{-^i}]$ and $\psi^k = \widehat{f}^{ik}(\psi^i)$) such that $\widehat{\alpha}^k[\Gamma^{-^k}] = \Gamma'^-$, $\widehat{\alpha}^k(\psi^k) = \psi'$ and $\Gamma^{-^k} \vdash_k \psi^k$ (this because $f^{ik} : l^i \longrightarrow l^k$ is a translation morphism). As $\alpha^j, \alpha^k, f^{jk}$ are $\mathcal{S}$-*monomorphisms* (by Proposition 2.12 and Fact 3.(iii)) and $\alpha^j = \alpha^k \circ f^{jk}$ we have, by Fact 2.(ii), that the formula algebra functions $\widehat{\alpha}^j, \widehat{\alpha}^k, \widehat{f}^{jk}$ are injectives and $\widehat{\alpha}^j = \widehat{\alpha}^k \circ \widehat{f}^{jk}$. From this we can conclude that $\psi^k = \widehat{f}^{ik}(\psi^j)$ and there is $\Gamma^{-^j} \subseteq_{fin} \Gamma^j$ such that $\Gamma^{-^k} = \widehat{f}^{jk}[\Gamma^{-^j}]$ (take $\Gamma^{-^j} = (\widehat{\alpha}_j)^{-1}[\Gamma'^-]$). As $\Gamma^{-^k} \vdash_k \psi^k$ and $f^{jk} : l^j \longrightarrow l^k$ is a conservative translation morphism, we have $\Gamma^{-^j} \vdash_j \psi^j$. Finally, as $\Gamma^{-^j} \subseteq \Gamma^j$ and $\vdash_j$ is increasing, we have $\Gamma^j \vdash_j \psi^j$. $\square$

Proposition 2.13. *Any logic is a directed colimit of finite type logics.*

Proof. Let $l = (\Sigma, \vdash) \in Obj(\mathcal{L})$ and take the set I of all $l' = (\Sigma', \vdash') \in Obj(\mathcal{L})$ such that $|\Sigma'| \subseteq_{fin} |\Sigma|$; $\vdash'$ is given by a finite set of axioms and a finite set of finitary inference rules; the signature morphism of inclusion $\Sigma' \hookrightarrow \Sigma$ is also a translation morphism $l' \hookrightarrow l$. Then:

(a) This diagram is "directed by inclusions" (clear);
(b) $(l, (l' \hookrightarrow l)_{l' \in I})$ is the colimit of this diagram.

This follows from the characterization of $\vdash$ in filtered colimits.

It is clear that the first two conditions in Fact 6.(i) are satisfied. Now consider $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$ such that $\Gamma \vdash \psi$: take $\Gamma^- \subseteq_{fin} \Gamma$ such that $\Gamma^- \vdash \psi$. Take all symbols $\in |\Sigma|$ that occur in formulas in $\Gamma^- \cup \{\psi\}$: this is a finite set $S \subseteq_{fin} |\Sigma|$, and take the unique subsignature $\Sigma' \hookrightarrow \Sigma$ such that $|\Sigma'| = S$. Take $l' = (\Sigma', \vdash')$ the unique logic that is generated (by substitutions) by the unique basic axiom (if $\Gamma^- = \emptyset$) or inference rule (if $\Gamma^- \neq \emptyset$) "from hypothesis Γ^- conclude ψ", then the inclusion $l' \hookrightarrow l$ is in fact a translation morphism and $\Gamma \vdash \psi$ iff there is $\Gamma^- \subseteq_{fin} \Gamma$ such that if $i' : l' \hookrightarrow l$ then $\Gamma^- \cup \{\psi\} \subseteq \widehat{i'}[F(\Sigma')]$ and $\Gamma^- \vdash' \psi$ (*in* l'). $\square$

Remark 2.14. Through an analogous argument we can prove that any logic is a filtered colimit of "conservative sublogics" with finite type underlying signature. However, the next proposition say that we have chosen the "correct definition" of finite type logic.

Proposition 2.15. *A logic is finitely presentable if and only if it is of finite type.*

Proof.
$(\Leftarrow)$. Let $l' = (\Sigma', \vdash')$ be a logic of finite type and consider $D : (I, \leq) \longrightarrow \mathcal{L}$, $(l^i \xrightarrow{f^{ij}} l^j)_{(i \leq j) \in I}$ a directed diagram of logics. Then the canonical arrow k :

$colim_{i\in I}\mathcal{L}(l', l^i) \longrightarrow \mathcal{L}(l', colim_{i\in I}l^i)$ is an isomorphism.

$$\left[\left(l' \xrightarrow{h^i} l^i\right), i)\right] \mapsto \left(l' \xrightarrow{h^i} l^i \xrightarrow{\gamma^i} colim_{i\in I}l^i\right).$$

Let us prove that k is surjective. Let $h : l' \longrightarrow colim_{i\in I}l^i$ be a $\mathcal{L}$-morphism. Because $|\Sigma'|$ is a finite set, as in Proposition 2.4, there exists $i \in I$ such that, for each $n \in \omega$, $h_n[\Sigma'_n] \subseteq \gamma^i_n[\Sigma^i_n]$. As $\vdash'$ is given by a finite set of axioms and a finite set of (finitary) inference rules and $(I, \leq)$ is a directed ordered set, there exists $j \in I$ such that $j \geq i$ and the finite image set of formulas in this chosen axioms and rules by $\widehat{h} : F(\Sigma') \longrightarrow F(colim_{i\in I}\Sigma^i)$ are contained in the set $\widehat{\gamma}^j[F(\Sigma^j)]$. Since h is a translation morphism whose codomain is a filtered colimit, then we can assume also, by the definition of $\vdash$ in filtered colimits, that j is such that the images of these axioms and inference rules under $\widehat{h}$ are in fact $\vdash_j$-derivable in l^j. So if we take for each $n \in \omega$ and each $c'_n \in \Sigma'_n$, $h^j_n : \Sigma'_n \longrightarrow \Sigma^j_n$ such that $h^j_n(c'_n) \in \Sigma^j_n$ with $h_n(c'_n) = [(h^j_n(c'_n), j)]$ then, $h^j : l' \longrightarrow l^j$ is a translation morphism (by Fact 5.(ii), because $\vdash' \leq (h^j)^\star(\vdash_j))$ and $h = \gamma^j \circ h^j$.

Now we prove that k is injective. This is analogous to the correspondent part of Proposition 2.4; in fact, here we need only the information that $|\Sigma'|$ is a finite set.

$(\Rightarrow)$. Let $l = (\Sigma, \vdash)$ be a finitely presentable logic. Then, by the proof of Proposition 2.13, the logic l is the colimit of the directed diagram of its finite type sublogics. Then, as l is a finitely presentable logic, there is l', a finite type sublogic of l, such that the identity translation morphism $id_l : l \longrightarrow l$ must factor through the (colimit) canonical inclusion $l' \hookrightarrow l$, because the canonical morphism $k : colim_{l'\in I}\mathcal{L}(l, l') \longrightarrow \mathcal{L}(l, colim_{l'\in I}l')$ is surjective, that is, there is a translation morphism $h' : l \longrightarrow l'$ such that $(l \xrightarrow{id_l} l) = (l \xrightarrow{h'} l' \hookrightarrow l)$. Then the $\mathcal{L}$-inclusion $l' \hookrightarrow l$ must be a $\mathcal{S}$-isomorphism. Then $l' \hookrightarrow l$ and $h' : l \longrightarrow l'$ have as subjacent the identity $\mathcal{S}$-morphism. So, by Fact 5.(ii), as $l' \hookrightarrow l$ is a translation morphism, then $\vdash' \leq \vdash$ and as $h' : l \longrightarrow l'$ is a translation morphism, then $\vdash \leq \vdash'$. Then we have $l = l'$ and l is a finite type logic. □

Theorem 2.16. *The category $\mathcal{L}$ is a finitely locally presentable category, that is, $\mathcal{L}$ is an accessible category that is cocomplete and complete.*

Proof. Direct consequence of the Propositions 2.13, 2.15 and 2.11. □

Corollary 2.17. (i) *The Yoneda functor $Y : \mathcal{L}_{fp} \longrightarrow Set^{(\mathcal{L}_{fp})^{op}}$ has an extension to a functor $Y' : \mathcal{L} \longrightarrow Set^{(\mathcal{L}_{fp})^{op}}$, $l \mapsto Y'(l) = \mathcal{L}(\iota(.), l)$ that is full and faithful;*

(ii) *Let $Flat(\mathcal{L}_{fp}, Set)$ be the full subcategory of $Set^{(\mathcal{L}_{fp})^{op}}$ whose objects are the functors that are filtered colimits of representable functors. Then $Flat(\mathcal{L}_{fp}, Set)$ is the "essential image" of Y', so its restriction functor $E : \mathcal{L} \longrightarrow Flat(\mathcal{L}_{fp}, Set)$ is an equivalence of categories;*

(iii) *$Flat(\mathcal{L}_{fp}, Set)$ coincides with the category of Set-valued functors that preserve finite limits;*
(iv) *Y' has a left adjoint.*

Proof. For (i) and (ii) see [9], Theorem 5.3.5 (p. 265) or [1], Theorem 2.26 (p. 83) or [22], Observation 1.6 (p. 46). For (iii) and (iv) see [1], Theorem 1.46 (p. 38). □

Remark 2.18. $\mathcal{L}$ is an "algebraic category" and is not a "topological category":

- $\mathcal{L}$ is an "algebraic category": all the usual categories in algebra are accessible;
- $\mathcal{L}$ has a "topological appeal": because its objects have a particular kind of closure operator and its morphisms are continuous functions relative to the closure operators and the set of consequence relations over any given signature, it is a complete lattice;
- $\mathcal{L}$ is not a "topological category": the category of commutative C^*-algebras is ω_1-locally presentable and has as dual the category of compact Hausdorff topological spaces. There is a general non-duality principle for categories locally presentables: If a category and its opposite are both locally presentable then they are equivalent to a complete lattice[6]: see Theorem 1.64 in [1] (p. 51).

3. The category of algebraizable logics $\mathcal{A}$

3.1. What is $\mathcal{A}$?

The category $\mathcal{A}$ is the category of algebraizable logics and translation morphisms that preserves algebraizing pair. An *algebraizable logic* is a logic $l = (\Sigma, \vdash) \in Obj(\mathcal{L})$ that is algebraizable in a Blok–Pigozzi sense: There is an ordered pair $((\delta \equiv \epsilon), \Delta)$, called an *algebraizing pair*, and a class K of Σ-structures (i.e. Σ-algebras), called an *equivalent algebraic semantic*, such that:

- $(\delta \equiv \epsilon)$ is a finite set of ordered pair of Σ-formulas $(\delta \equiv \epsilon) = \{(\delta_r, \epsilon_r) : r < s\}$, called the *set of defining equations*, such that $\{\delta_r, \epsilon_{r'} : r \leq r' < s\} \subseteq F(\Sigma)[1]$;
- Δ is a finite set of Σ-formulas $\Delta = \{\Delta_u : u < v\}$, called the *set of equivalence formulas*, such that, $\{\Delta_u : u < v\} \subseteq F(\Sigma)[2]$;

and $((\delta \equiv \epsilon), \Delta)$ satisfies conditions (i) and (ii) (and/or conditions (i)′ and (ii)′) below, with $\Gamma \cup \Theta \cup \{\psi, \varphi, \zeta, \eta, \vartheta\} \subseteq F(\Sigma)$:

(i) $\Gamma \vdash \varphi \Leftrightarrow \{(\delta(\psi) \equiv \epsilon(\psi)) : \psi \in \Gamma\} \vDash_K (\delta(\varphi) \equiv \epsilon(\varphi))$;
(ii) $(\varphi \equiv \psi) \vDash_K \vDash (\delta(\varphi \Delta \psi) \equiv \epsilon(\varphi \Delta \psi))$;[7]
(i)′ $\Theta \vDash_K (\varphi \equiv \psi) \Leftrightarrow \{\zeta \Delta \eta : (\zeta \Delta \eta) \in \Theta\} \vdash \varphi \Delta \psi$;
(ii)′ $\vartheta \dashv\vdash \delta(\vartheta) \Delta \epsilon(\vartheta)$.

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[6] This result connects, in some sense, the three fundamental species of structures of Bourbaki...
[7] This is an abbreviation for $(\varphi \equiv \psi) \vDash_K \{\delta_r(\varphi \Delta_u \psi) \equiv \epsilon_r(\varphi \Delta_u \psi) : r < s, u < v\}$ and $\{\delta_r(\varphi \Delta_u \psi) \equiv \epsilon_r(\varphi \Delta_u \psi) : r < s, u < v\} \vDash_K (\varphi \equiv \psi)$.

If $l = (\Sigma, \vdash), \Sigma' = (\Sigma', \vdash')$ are algebraizable logics, then a morphism $f : \Sigma \longrightarrow \Sigma'$ is a *translation morphism* that also "preserves the algebraizing pairs": if $((\delta \equiv \epsilon), \Delta)$ is an algebraizing pair for l, then $((\widehat{f}[\delta] \equiv \widehat{f}[\epsilon]), \widehat{f}[\Delta])$ is an algebraizing pair for l', where $(\widehat{f}[\delta] \equiv \widehat{f}[\epsilon]) = \{(\widehat{f}(\delta_r), \widehat{f}(\epsilon_r)) : r < s\}$ and $\widehat{f}[\Delta] = \{\widehat{f}(\Delta_u) : u < v\}$. By Remark 2.1, we have $(\widehat{f}[\delta] \equiv \widehat{f}[\epsilon]) \subseteq F(\Sigma')[1]$ and $\widehat{f}[\Delta] \subseteq F(\Sigma')[2]$.

Composition and identities are similar to $\mathcal{L}$.

Proposition 3.1. *For each $l \in \mathcal{A}$, let $((\delta_i \equiv \epsilon_i), \Delta_i)$, an* algebraizing pair*, and K_i an* equivalent algebraic semantic*, for each $i \in \{0, 1\}$. For any class K' of Σ-algebras let us denote $(K')^Q$ the Σ-quasivariety generated by K'. Then some uniqueness conditions holds: on quasivariety semantics: $(K_0)^Q = (K_1)^Q$; on equivalence formulas: $\Delta_0 \dashv_l \vdash \Delta_1$; on defining equations: $(\delta_0 \equiv \epsilon_0) \vDash_K \vDash (\delta_1 \equiv \epsilon_1)$ (where $K \doteq (K_0)^Q = (K^1)^Q$).*

Proof. Theorem 2.15 in [8]. □

Remark 3.2. By Proposition 3.1, a $\mathcal{L}$-morphism $f : (\Sigma, \vdash) \longrightarrow (\Sigma', \vdash')$ between algebraizable logics l, l' is a $\mathcal{A}$-morphism if and only if there exists $((\delta \equiv \epsilon), \Delta)$, an algebraizing pair for l, such that $((\widehat{f}[\delta] \equiv \widehat{f}[\epsilon]), \widehat{f}[\Delta])$ is an algebraizing pair for l'. So we believe that the terminology "category of *algebraizable* logics" fits better than "category of *algebrized* logics", as the notion of morphism between $\mathcal{A}$-objets does not depend of a preservation of any particular choice of algebraizing pairs of the logics source and target.

The theorem below (Theorem 4.7 in [8]) gives a useful characterization of algebraizable logics through an algebraizing pair.

Theorem 3.3. *Let $l = (\Sigma, \vdash)$ a logic and $\Delta \subseteq_{fin} F(\Sigma)[2]$, $(\delta \equiv \epsilon) \subseteq_{fin} (F(\Sigma)[1] \times F(\Sigma)[1])$ such that the conditions below are satisfied*

(a) $\vdash \varphi\Delta\varphi$[8], *for all $\varphi \in F(\Sigma)$;*
(b) $\varphi\Delta\psi \vdash \psi\Delta\varphi$, *for all $\varphi, \psi \in F(\Sigma)$;*
(c) $\varphi\Delta\psi, \psi\Delta\vartheta \vdash \varphi\Delta\vartheta$, *for all $\varphi, \psi, \vartheta \in F(\Sigma)$;*
(d) $\varphi_0\Delta\psi_0, \ldots, \varphi_{n-1}\Delta\psi_{n-1} \vdash c(\varphi_0, \ldots, \varphi_{n-1})\Delta c(\psi_0, \ldots, \psi_{n-1})$, *for all $c \in \Sigma_n$ and all $\varphi_0, \psi_0, \ldots, \varphi_{n-1}, \psi_{n-1} \in F(\Sigma)$;*
(e) $\vartheta \dashv \vdash \delta(\vartheta)\Delta\epsilon(\vartheta)$, *for all $\vartheta \in F(\Sigma)$.*

Then l is an algebraizable logic with Δ as equivalence formulas and $(\delta \equiv \epsilon)$ as defining equations.

It follows easily that:

Corollary 3.4. *If $f : l \twoheadrightarrow l'$ is an $\mathcal{L}$-epimorphism where l is an* algebraizable logic *then l' is an* algebraizable logic *and f is a $\mathcal{A}$-epimorphism.*

We might ask if there are algebraizable logics with a given number of equivalence formulas and/or of defining equations. The example below gives an affirmative to the first question.

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[8] That is, $\vdash \varphi\Delta_u\varphi$, for all $u < v$.

Example. For each natural number $k > 1$ we consider the following *algebraizable logic* $l_k = (\Sigma_k, \vdash_k)$:

Σ_k : as exactly two connectives, c^1 a unary connective and c^k a k-ary connective.

$\vdash_k$: is the consequence operator *determined* by the following axioms and inference rules:

First let us write:
$\Delta_k(x_0, x_1) \doteq \{c^k(x_0, x_1, \ldots, x_1), c^k(x_0, x_0, x_1, \ldots, x_1), \ldots, c^k(x_0, \ldots, x_0, x_1)\}$ a set with $k-1$-formulas in $F(\Sigma)[2]$;
$(\epsilon(x_0) \equiv \delta(x_0)) \doteq \{(x_0, c^1(x_0))\}$ a unitary set with a pair of formulas in $F(\Sigma)[1]$

[*axiom:*]

- $\vdash_k \;\; c^k(x_0, \ldots, x_0)$

[*inference rules:*]

- $\Delta_k(x_0, x_1) \;\vdash_k\; \Delta_k(x_1, x_0)$ [9]
- $\Delta_k(x_0, x_1), \Delta_k(x_1, x_2) \;\vdash_k\; \Delta_k(x_0, x_2)$
- $\Delta_k(x_0, x_1) \;\vdash_k\; \Delta_k\left(c^1(x_0), c^1(x_1)\right)$
- $\Delta_k(x_0, x_1), \Delta_k(x_2, x_3), \ldots, \Delta_k(x_{2k-2}, x_{2k-1}) \quad \vdash_k \; \Delta_k(c^k(x_0, x_2, \ldots, x_{2k-2}), c^k(x_1, x_3, \ldots, x_{2k-1}))$
- $x_0 \;\vdash_k\; \Delta_k\left(\epsilon(x_0), \delta(x_0)\right)$
- $\Delta_k\left(\epsilon(x_0), \delta(x_0)\right) \;\vdash_k\; x_0$

It follows from Theorem 3.3 that l_k is an algebraizable logic with algebraizing pair $((\epsilon \equiv \delta), \Delta_k)$. Besides, if $\Delta'(x_0, x_1)$ is a set of formulas in $F(\sigma)[2]$ such that $x_0 \Delta' x_1 \vdash_k x_0 \Delta_k x_1$ then $\Delta_k \subseteq \Delta'$ so, by Proposition 3.1, $k-1$ is the least number of equivalence formulas for the logic l_k.

The example above suggests the:

Definition 3.5. (i) Let $l = (\Sigma, \vdash)$ be an algebraizable logic. We write $n_{eqv} = min\{k \in \mathbb{N} :$ there is an algebraizable pair $((\delta, \epsilon), \Delta)$ of l such that $card(\Delta) = k\}$; $n_{eqt} = min\{k \in \mathbb{N} :$ there is an algebraizable pair $((\delta \equiv \epsilon), \Delta)$ of l such that such that $card((\delta \equiv \epsilon)) = n\}$.

(ii) Let I be a small category and $D : I \longrightarrow \mathcal{A}$ a diagram; we write $D(i) = l^i = (\Sigma^i, \vdash_i)$, for each $i \in I$. We say that diagram D of algebraizable logics is *bounded* if both the sets of natural numbers $\{n^i_{eqv} : i \in I\}$ and $\{n^i_{eqt} : i \in I\}$ have an upper bound in $\mathbb{N}$.

Remark 3.6. (i) For each signature Σ, $l_\top = (\Sigma, \vdash_{top})$, the greatest logic over Σ, is algebraizable and $n_{eqv} = n_{eqt} = 0$.

(ii) Let l, l' algebraizable logics. Then $n'_{eqv} \leq n_{eqv}$ and $n'_{eqt} \leq n_{eqt}$ are necessary conditions for the existence of a $\mathcal{A}$-morphism $f : l \;\longrightarrow\; l'$.

(iii) A diagram $D : I \;\longrightarrow\; \mathcal{A}$ of algebraizable logics is bounded if and only if $\{D(i) : i \in obj(I)\}$, the *discrete diagram* underlying to D, is bounded.

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[9] That are $k-1$ distinct inference rules!

3.2. Filtered Colimits in $\mathcal{A}$

3.2.1. Obtaining the filtered colimits.

Theorem 3.7. *The category $\mathcal{A}$ has all filtered colimits and the (obvious) underlying functor $U : \mathcal{A} \longrightarrow \mathcal{L}$ creates all such colimits.*

Proof. Let $(I, \leq)$ be a directed ordered set and $D : (I, \leq) \longrightarrow \mathcal{A}$, $(l^i \xrightarrow{f^{ij}} l^j)_{(i \leq j) \in I}$ a diagram. Consider $(l, (\gamma^i)_{i \in I})$ the colimit in $\mathcal{L}$ of the underlying diagram $((I, \leq) \xrightarrow{D} \mathcal{A} \xrightarrow{U} \mathcal{L}$ and choose an $i \in I$ and an algebraizing pair $((\delta^i \equiv \epsilon^i), \Delta^i) = (\{(\delta^i_r \equiv \epsilon^i_r) : r < s\}, \{\Delta^i_u : u < v\})$ of l^i and take $((\delta \equiv \epsilon), \Delta) = (\widehat{\gamma}^i[\delta^i] \equiv \widehat{\gamma}^i[\epsilon^i], \widehat{\gamma}^i[\Delta^i])$. Then l is an algebraizable logic with algebraizing pair $((\delta \equiv \epsilon), \Delta)$ and $(l, (\gamma^i)_{i \in I})$ is the colimit of D in $\mathcal{A}$. In fact, this follows from (a), (b) and (c) below:

(a) $l = (\Sigma, \vdash) \in Obj(\mathcal{A})$.[10] To check this, we invoke the characterization of algebraizable logics through an algebraizing pair in Theorem 3.3:

(a$_1$) $\vdash \varphi \Delta \varphi$[11], for all $\varphi \in F(\Sigma)$: By definition of Σ and by Fact 3.(ii), for each $u < v$ there is an $j_u \in I$ and a $\varphi^{j_u} \in F(\Sigma^{j_u})$ such that $\widehat{\gamma}^{j_u}(\varphi^{j_u}) = \varphi$. Because $(I, \leq)$ is a directed ordered set and $D : (I, \leq) \longrightarrow \mathcal{A}$ is a diagram, there exists $k \geq i, j_0, \ldots, j_{v-1}$ such that $\widehat{\gamma}^k(\varphi^k) = \varphi$. As $f^{ik} : l^i \longrightarrow l^k$ is an $\mathcal{A}$-morphism, then $((\widehat{f}^{ik}[\delta^i] \equiv \widehat{f}^{ik}[\epsilon^i]), \widehat{f}^{ik}[\Delta^i])$ is an algebraizing pair for l^k. By Theorem 3.3 , $\vdash_k \varphi^k \widehat{f}^{ik}(\Delta^i_u)\varphi^k$, for all $u < v$. As $\gamma^k : l^k \longrightarrow l$ is an $\mathcal{L}$-morphism, then $\vdash \widehat{\gamma}^k(\varphi^k)\widehat{\gamma}^k(\widehat{f}^{ik}(\Delta^i_u))\widehat{\gamma}^k(\varphi^k)$, for all $u < v$. As $(\gamma^j : l^j \longrightarrow l)_{j \in I}$ is a commutative cocone over the diagram $((I, \leq) \longrightarrow \mathcal{A} \longrightarrow \mathcal{L})$, then $\vdash \widehat{\gamma}^k(\varphi^k)\widehat{\gamma}^i(\Delta^i_u)\widehat{\gamma}^k(\varphi^k)$, for all $u < v$. Therefore, for all $u < v$, $\vdash \varphi \Delta_u \varphi$, because we have taken $\Delta = \widehat{\gamma}^i[\Delta^i]$.

(a$_2$) $\varphi \Delta \psi \vdash \psi \Delta \varphi$, for all $\varphi, \psi \in F(\Sigma)$: is analogous to (a$_1$)

(a$_3$) $\varphi \Delta \psi, \psi \Delta \vartheta \vdash \varphi \Delta \vartheta$, for all $\varphi, \psi, \vartheta \in F(\Sigma)$: is analogous to (a$_1$)

(a$_4$) $\varphi_0 \Delta \psi_0, \ldots, \varphi_{n-1} \Delta \psi_{n-1} \vdash c(\varphi_0, \ldots, \varphi_{n-1}) \Delta c(\psi_0, \ldots, \psi_{n-1})$, for all $c \in \Sigma_n$ and all $\varphi_0, \psi_0, \ldots, \varphi_{n-1}, \psi_{n-1} \in F(\Sigma)$: We can find $k \geq i$ and $c^k \in (\Sigma^k)_n$, $\varphi^k_0, \psi^k_0, \ldots\ \varphi^k_{n-1}, \psi^k_{n-1} \in F(\Sigma^k)$ such that $\gamma^k(c^k) = c$ and, for all $m < n$, $\widehat{\gamma}^k(\varphi^k_m) = \varphi_m$, $\widehat{\gamma}^k(\psi^k_m) = \psi_m$. As $f^{ik} : l^i \longrightarrow l^k$ is an $\mathcal{A}$-morphism, then $((\widehat{f}^{ik}[\delta^i] \equiv \widehat{f}^{ik}[\epsilon^i]), \widehat{f}^{ik}[\Delta^i])$ is an algebraizing pair for l^k. Hence by Theorem 3.3 $\varphi^k_0 \widehat{f}^{ik}(\Delta^i_u)\psi^k_0, \ldots, \varphi^k_{n-1}\widehat{f}^{ik}(\Delta^i_u)\psi^k_{n-1} \vdash_k c^k(\varphi^k_0, \ldots, \varphi^0_{n-1})\widehat{f}^{ik}(\Delta^i_u)c^k(\psi^k_0, \ldots, \psi^0_{n-1})$, for all $u < v$. As $\gamma^k : l^k \longrightarrow l$ is an $\mathcal{L}$-morphism, then $\widehat{\gamma}^k(\varphi^k_0)\widehat{\gamma}^k(\widehat{f}^{ik}(\Delta^i_u))\widehat{\gamma}^k(\psi^k_0), \ldots, \widehat{\gamma}^k(\varphi^k_{n-1})$ $\widehat{\gamma}^k(\widehat{f}^{ik}(\Delta^i_u))\widehat{\gamma}^k(\psi^k_{n-1}) \vdash \widehat{\gamma}^k(c^k(\varphi^k_0, \ldots, \varphi^0_{n-1}))\widehat{\gamma}^k(\widehat{f}^{ik}(\Delta^i_u))\widehat{\gamma}^k(c^k(\psi^k_0, \ldots,$ $\psi^0_{n-1}))$, for all $u < v$. As $(\gamma^j : l^j \longrightarrow l)_{j \in I}$ is a commutative cocone over the diagram $((I, \leq) \longrightarrow \mathcal{A} \longrightarrow \mathcal{L})$, then $\widehat{\gamma}^k(\varphi^k_0)\widehat{\gamma}^i(\Delta^i_u)\widehat{\gamma}^k(\psi^k_0), \ldots,$ $\widehat{\gamma}^k(\varphi^k_{n-1})\widehat{\gamma}^i(\Delta^i_u)\widehat{\gamma}^k(\psi^k_{n-1}) \vdash \widehat{\gamma}^k(c^k(\varphi^k_0, \ldots, \varphi^0_{n-1}))\widehat{\gamma}^i(\Delta^i_u)\widehat{\gamma}^k(c^k(\psi^k_0, \ldots,$

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[10] This is independent of the chosen $i \in I$ and the algebraizing pair $((\delta^i \equiv \epsilon^i), \Delta^i)$ of l^i.

[11] That is, $\vdash \varphi \Delta_u \varphi$, for all $u < v$.

$\psi^0_{n-1}))$, for all $u < v$. Therefore, for all $u < v$, $\varphi_0 \Delta_u \psi_0, \ldots, \varphi_{n-1}\Delta_u \psi_{n-1} \vdash c(\varphi_0, \ldots, \varphi_{n-1})\Delta_u c(\psi_0, \ldots, \psi_{n-1})$ because we have taken $\Delta = \widehat{\gamma}^i[\Delta^i]$.

(a$_5$) $\vartheta \dashv\vdash \delta(\vartheta)\Delta\epsilon(\vartheta)$, for all $\vartheta \in F(\Sigma)$: We can find $k \geq i$ such that $\widehat{\gamma}^k(\vartheta^k) = \vartheta$. As $f^{ik} : l^i \longrightarrow l^k$ is an $\mathcal{A}$-morphism, then $((\widehat{f}^{ik}[\delta^i] \equiv \widehat{f}^{ik}[\epsilon^i]), \widehat{f}^{ik}[\Delta^i])$ is an algebraizing pair for l^k. Hence by Theorem 3.3 $\vartheta^k \dashv_k\vdash (\widehat{f}^{ik}(\delta^i_r))(\vartheta^k)\ (\widehat{f}^{ik}(\Delta^i_u))\ (\widehat{f}^{ik}(\epsilon^i_r))(\vartheta^k)$, for all $u < v$ and $r < s$. As $\gamma^k : l^k \longrightarrow l$ is an $\mathcal{L}$-morphism, then for each $u < v$ and $r < s$ $\widehat{\gamma}^k(\vartheta^k) \dashv\vdash \widehat{\gamma}^k((\widehat{f}^{ik}(\delta^i_r))(\vartheta^k))\ \widehat{\gamma}^k(\widehat{f}^{ik}(\Delta^i_u))\ \widehat{\gamma}^k((\widehat{f}^{ik}(\epsilon^i_r))(\vartheta^k))$. As $(\gamma^j : l^j \longrightarrow l)_{j\in I}$ is a commutative cocone over the diagram $((I, \leq) \longrightarrow \mathcal{A} \longrightarrow \mathcal{L})$, then for each $u < v$ and $r < s$, $\widehat{\gamma}^k(\vartheta^k) \dashv\vdash \widehat{\gamma}^i(\delta^i_r)\widehat{\gamma}^k(\vartheta^k)\widehat{\gamma}^i(\Delta^i_u)\ \widehat{\gamma}^i(\epsilon^i_r)\widehat{\gamma}^k(\vartheta^k)$. Therefore, for all $u < v$ and $r < s$, $\vartheta \dashv\vdash \delta_r(\vartheta)\Delta_u\epsilon_r(\vartheta)$, because we have taken $(\delta \equiv \epsilon) = (\widehat{\gamma}^i[\delta^i] \equiv \widehat{\gamma}^i[\epsilon^i])$ and $\Delta = \widehat{\gamma}^i[\Delta^i]$.

(b) $\gamma^j \in \mathcal{A}(l^j, l)$, for all $j \in I$: As $(\gamma^j : l^j \longrightarrow l)_{j\in I}$ is a commutative cocone over the diagram $((I, \leq) \longrightarrow \mathcal{A} \longrightarrow \mathcal{L})$, to show that $\gamma^j : l^j \longrightarrow l$ is an $\mathcal{A}$-morphism it is enough to find $k \geq j$ such that $\gamma^k : l^k \longrightarrow l$ is an $\mathcal{A}$-morphism because, in this case, γ^j is a composition of $\mathcal{A}$-morphisms: $\gamma^j = \gamma^k \circ f^{jk}$. So, as $(I, \leq)$ is a directed ordered set, it is enough to show that $\gamma^k : l^k \longrightarrow l$ is a $\mathcal{A}$-morphism, for all $k \geq i$. Take a $k \geq i$: because $f^{ik} : l^i \longrightarrow l^k$ is an $\mathcal{A}$-morphism, we have that $((\widehat{f}^{ik}[\delta^i] \equiv \widehat{f}^{ik}[\epsilon^i]), \widehat{f}^{ik}[\Delta^i])$ is an algebraizing pair for l^k; because $\gamma^i = \gamma^k \circ f^{ik}$ and, by definition, $((\delta \equiv \epsilon), \Delta) = ((\widehat{\gamma}^i[\delta^i] \equiv \widehat{\gamma}^i[\epsilon^i]), \widehat{\gamma}^i[\Delta^i])$, we have that the $\mathcal{L}$-morphism γ^k takes an algebraizing pair of l^k onto an algebraizing pair of l (by item (a) just above), hence by Remark 3.2, $\gamma^k : l^k \longrightarrow l$ is an $\mathcal{A}$-morphism.

(c) If $(l', (\alpha^j)_{j\in I})$ is a commutative cocone over the diagram D then the unique *translation morphism* $\alpha : l \longrightarrow l'$ such that $\alpha^j = \alpha \circ \gamma^j$, $j \in I$ is an $\mathcal{A}$-morphism: By definition of l and item (a) above, $((\delta \equiv \epsilon), \Delta) = ((\widehat{\gamma}^i[\delta^i] \equiv \widehat{\gamma}^i[\epsilon^i]), \widehat{\gamma}^i[\Delta^i])$ is an algebraizing pair of l, so by Remark 3.2, to prove that $\alpha : l \longrightarrow l'$ is an $\mathcal{A}$-morphism it is enough to show that $((\widehat{\alpha}[\delta] \equiv \widehat{\alpha}[\epsilon]), \widehat{\alpha}[\Delta])$ is an algebraizing pair of l'. As $\alpha^i : l^i \longrightarrow l'$ is an $\mathcal{A}$-morphism, then $((\widehat{\alpha}^i[\delta^i] \equiv \widehat{\alpha}^i[\epsilon^i]), \widehat{\alpha}^i[\Delta^i])$ is an algebraizing pair of l'. Finally, as $\alpha^i = \alpha \circ \gamma^i$, we have that the $\mathcal{L}$-morphism α takes an algebraizing pair of l onto an algebraizing pair of l', so by Remark 3.2, $\alpha : l \longrightarrow l'$ is an $\mathcal{A}$-morphism. $\square$

3.2.2. On the algebraic semantics for filtered colimits.

Fact 7. *About structures and morphisms*:

(i) For each signature morphism $\Sigma \xrightarrow{f} \Sigma'$, there is a functor $\Sigma{-}Str \xrightarrow{\bar{f}} \Sigma'{-}Str$ between categories of structures over the signatures such that:

- $\mathcal{A}' = (A', ({c'_n}^{\mathcal{A}'} : A'^n \to A')_{c'_n \in \Sigma'_n, n\in\omega}) \mapsto \bar{f}(\mathcal{A}') = (A', (c_n^{\bar{f}(\mathcal{A}')} : A'^n \to A')_{c_n\in\Sigma_n, n\in\omega})$, where $(c_n^{\bar{f}(\mathcal{A}')} : A'^n \to A') = ((f_n(c_n))^{\mathcal{A}'} : A'^n \to A')$, $c_n \in \Sigma_n, n \in \omega$;

  - $(\mathcal{A}' \xrightarrow{h} \mathcal{B}') \mapsto (\bar{f}(\mathcal{A}') \xrightarrow{h} \bar{f}(\mathcal{B}'))$.

(ii) Connectives $(c'_n \in \Sigma'_n, n \in \omega)$ and (propositional) Σ'-formulas $(\varphi'(x_0, \ldots, x_{n-1}))$ give functions $A'^n \longrightarrow A'$. The first order atomic formulas are equations between Σ'-terms (= propositional Σ'-formulas);

(iii) Let $\mathcal{A}'$ be a Σ'-structure. Then, for every first order valuation in A':
  - For each Σ-equation $\varphi(x_0, \ldots, x_{n-1}) \equiv \psi(x_0, \ldots, x_{n-1})$: $A' \vDash_{\Sigma'-Str} (\widehat{f}(\varphi)[a'_0, \ldots, a'_{n-1}]) \equiv (\widehat{f}(\psi)[a'_0, \ldots, a'_{n-1}]) \Leftrightarrow \bar{f}(A') \vDash_{\Sigma-Str} (\varphi)[a'_0, \ldots, a'_{n-1}]) \equiv (\psi)[a'_0, \ldots, a'_{n-1}])$;
  - For each first order Σ-formula $\mathcal{P}(x_0, \ldots, x_{n-1})$: $A' \vDash_{\Sigma'-Str} (\widehat{f}(\mathcal{P}))[a'_0, \ldots, a'_{n-1}] \Leftrightarrow \bar{f}(A') \vDash_{\Sigma-Str} (\mathcal{P})[a'_0, \ldots, a'_{n-1}]$.

Proposition 3.8. *On the quasivariety semantics for filtered colimits: Let* $(I, \leq)$ *be a directed ordered set and* $D : (I, \leq) \longrightarrow \mathcal{A}$, $(l^i \xrightarrow{f^{ij}} l^j)_{(i \leq j) \in I}$ *a diagram. Consider* $(l, (\gamma^i)_{i \in I})$ *the colimit in* $\mathcal{A}$. *For each* $i \in I$, *take* K_i *the unique quasivariety-semantics for* l_i *and let* K *be the unique quasivariety-semantics for* l. *Then, for all* $\mathcal{A} \in \Sigma - Str$*:*

$$\mathcal{A} \in K \Leftrightarrow \bar{\gamma}^i(\mathcal{A}) \in K_i, \quad \textit{for all } i \in I.$$

Proof. This follows from Theorem 2.17 in [8], by the construction in Theorem 3.7 and the Fact 7 above. □

3.3. More on colimits and limits in $\mathcal{A}$

3.3.1. Colimits.

Proposition 3.9. *The category* $\mathcal{A}$ *has colimits of all non-empty diagrams and the (obvious) underlying functor* $U : \mathcal{A} \longrightarrow \mathcal{S}$ *creates all such colimits.*[12]

Proof. (Sketch) Let $I \neq \emptyset$ be a nonempty category and $D : I \longrightarrow \mathcal{A}$ a diagram. We write $D(i) = l^i = (\Sigma^i, \vdash_i)$ for each $i \in Obj(I)$. Consider $(\Sigma, (\gamma^i)_{i \in Obj(I)})$ the colimit in $\mathcal{S}$ of the underlying diagram $(I \xrightarrow{D} \mathcal{A} \xrightarrow{U} \mathcal{S})$. As $\mathcal{L}_\Sigma$ is a complete lattice, take $\vdash$ as the least consequence relation over the signature Σ such that:

- For each $i \in Obj(I)$ and $\Theta^i \cup \{\psi^i\} \subseteq F(\Sigma^i)$, if $\Theta^i \vdash_i \psi^i$ then $\widehat{\gamma}^i[\Theta^i] \vdash \widehat{\gamma}^i(\psi^i)$;
- For each $i, j \in Obj(I)$ and each algebraizing pairs $((\delta^i \equiv \epsilon^i), \Delta^i)$ of l^i and $((\delta^j \equiv \epsilon^j), \Delta^j)$ of l^j, then: $\widehat{\gamma}^i[\Delta^i] \dashv\vdash \widehat{\gamma}^j[\Delta^j]$.

Now choose any $i \in Obj(I)$ and any algebraizing pair $((\delta^i \equiv \epsilon^i), \Delta^i)$ of l^i and take $((\delta \equiv \epsilon), \Delta) = ((\widehat{\gamma}^i[\delta^i] \equiv \widehat{\gamma}^i[\epsilon^i]), \widehat{\gamma}^i[\Delta^i])$. Then l is an algebraizable logic with algebraizing pair $((\delta \equiv \epsilon), \Delta)$ and $(l, (\gamma^i)_{i \in Obj(I)})$ is the colimit of D in $\mathcal{A}$. □

Remark 3.10. The restriction on the proposition above is essential, i.e., the category $\mathcal{A}$ does not have initial object. In fact, $\mathcal{A}$ does not have *weak initial objects:* as there is a *unbounded family of algebraizable logics* (see, for instance, Example 3.1)

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[12] The construction in this result is similar to the one in [16], but they work with another notion of morphism.

and if l^- is any algebraizable logic then, by Remark 3.6.(b), the subset of an unbounded family of the algebraizable logics that are target of some $\mathcal{A}$-morphism with source l^- is a *proper subset.*

3.3.2. Products. As we saw in the Remark 3.10 above there is no (projective) cone over an unbounded diagrams. But there are universal commutative cones in the case of discrete bounded diagrams:

Proposition 3.11. *The category $\mathcal{A}$ has products of all bounded families and the (obvious) inclusion functor $J : \mathcal{A} \longrightarrow \mathcal{L}$ creates all such limits.*[13]

Proof. (Sketch) Let I be a set and $D : I \longrightarrow \mathcal{A}$ a bounded discrete diagram. We write $D(i) = l^i = (\Sigma^i, \vdash_i)$, for each $i \in I$. Consider $(\Sigma, (\pi^i)_{i\in I})$ the product in $\mathcal{L}$ of the underlying diagram $(I \xrightarrow{D} \mathcal{A} \xrightarrow{J} \mathcal{L})$:

- If $I = \emptyset$: observe that the empty family can not be unbounded and the terminal logic in $\mathcal{L}$ is algebraizable, so it is the terminal algebraizable logic (in $\mathcal{A}$);
- If $I \neq \emptyset$: then, with the notation just above, choose $n, k \in \mathbb{N}$ and any algebraizing pair $((\delta^i \equiv \epsilon^i), \Delta^i)$ of l^i such that $card((\delta_i \equiv \epsilon_i)) \leq n$ and $card(\Delta_i) \leq k$, for each $i \in I$. We take $((\delta \equiv \epsilon), \Delta)$, with $card((\delta \equiv \epsilon)) = n$ and $card(\Delta) = k$, as the finite sets of formulas in the product signature that are obtained as I-indexed sequences of the corresponding finite set of formulas in the chosen algebraizing pair $((\delta^i \equiv \epsilon^i), \Delta^i)$ (with some repetition, when necessary). Then l is an algebraizable logic with algebraizing pair $((\delta \equiv \epsilon), \Delta)$ and $(l, (\pi^i)_{i\in I})$ is the product of D in $\mathcal{A}$. □

3.4. $\mathcal{A}$ is an accessible category

Theorem 3.12. *The category $\mathcal{A}$ is a finitely accessible category.*

Proof. (Sketch) Through analogous arguments to others in the category $\mathcal{L}$, we obtain that:

- Each algebraizable logic is the colimit of a directed diagram of finite type logics that are also algebraizable;
- An algebraizable logic is finitely presentable in $\mathcal{A}$ if and only if it is a finite type logic that are also algebraizable if and only if it is a logic finitely presentable in $\mathcal{L}$ that is also an algebraizable logic.

It follows that any algebraizable logic is the direct colimit of finitely presentable algebraizable logics, that is, $\mathcal{A}$ is a finitely accessible category. □

Remark 3.13. Note that $\mathcal{A}$ is a finitely accessible category but, on the contrary to $\mathcal{S}$ and $\mathcal{L}$, $\mathcal{A}$ is not a finitely locally presentable category because it is not cocomplete, as it has no initial object, and is not complete, as it has no commutative cone over unbounded diagrams. These results are related with a general fact about accessible

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[13] The construction in this result is similar to the one in [11], but they work with another notion of morphism.

categories: an accessible category has all non-empty colimits if and only if it has limits of diagrams that are base of some commutative cone[14] (see [2, 3]).

3.5. Application: Remote algebrization revisited

As another consequence of the theory of accessible categories to our context, we obtain a general (for any logic) weak form of the concept of remote algebrization firstly introduced in [12]: since $\mathcal{E} : \mathcal{A} \longrightarrow \mathcal{L}$ is an accessible functor, that is, $\mathcal{L}$ and $\mathcal{A}$ are accessible categories and $\mathcal{E}$ preserves filtered colimits then, by Proposition 6.1.2 in [21], for each logic l there is a set I and a family of $\mathcal{L}$-arrows $\eta^i : l \longrightarrow \mathcal{E}(l^i)$, where the l^i are $\mathcal{A}$-objects such that, for each $\mathcal{L}$-arrow of l to an algebraizable logic l', $f : l \longrightarrow \mathcal{E}(l')$, there exists $i \in I$ and an $\mathcal{A}$-morphism $f^i : l^i \longrightarrow l'$ such that $f = f^i \circ \eta^i$.

It follows that for any logic l there exists a $\mathcal{L}$-arrow $\eta : l \longrightarrow \prod_{i\in I} l^i$ (take $\eta = (\eta^i)_{i\in I}$) such that, for any $\mathcal{L}$-morphism $f : l \longrightarrow l'$ with $l' \in Obj(\mathcal{A})$, there is an $\mathcal{L}$-morphism $\bar{f} : \prod_{i\in I} l^i \longrightarrow l'$ such that $f = \bar{f} \circ \eta$. Consequently, the logic l is remotely algebraizable if an only if the $\mathcal{L}$-morphism η is a conservative translation. It seems to be an interesting problem to characterize the class of logics that have such a “canonical” family (in case that the existential conditions above are substituted by “there is only one”). In this situation it follows easily that the set I and the $\mathcal{L}$-arrows $\eta^i : l \longrightarrow l^i$ are unique up to isomorphism. This will provide a kind of “algebraizable spectre” of that logic.

4. Fibrings, coverings and sheaves

4.1. Fibrings and coverings

Now that we have described the objects of $\mathcal{L}$ as colimits of the essentially small category $\mathcal{L}_{fp}$ and thus gained an embedding of $\mathcal{L}$ into $Set^{(\mathcal{L}_{fp})^{op}}$ (which is the cocompletion of $\mathcal{L}_{fp}$), we are in the position to pursue the intuition, mentioned in the introduction, that a logic l, whose consequence relation is generated by the images of translations of other logics l_i into l, can be considered to be “covered” by the l_i. To this end we introduce Grothendieck topologies on $\mathcal{L}_{fp}$ and investigate how the related sheaf theoretic notions apply to the logics in $\mathcal{L}$. The overall picture is: We are relating the categories displayed in the following diagram, where $Flat((\mathcal{L}_{fp})^{op}, Set)$ denotes the category of functors in $Set^{(\mathcal{L}_{fp})^{op}}$ that are filtered colimits of representables, where k, i denote the inclusion functors from sheaves into separated presheaves and from the latter to presheaves (we will recall these notions below) and where a and s are the associated sheaf and separated presheaf functors. a and s are left adjoint to k and i respectively. j denotes the inclusion functor of flat presheaves into general presheaves and, by the theory of locally presentable categories, there is a left adjoint to j which we call e. Further we define $L := e \circ i \circ k$ and $S := a \circ s \circ j$, L and S are adjoint to each other being a composition of adjoints; given any Grothendieck topology on $\mathcal{L}_{fp}$ we thus get an

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[14] We thank the referee who pointed to us this result and references.

adjunction between $Flat((\mathcal{L}_{fp})^{op}, Set)$ and the category of sheaves. Further, by the theory of accessible categories, $\mathcal{L}$ and $Flat((\mathcal{L}_{fp})^{op}, Set)$ are equivalent (by an equivalence denoted E in the diagram). In the following we shall make free use of these functors and their properties.

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "Y'"] \arrow[dr, "E"', "\simeq"] & Set^{(\mathcal{L}_{fp})^{op}} \arrow[r, shift left, "s"] \arrow[d, shift left, "e"] \arrow[dl, phantom, "\circlearrowleft"] & Sep(\mathcal{L}_{fp}, Fib) \arrow[l, shift left, "i"] \arrow[r, shift left, "a"] \arrow[dl, phantom, "\circlearrowleft"] & Sh(\mathcal{L}_{fp}, Fib) \arrow[l, shift left, "k"] \arrow[dll, shift right, "L"'] \\
 & Flat((\mathcal{L}_{fp})^{op}, Set) \arrow[u, shift left, "j"] \arrow[urr, shift right, "S"'] & &
\end{tikzcd}$$

First we will take a look at some possible notions of covering.

Definition 4.1. Let $l = (\Sigma, \vdash)$ be a finitely presentable logic and H a set of translation morphisms with codomain l and with domain a finitely presentable logic:

(i) H is a covering(i) of l iff for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma) \Gamma \vdash \psi \Leftrightarrow$ there is $\Gamma^- \subseteq_{fin} \Gamma$ and there exists $h \in H$ such that $\Gamma^- \cup \{\psi\} \subseteq \widehat{h}[F(\Sigma^{dom(h)})]$ and there is $\Gamma^{-^h} \cup \{\psi^h\} \subseteq_{fin} F(\Sigma^{dom(h)})$ such that $\widehat{h}[\Gamma^{-^h}] = \Gamma^-$, $\widehat{h}(\psi^h) = \psi$ and $\Gamma^{-^h} \vdash_{dom(h)} \psi^h$.

Since we have coproducts in $\mathcal{L}$, we can also express properties of families of morphisms with common codomain by means of the induced arrow from the coproduct of the occurring domains:

(ii) H is a covering(ii) of l iff the canonical translation morphism $c_H : \coprod_{h \in H} dom(h) \longrightarrow l$, the unique arrow such that for all $h \in H$, $(dom(h) \overset{h}{\rightarrow} \longrightarrow l) = (dom(h) \xrightarrow{i_h} \coprod_{h \in H} dom(h) \xrightarrow{c_H} l)$, is such that $\vdash = c_{H\star}(\vdash_H)$.

(iii) H is a covering(iii) of l iff the canonical translation morphism $c_H : \coprod_{h \in H} dom(h) \longrightarrow l$, the unique arrow such that for all $h \in H$, $(dom(h) \overset{h}{\rightarrow} \longrightarrow l) = (dom(h) \xrightarrow{i_h} \coprod_{h \in H} dom(h) \xrightarrow{c_H} l)$, is such that c_H is a conservative translation morphism that is also an $\mathcal{L}$-epimorphism.

(iv) H is a covering(iv) of l iff the morphism c_H of (ii) is an isomorphism.

The latter is equivalent to saying that H covers l iff l is a fibring in the sense of [24] of the domains of the morphisms in H and thus suggests the more general definition:

(v) Given any notion of fibring, define H to be a covering of l iff l is the result of fibring (in the given sense) the domains of the arrows in H.

Remark 4.2. Adopting either of the above covering notions[15] (or any other notion which gives conditions only involving the consequence relations), we have some space for "fine tuning" according to the intended meaning of "covering":

(i) We can require covering families to be epimorphic families (or, equivalently, to be c_H in (ii) to be epic). On the logical side, by Proposition 2.12, this means that every generating connective of the underlying language of the covered logic will occur in the image of some covering morphism and so rules out coverings by proper linguistic fragments as given by the one-element covering family $(\{\rightarrow\}, \textit{modus ponens}) \longrightarrow (\{\neg, \rightarrow\}, \textit{modus ponens})$. On the sheaf-theoretical side it implies that every representable functor will be a separated presheaf (see next subsection).

(ii) We can require the members of a covering family to be monos, thus (again by Proposition 2.12) allowing only coverings by proper linguistic fragments but not, for example, $(\{\wedge_1, \wedge_2\}, \langle A \wedge_1 B \vdash A, A \wedge_2 B \vdash B\rangle) \longrightarrow (\{\wedge\}, \langle A \wedge B \vdash A, A \wedge B \vdash B\rangle)$ (where the expressions $\langle \dots \rangle$ denote the consequence relations generated by the given inference rules and/or axioms; see Remark 2.8). This condition amounts to allowing only proper decompositions of logics as coverings and seems appropriate to treat the splitting of logics.

4.2. Sheaves

A notion of covering (i.e., a mapping Cov associating to each object X a collection $Cov(X)$ of families of morphisms to X) gives rise to a unique Grothendieck topology on $\mathcal{L}_{fp}$.[16] The covering sieves of an object X in this Grothendieck topology are the compositional right ideals of pullbacks to X of covering families in $\mathcal{L}_{fp}$ and will also be denoted by $Cov(X)$.

The purpose of this section is to investigate what it means for a logic to be a separated presheaf or a sheaf.

We first recall the relevant definitions.

Definition 4.3. Given a presheaf $F \in |Set^{(\mathcal{L}_{fp})^{op}}|$, an object $X \in |\mathcal{L}_{fp}|$ and a covering sieve $S = \{f_i : X_i \to X \mid i \in I\} \in Cov(X)$, a compatible family is a family $\{s_i \in F(X_i) \mid i \in I\}$ such that $F(f : X_i \to X_j)(s_j) = s_i$ for all $i, j \in I$. A presheaf F is called a separated presheaf (a sheaf, respectively) if, for each such $X \in |\mathcal{L}_{fp}|$, $S \in Cov(X)$ and compatible family $\{s_i \in F(X_i) \mid i \in I\}$, there is at most one (exactly one, respectively) $s \in F(X)$ such that $s_i = F(f_i : X_i \to X)(s)$ for all $i \in I$.

Remark 4.4. As we see in the diagram above, there is a pair of functors $\mathcal{L} \overset{L'}{\underset{S'}{\leftrightarrows}} Sh(\mathcal{L}_{fp}, Fib)$ such that $S' : \mathcal{L} \longrightarrow Sh(\mathcal{L}_{fp}, Fib)$ preserves finite limits and filtered

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[15] Some of these covering notions are obviously related: covering(iv) is the strongest notion between (i), (ii), (iii) and (iv); covering(iii) and covering(i) are stronger than covering(ii).

[16] Notation for that kind of site: $(\mathcal{L}_{fp}, Fib)$.

colimits.[17] As we know that every logic is a filtered colimit of finitely presentable logic, we get a "good codification" of $\mathcal{L}$ in $Sh(\mathcal{L}_{fp}, Fib)$ in the sense that codification preserves the glue (the filtered colimits) between fundamental bricks (the FP-logics). We also know that the category $\mathcal{L}$ of all logics has nice categorial properties, but the category has no nice logical properties! This "defect" of $\mathcal{L}$ is mitigated by the codification into the category of sheaves, a complete, cocomplete, locally finitely presentable category that does have nice logical properties: as an elementary topos it has exponential objects and a classifying object.

Now we will take a closer look at the notions of sheaf and separated sheaf and what they say about the behavior of a logic; we rephrase these definitions into statements about F in the category $Set^{(\mathcal{L}_{fp})^{op}}$.

Under the Yoneda lemma isomorphism, elements $s_i \in F(X_i), s \in F(X)$ correspond to unique natural transformations $\tau_i : Hom(-, X_i) \to F, \tau : Hom(-, X) \to F$ respectively and $F(f : X_i \to X_j)(s_j) = s_i$, $F(f : X_i \to X)(s) = s_i$ translate to $\tau_i = \tau_j \circ Hom(-, f)$, $\tau_i = \tau \circ Hom(-, f_i)$ respectively, so that we get the following characterization: F is a sheaf iff for each $X \in |\mathcal{L}_{fp}|, S \in Cov(X)$ the Yoneda embedded cone $Y(S)$ is universal for F in the sense that for every compatible family $\{\tau_i \mid i \in I\}$ of morphisms from the domains of $Y(S)$, there exists a unique arrow as in the following diagram:

$$\begin{tikzcd}
{} \arrow[dr] & & {} \arrow[dl] & & {} \arrow[dr] & & {} \arrow[dl] \\
& Hom(-, X_i) \arrow[rr, "g_i"'] \arrow[ddrr, "\tau_i"'] & & Hom(-, X) \arrow[dd, "\exists!"] & & Hom(-, X_j) \arrow[ll, "g_j"] \arrow[ddll, "\tau_j"] & \\
& & \circlearrowleft & & \circlearrowleft & & \\
& & & F & & &
\end{tikzcd}$$

So that F is a sheaf roughly says that "for F all (Yoneda embedded) covering sieves are colimit cones". From this formulation it is immediate to see the following equivalences which we note in passing:

Fact 8. (i) The representable functors are separated presheaves iff all covering families are epimorphic.

(ii) The representable functors are sheaves iff all covering sieves are colimit cones.

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[17] Because $Y' = j \circ E : \mathcal{L} \longrightarrow Set^{(\mathcal{L}_{fp})^{op}}$ preserves limits and filtered colimits and the "associated sheaf functor" $a \circ s : Set^{(\mathcal{L}_{fp})^{op}} \longrightarrow Sh(\mathcal{L}_{fp}, Fib)$ preserves finite limits and colimits.

So separatedness for a logic l means that a translation h from a finitary logic l' into l is completely determined by the translations obtained by composing with the morphisms of a covering family of l' and the sheaf property says that, given translations from the domains of a covering family of l', such a translation $l' \to l$ in fact always exists.

Example. A logic which is not a separated presheaf: take as (generating) coverings all families $\{f_i : l_i \to l \mid i \in I\}$ such that $\vdash_l \leq sup\{f_i^\star(\vdash_{l_i}) \mid i \in I\}$. Let l_1 be the logic with signature given by one connective c and the minimal consequence relation, and l_2 the logic with the signature consisting of two connectives c_1, c_2 with the same arity as c and the minimal consequence relation. Then the signature morphism mapping c to c_1 is clearly a morphism of logics and moreover gives a (one-element) covering family of l, but we have two different endomorphisms of l making the diagram below commute, namely the identity and the morphism that maps both c_1 and c_2 to c_1:

$$\begin{tikzcd}
(\{c\}, \vdash_{min}) \arrow[rr, "c \mapsto c_1"] \arrow[ddr, "c \mapsto c_1"'] & & (\{c_1, c_2\}, \vdash_{min}) \\
& \circlearrowleft & \\
& (\{c_1, c_2\}, \vdash_{min}) \arrow[uur, shift left, "id"] \arrow[uur, shift right, "t"'] &
\end{tikzcd}$$

This example (and lots of similar ones) suggests that separatedness is a kind of “Occam’s Razor property” in the sense that a separated logic has no redundant (e.g., doubly occurring or interchangeable) connectives. Thus the separation functor s (the left adjoint to the inclusion of separated presheaves into presheaves) would give a procedure to cut down redundancies in the presentation of a logic — and in fact s is defined by a quotient construction.[18] Be careful to note that we do not know a priori whether the application of s actually yields a presheaf corresponding to a logic; this has to be investigated separately after fixing a specific notion of covering. However, at any rate we get an equivalence relation between logics given by $l \sim l'$ iff $s(l) \cong s(l')$ which, in the spirit of the above, could be understood to hold when two (presentations of) logics have the same essence after cleaning up the redundancies. In the same vein, the sheaf property for a logic l indicates a good behavior with respect to translations from finitary logics into l and the sheafification functor yields an equivalence relation between logics.

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[18] For F a presheaf, $s(F)$ is defined by $s(F)(X) := F(X)/\sim$, where $a \sim b$ iff there is a covering $\{f_i \mid i \in I\}$ such that $F(f_i)(a) = F(f_i)(b)$ for all i.

5. Operads and the need for better categories of logics

Our chosen framework, the category $\mathcal{S}$ of signatures and the category $\mathcal{L}$ of logics, excludes many kinds of logics: the propositional logics with infinitary connectives, the non-reflexive logics, the linear logics and the relevant logics. Our category $\mathcal{S}$ of signatures is, for its simplicity and good categorial properties, often chosen as a test bed for categorial constructions like combination of logics, as is the case here. In practice it leads, however, to a much restricted notion of morphism between logics: for example, two presentations of classical propositional logic taking as primitive connectives (i.e., signatures) $\{\neg, \rightarrow\}, \{\neg, \vee\}$ respectively, do not admit any morphism between each other (since it would have to take $\rightarrow$ to $\vee$) while they could intuitively be expected to be isomorphic. In terms of the introduction, this means that our categories give an unsatisfactory (partial) answer to the identity problem.

To remedy this defect, [10] and others have taken as signature morphisms the substitutions from Section 2.1.2, thus allowing to take $\rightarrow$ to the derived connective $(\neg\ (-) \vee -)$. The resulting category $\mathcal{S}$ of signatures has, however, bad categorial properties — for instance, it does not have all pullbacks nor all colimits, which implies that a category of logics built above it (and thus coming with a limit creating forgetful functor to $\mathcal{S}$) can not be accessible. The reason for these categorial insufficiencies lies in the fact that $F(\Sigma)$ (the language freely generated by Σ) is the absolutely free algebra over its signature, so $\mathcal{S}$ is a category of free algebras and colimits of free structures are hardly free again. That formal languages are free algebras over some signature seems to be a crucial feature in logic, so how can we overcome this difficulty?

It could be helpful to rephrase the situation using the language of operads: An operad is a multicategory (a structure coming from proof theory and thus maybe known to logicians) with only one object. Such a structure can be seen as an axiomatization of the behavior of a collection of finitary operations on a set, closed under the formation of derived operations. Thus it consists of a set of "operations" of finite arity and bears a structure given by substitution of operations into other operations. A morphism of operads is a function between the sets of operations preserving arities and commuting with substitution. An algebra over an operad is an interpretation of the sets of operations as actual operations on some set; more formally, it is an operad morphism into the operad of all finitary operations on some set. An introduction to operads would go beyond the scope of this article, for this we refer to [6], which, although treating much more general operads than we are needing here, gives an excellent intuition of this notion. It is also possible to describe an operad by giving generating operations and relations between them, and accordingly there are free operads generated by a series of sets of n-ary operations — those with no relations. The formation of an absolutely free algebra over a signature can now be described in two steps: First form the free operad over the signature, then the free algebra over that operad. The substitutions from above are just morphisms between (free) operads which of course induce a function between the corresponding free operad-algebras. The above category of signatures is thus just the category of free operads and arbitrary operad morphisms. The operad-algebras only come into play when on wants to define a category of logics above the signatures: Since consequence relations are defined on sets (of formulas) we now need to form the free algebra over the operad to get such a set.

To escape from the dilemma that categories of free structures are bad behaved we propose a simple step: One could take as a category of signatures the category of *all* operads which is known to have good categorial properties. In particular it is locally presentable, and thus possibly allows to reproduce the results of this article. This would lead to a category of logics over languages with possibly interdefinable connectives (since now the operations can satisfy relations) and would thus include a common practice in logic into the formal treatment. To build a category of logics above this category of signatures one could proceed as before: Formally one could define an object in this category as a pair consisting of an operad $\mathcal{O}$ and a consequence relation (possibly satisfying some further conditions) on the underlying set of the associated free $\mathcal{O}$-algebra. A morphism would be a morphism of operads such that the induced function between the associated free algebras preserves consequence (again possibly satisfying further conditions).

The step from free operads to arbitrary ones may seem to go away from the usual intuitions of logic, where one is used to the sets of formulas having the structure of an absolutely free algebra, and at first one may feel to have thus introduced pathological objects into one's category of logics. But first, the informal use of interdefinable connectives in logical practice does not cause any problems, and if the language of operads is an adequate formalization of this practice, which it seems, this gives a reason to hope not to encounter difficulties on the formal side as well. Second, on a more abstract level, it is a highly successful mathematical practice to admit pathological objects in a category in order to make (the global properties of) the category itself less pathological — the passage from manifolds to C^∞-schemes in Differential Geometry illustrates well this point, as does the functorial approach to algebraic geometry, where one passes by the Yoneda embedding from schemes into a category of functors where most objects have no geometric appeal at all.

6. Conclusion

The main concrete purpose of this work was to bring the theory of accessible categories into the study of logics. Let us summarize what we have achieved for this goal and point out some further consequences:

1. The category of finitary structural logics is locally finitely presentable.
2. The category of algebraizable logics is accessible but not locally presentable.
3. We have identified which are the finitely presentable logics: They are precisely those whose signature has a finite number of symbols and whose consequence relation can be generated by a finite number of axioms and deduction rules.

Thus the categorical notion of finite presentability makes sense in the context of logics — a fact which was not a priori clear and encourages further studies of the accessibility of categories of logics.

4. There exist direct and inverse image logics.
5. It is a question of Universal Logic whether there exist irreducible logics, i.e., logics which cannot be obtained by combining proper non-trivial sublogics, and, if so, what characterizes such logics. We know that, in a λ-accessible category of logics, any logic is a colimit of λ-presentable logics, and therefore it receives morphisms from its colimit cocone. Taking the images of these morphisms we obtain sublogics whose combination gives the original logic. This indicates that an irreducible logic would have to be λ-presentable.

   The question about irreducible logics is, of course, aimed at finding the fundamental building bricks out which any other logic can be constructed. In a λ-accessible category of logics it would suffice to show that every λ-presentable logic can be built out of irreducible ones, the rest being given by colimits (i.e. combinations) of these.
6. Another question of Universal Logic is to what extent Logic (as a field of study) has an algebraic character. The notions of accessibility and local presentability bring new aspects into this question. Of course it is still not an entirely settled question how well local presentability really corresponds to algebraicity.
7. About the metatheory of logics: That the categories studied here are accessible, implies, by the model theoretical characterizations of accessible categories mentioned in the introduction, that they are categories of models of first order theories. The locally finitely presentable category $\mathcal{L}$ is even a category of models of a theory in usual classical first order logic, which is somewhat astonishing, given the fact that the definition of a consequence relation involves the powerset. It is possible that there is a reasonable model theory for logics which form a locally finitely presentable category.

For the moment the main task for starting the exploitation of the theory of accessible categories is to find “serious” categories of logics, not suffering from the defects pointed out in the previous section — the possible use of operads suggested there could do the job. Independently of that we hope to have convinced the reader that global properties of categories of logics are not an esoteric subject matter, but make themselves felt in concrete situations and are worthwhile to care about.

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