A propositional logic for Tarski's consequence operator

Hércules de Araujo Feitosa
UNESP - FC - Bauru
haf@fc.unesp.br

Mauri Cunha do Nascimento
UNESP - FC - Bauru
mauri@fc.unesp.br

Maria Claudia Cabrini Grácio
UNESP - FFC - Marília
cabrini@marilia.unesp.br

Abstract: This paper presents the TK-algebras associated to the Tarski's consequence operator and introduces the TK Logic. So it shows the adequacy (soundness and completeness) of TK Logic relative to the algebraic model given by the TK-algebras.

Introduction

In (Grácio, Feitosa, Nascimento, 200_) it was introduced an extended logic to capture the concept of Tarski's deductive system. The central notion of consequence operator of Tarski is expressed through a new generalized quantifier included in the first-order language. Syntactical elements of the Tarski's consequence operators are involved and they are interpreted by open sets of the Tarski's deductive system. In that reference emerged the question about the possibility of representing such deductive systems only in propositional environments. This paper introduces the TK-algebras that correspond to the semantical structures in which notions of Tarski's operators will be interpreted at the propositional TK Logic.

1. Tarski's consequence operators and logic

We define an *almost topological space* as a pair (S, θ) such that S is a nonempty set and $\theta \subseteq \mathcal{P}(S)$ satisfies the following condition:

(QT_1) if $B \subseteq \theta$, then $\cup B \in \theta$.

The collection θ is called an *almost topology* and each member of θ is called an *open set* of (S, θ). A set in $\mathcal{P}(S)$ is called a *closed* of (S, θ) when its complement is an open set.

Proposition 1.1: In an almost topological space (S, θ) the set $\varnothing$ is an open set and S is a closed set. ■

Proposition 1.2: In an almost topological space any intersection of closed sets is still a closed set of (S, θ). ■

Tarski, in 1935, looking for a general characterization to the notion of logic, introduced the concept of consequence operator in a version similar to the following one.

A *consequence operator* on S is a function $\mathsf{C}: \mathcal{P}(S) \to \mathcal{P}(S)$ such that, for every A, B $\subseteq$ S:

(C_1) $A \subseteq \mathsf{C}(A)$

(C_2) $A \subseteq B \Rightarrow \mathsf{C}(A) \subseteq \mathsf{C}(B)$

(C_3) $\mathsf{C}(\mathsf{C}(A)) \subseteq \mathsf{C}(A)$.

Of course, for every consequence operator C, by (C_1) and (C_3), the equality $\mathsf{C}(\mathsf{C}(A)) = \mathsf{C}(A)$ holds.

A consequence operator C on S is *finitary* when, for every $A \subseteq S$, $\mathsf{C}(A) = \cup\{\mathsf{C}(A_0) / A_0$ is a finite subset of $A\}$.

A *Tarski's deductive system* is a pair (S, C) such that S is a set and C is a consequence operator on S.

Let C be a consequence operator on S. The set A is called a *closed* in (S, C) when $\mathsf{C}(A) = A$, and A is called an *open* when its complement relative to S, denoted by A^{C}, is a closed of (S, C).

Proposition 1.3: In (S, C) any intersection of closed sets is also a closed set. ■

Clearly, $\mathsf{C}(\varnothing)$ and S correspond to least and greatest closed sets, respectively, associated to the consequence operator C.

A deductive system (S, C) is *vacuous* when $\mathsf{C}(\varnothing) = \varnothing$.

We can see that every topological space is a deductive system. However, the converse is not true, since in general, in a deductive system $\mathsf{C}(\varnothing) \neq \varnothing$. Topological spaces are examples of vacuous deductive systems and not much interesting from the logical point of view.

The question is to know if we can define a deductive system in a similar way that one in topology. The answer is affirmative and in a surprising way, because we can verify that the concepts of deductive system and almost topology are equivalents.

Let (S, θ) be an almost topological space. We verify that (S, θ) is a deductive system if we define, for every $A \subseteq S$, the *closure* of A as $\mathsf{C}(A) = \cap\{X \subseteq S / X$ is closed and $A \subseteq X\}$.

Proposition 1.4: In any almost topological space (S, θ), for every $A \subseteq S$, the set $\mathsf{C}(A)$ is a closed set. ■

Theorem 1.5: Let (S, θ) be an almost topological space and let $\mathsf{C}(A)$ be defined as above, for every $A \subseteq S$. Then (S, C) is a deductive system. ■

On the other hand, if (S, C) is a deductive system, let us consider $\theta = \{X \subseteq S / X$ is open$\}$.

Theorem 1.6: Let (S, C) be a deductive system. Then (S, θ) is an almost topological space. ■

For the logic of Tarski's consequence operators, we consider a classical first-order language $L^{\tau}_{\omega\omega}$ with type of similarity τ containing symbols for predicates, functions and constants, closed for the connectives ∧, ∨, →, ¬ and for the quantifiers ∃ and ∀.

By $L^{\tau}_{\omega\omega}(Q)$ we denote the extension of $L^{\tau}_{\omega\omega}$ obtained by including the extended quantifier Q. The formulas (and sentences) of $L^{\tau}_{\omega\omega}(Q)$ are the same of $L^{\tau}_{\omega\omega}$ plus those generated by the following clause: if **A** is a formula in $L^{\tau}_{\omega\omega}(Q)$, then (Qx)**A** is also a formula in $L^{\tau}_{\omega\omega}(Q)$.

The notions of free and bound variables in a formula, as well as other syntactical notions, are naturally extended for the quantifier Q.

We denote by **A**(t / x) the result of substituting all free occurrences of a variable x by a term t in the formula **A**. For simplicity, as in $L^{\tau}_{\omega\omega}$, when there is no danger of confusion, we just write **A**(t) instead of **A** (t / x).

The axioms of the logic of Tarski's consequence operators denoted by $\mathcal{L}^{\tau}_{\omega\omega}(Q)$ are those of $\mathcal{L}^{\tau}_{\omega\omega}$, including the identity axioms, plus the following specific axioms for the quantifier Q:

(Ax_1) (∀y)(Qx)**A**(x, y) → (Qx)(∃y)**A**(x, y)

(Ax_2) (∀x)(**A**(x) ↔ **B**(x)) → ((Qx)**A**(x) ↔ (Qx)(**B**(x))

(Ax_3) (Qx)**A**(x) → (Qy)**A**(y), when y is free for x in **A**(x).

The intuitive interpretation for these axioms is as follows:

(Ax_1) The union of open sets is an open set;

(Ax_2) If the sets A and B have the same elements, then A is an open set iff B is an open set;

(Ax_3) This axiom permits the change of variable bounded by the quantifier Q.

The basic logical rules of the system $\mathcal{L}^{\tau}_{\omega\omega}(Q)$ are the usual rules of classical logic:

Modus Ponens (MP): **A**, **A**→**B** ⊢ **B**

Generalization (Gen): **A** ⊢ (∀x)**A**.

Usually syntactical concepts such as proof, theorem, consistency and others for $\mathcal{L}^{\tau}_{\omega\omega}(Q)$ are also appropriately adapted from classical first-order logic.

We now introduce the TK-algebras and after a propositional logic associated to those algebras.

2. Introducing the TK-algebras

A *TK-algebra* is a sextuple $\mathcal{A}$ = (A, 0, 1, ∨, ~, •) such that (A, 0, 1 ∨, ~) is a Boolean algebra and • is a new operator, called *operator of Tarski*, such that:

(i) $a \vee \bullet a = \bullet a$;

(ii) $\bullet a \vee \bullet(a \vee b) = \bullet(a \vee b)$

(iii) $\bullet(\bullet a) = \bullet a$.

Since we are working with a Boolean algebra, the item (i) of the above definition asserts that, for every $a \in$ A, $a \leq \bullet a$ and we can define in the algebra $a \rightarrowtail b =_{df} \sim a \vee b$.

Proposition 2.1: In a TK-algebra it is valid:

(i) $\sim\bullet a \leq \sim a \leq \bullet\sim a$

(ii) $a \leq b \Rightarrow \bullet a \leq \bullet b$.

Proof: (ii) $a \leq b \Rightarrow a \vee b = b \Rightarrow \bullet(a \vee b) = \bullet b \Rightarrow \bullet a \vee \bullet(a \vee b) = \bullet(a \vee b) = \bullet b \Rightarrow \bullet a \leq \bullet b$. ■

Proposition 2.2: In a TK-algebra the following assertions are valid:

(i) $\bullet(a \wedge b) \leq \bullet a \wedge \bullet b$

(ii) $\bullet a \vee \bullet b \leq \bullet(a \vee b)$.

Proof: (i) $a \wedge b \leq a$ and $a \wedge b \leq b \Rightarrow \bullet(a \wedge b) \leq \bullet a$ and $\bullet(a \wedge b) \leq \bullet b \Rightarrow \bullet(a \wedge b) \leq \bullet a \wedge \bullet b$.

(ii) is similar to (i). ■

Proposition 2.3: In a TK-algebra it is valid:

(i) $\bullet(\bullet a \wedge \bullet b) = \bullet a \wedge \bullet b$

Proof: It is enough to verify that $\bullet(\bullet a \wedge \bullet b) \leq \bullet a \wedge \bullet b$. But, $\bullet(\bullet a \wedge \bullet b) \leq \bullet\bullet a \wedge \bullet\bullet b = \bullet a \wedge \bullet b$. ■

We have a new operation in a TK-algebra, dual of $\bullet$:

$$\circ a =_{df} \sim\bullet\sim a.$$

Proposition 2.4: In a TK-algebra, the following conditions are valid:

(i) $\circ a \leq a$

(ii) $\circ(a \wedge b) \leq \circ a$

(iii) $\circ a \leq \circ\circ a$

(iv) $a \leq b \Rightarrow \circ a \leq \circ b$ ■

An element $a \in \mathcal{A}$ is *closed* when $\bullet a = a$ and $a \in \mathcal{A}$ is *open* when $\circ a = a$.

Proposition 2.5: (i) If a is open, then $a \leq b \Leftrightarrow a \leq \circ b$

(ii) If b is closed, then $a \leq b \Leftrightarrow \bullet a \leq b$. ■

An algebra $\mathcal{A}$ is *non-degenerate* when its universe A has at least two elements.

Proposition 2.6: For each TK-algebra $\mathcal{A} = (A, 0, 1, \vee, \sim, \bullet)$ there is a monomorphism h of $\mathcal{A}$ into an almost topological space of sets defined in $\mathcal{P}(\mathcal{P}(A))$.

Proof: Through the Stone's isomorphism, we know that for each Boolean algebra $\mathcal{A} = (A, 0, 1, \vee, \sim)$ there is a monomorphism h of A into a field of subsets of $\mathcal{P}(A)$.

Next, we introduce an almost topology in $\mathcal{P}(\mathcal{P}(A))$ in the following way. For each set $X \subseteq A$, we define:

$$C(X) = \cap_{a \in A} \{h(a) \,/\, X \subseteq h(a) \text{ and } a = \bullet a\}.$$

We must show that:

(i) $X \subseteq C(X)$

(ii) $X \subseteq Y \Rightarrow C(X) \subseteq C(Y)$

(iii) $\mathsf{C}(\mathsf{C}(X)) \subseteq \mathsf{C}(X)$
(iv) $h(\bullet a) = \mathsf{C}(h(a))$.

(i) Direct consequence of definition.

(ii) Suppose that $\mathsf{C}(X) \not\subset \mathsf{C}(Y)$. Then there is z such that $z \in \mathsf{C}(X)$ and $z \notin \mathsf{C}(Y)$. So, for some $a \in A$, $z \notin h(a)$ with $Y \subseteq h(a)$ and $\bullet a = a$. Since $X \subseteq Y \subseteq h(a)$ and $\bullet a = a$, then $z \notin \mathsf{C}(X)$, and it contradicts the hypothesis.

(iii) $x \in \mathsf{C}(\mathsf{C}(X)) \Rightarrow x \in \cap_{a\in A} \{h(a) / C(X) \subseteq h(a) \text{ e } a = \bullet a\}$. As $X \subseteq \mathsf{C}(X)$, it follows that $x \in \cap_{a\in A} \{h(a) / X \subseteq h(a) \text{ and } a = \bullet a\} = C(X)$ and, therefore, $\mathsf{C}(\mathsf{C}(X)) \subseteq \mathsf{C}(X)$.

(iv) $h(\bullet a) \subseteq \mathsf{C}(h(a))$:

Consider that $h(a) \subseteq h(b)$ and $b = \bullet b$. Since h is a Boolean monomorphism, $a \leq b$ and $\bullet a \leq \bullet b$. But, since $\bullet b = b$, then $\bullet a \leq b$ and $h(\bullet a) \subseteq h(b)$. Concluding, for each $b \in A$ such that $\bullet b = b$ and $h(a) \subseteq h(b)$, results that $h(\bullet a) \subseteq h(b)$, that is, $h(\bullet a) \subseteq \mathsf{C}(h(a))$.

$\mathsf{C}(h(a)) \subseteq h(\bullet a)$:

$a \leq \bullet a \Rightarrow h(a) \subseteq h(\bullet a) \Rightarrow \mathsf{C}(h(a)) \leq \mathsf{C}(h(\bullet a))$. Since $h(\bullet a) \subseteq h(\bullet a)$ and $\bullet a = \bullet\bullet a$, then $\mathsf{C}(h(\bullet a)) = h(\bullet a)$ and $\mathsf{C}(h(a)) \subseteq h(\bullet a)$. ■

In next section we introduce a new logic associated with the concept of Tarski's consequence operator.

3. TK Logic

TK propositional logic is determined over a propositional language $L(\neg, \vee, \rightarrow, \blacklozenge, p_1, p_2, p_3, \ldots)$ as follows:

Axioms:

Classical Propositional Calculus Axioms (CPC) +
Ax_{TK1} $\mathbf{A} \rightarrow \blacklozenge\mathbf{A}$
Ax_{TK2} $\blacklozenge\blacklozenge\mathbf{A} \rightarrow \blacklozenge\mathbf{A}$.

Deduction Rules:

MP: *Modus Ponens*

R♦: $\mathbf{A} \rightarrow \mathbf{B} \vdash \blacklozenge\mathbf{A} \rightarrow \blacklozenge\mathbf{B}$.

Proposition 3.1: $\vdash \blacklozenge\mathbf{A} \rightarrow \blacklozenge(\mathbf{A}\vee\mathbf{B})$.

Proof:

1. $\mathbf{A} \rightarrow (\mathbf{A}\vee\mathbf{B})$	Tautology	
2. $\blacklozenge\mathbf{A} \rightarrow \blacklozenge(\mathbf{A}\vee\mathbf{B})$	R♦ in 1.	■

Proposition 3.2: $\mathbf{A} \vdash \blacklozenge\mathbf{A}$.

Proof:

1. $\mathbf{A}$	Premise	
2. $\mathbf{A} \rightarrow \blacklozenge\mathbf{A}$	Ax_{TK1}	
3. $\blacklozenge A$	MP in 1 and 2.	■

Like in algebra, we can define the dual operator of ♦ in the following way:

$$\Diamond\mathbf{A} \Leftrightarrow_{df} \neg\blacklozenge\neg\mathbf{A}.$$

Proposition 3.3: $\mathbf{A} \to \mathbf{B} \vdash \Diamond\mathbf{A} \to \Diamond\mathbf{B}$.

Proof:

1. $\mathbf{A} \to \mathbf{B}$	p.
2. $(\mathbf{A} \to \mathbf{B}) \to (\neg\mathbf{B} \to \neg\mathbf{A})$	Tautology
3. $\neg\mathbf{B} \to \neg\mathbf{A}$	MP in 1 and 2
4. $\blacklozenge\neg\mathbf{B} \to \blacklozenge\neg\mathbf{A}$	$\mathbf{R}\blacklozenge$ in 3
5. $(\blacklozenge\neg\mathbf{B} \to \blacklozenge\neg\mathbf{A}) \to (\neg\blacklozenge\neg\mathbf{A} \to \neg\blacklozenge\neg\mathbf{A})$	Tautology
6. $\neg\blacklozenge\neg\mathbf{A} \to \neg\blacklozenge\neg\mathbf{A}$	MP in 4 and 5
7. $\Diamond\mathbf{A} \to \Diamond\mathbf{B}$	Definition of $\Diamond$ in 6. ■

Proposition 3.4: $\vdash \Diamond\mathbf{A} \to \mathbf{A}$.

Proof:

1. $\neg\mathbf{A} \to \blacklozenge\neg\mathbf{A}$	Ax_{TK1}
2. $\neg\blacklozenge\neg\mathbf{A} \to \neg\neg\mathbf{A}$	Tautology in 1
3. $\Diamond\mathbf{A} \to \mathbf{A}$	DN and definition of $\Diamond$ in 2. ■

Proposition 3.5: $\vdash \Diamond\mathbf{A} \to \Diamond\Diamond\mathbf{A}$.

Proof:

1. $\blacklozenge\blacklozenge\neg\mathbf{A} \to \blacklozenge\neg\mathbf{A}$	Ax_{TK2}
2. $\neg\blacklozenge\neg\mathbf{A} \to \neg\blacklozenge\blacklozenge\neg\mathbf{A}$	Tautology in 1
3. $\neg\blacklozenge\neg\mathbf{A} \to \neg\blacklozenge\neg\neg\blacklozenge\neg\mathbf{A}$	DN in 2
4. $\Diamond\mathbf{A} \to \Diamond\Diamond\mathbf{A}$	Definition of $\Diamond$ in 3. ■

We could, alternatively, consider the operator $\Diamond$ as primitive and substitute the axioms Ax_{TK1} and Ax_{TK2} by the following:

Ax_{TK1*} $\quad \Diamond\mathbf{A} \to \mathbf{A}$

Ax_{TK2*} $\quad \Diamond\mathbf{A} \to \Diamond\Diamond\mathbf{A}$,

and the rule $R\blacklozenge$ by the rule $R\Diamond$:

$R\Diamond$ $\quad \mathbf{A} \to \mathbf{B} \vdash \Diamond\mathbf{A} \to \Diamond\mathbf{B}$.

The next objective of this paper is to show that the variety of TK-algebras consists in an appropriate algebraic semantic for the TK Logic.

4. The algebraic adequacy

Next, we will indicate the set of propositional variables of TK by **Var**(TK), the set of its formulas by **For**(TK) and a generic TK-algebra by $\mathcal{A}$.

The propositional logical system TK is determined by a pair (L, C), where L is the propositional language of TK and C is a consequence operator on **For**(TK) given by axioms and deduction rules of TK.

Thus, for $\Gamma \subseteq$ **For**(TK), considering **Ax** as the set of axioms of TK, then $C(\Gamma) = \{\mathbf{B} / \Gamma \cup \mathbf{Ax} \vdash \mathbf{B}\}$. We denote that **B** is derivable in TK or is a theorem of TK when $\mathbf{B} \in C(\varnothing)$.

A TK-*theory* is a triple **T** = (L, C, Δ) in which L is the language of TK, C is its consequence operator and Δ is a set of non-logical axioms.

When $\Delta = \emptyset$, then $\mathbf{T} = (\mathrm{L}, \mathsf{C}, \Delta) = \mathrm{TK}$. We denote that **B** is a theorem of **T** by $\vdash_{\mathbf{T}} \mathbf{B}$.

A formula $\mathbf{B} \in \mathbf{For}(\mathrm{TK})$ is *refutable* in **T** when $\neg\mathbf{B}$ is a theorem of **T**. Otherwise, **B** is *irrefutable*.

A *restrict valuation* is a function v^: $\mathbf{Var}(\mathrm{TK}) \to \mathcal{A}$, that interprets each variable of TK in an element of $\mathcal{A}$.

A *valuation* is a function v: $\mathbf{For}(\mathrm{TK}) \to \mathcal{A}$, that extends natural and uniquely v^ as follows:

$v(\mathrm{p}) = v\hat{}(\mathrm{p})$
$v(\neg\mathbf{A}) = {\sim}v(\mathbf{A})$
$v(\mathbf{A}\vee\mathbf{B}) = v(\mathbf{A})\vee v(\mathbf{B})$
$v(\mathbf{A}\to\mathbf{B}) = v(\mathbf{A}) \rightarrowtail v(\mathbf{B})$
$v(\blacklozenge\mathbf{A}) = \bullet v(\mathbf{A})$.

As usual, operator symbols of left members represent logical operators and the right ones represent algebraic operators.

Let $\mathcal{A}$ be a TK-algebra. A valuation v: $\mathbf{For}(\mathrm{TK}) \to \mathcal{A}$ is a model for a set $\Delta \subseteq \mathbf{For}(\mathrm{TK})$ when $v(\mathbf{A}) = 1$, for each formula $\mathbf{A} \in \Delta$. In particular, a valuation v: $\mathbf{For}(\mathrm{TK}) \to \mathcal{A}$ is a model for $\mathbf{A} \in \mathbf{For}(\mathrm{TK})$ when $v(\mathbf{A}) = 1$.

A formula **A** is *valid* in a TK-algebra $\mathcal{A}$ when each valuation v: $\mathbf{For}(\mathrm{TK}) \to \mathcal{A}$ is a model for **A**.

A formula **A** is *TK-valid*, what is denoted by $\vDash \mathbf{A}$, when it is valid in every TK-algebra.

Let $\mathbf{T} = (\mathrm{L}, \mathsf{C}, \Delta)$ be a TK-theory and $\mathcal{B} = (\mathbf{For}, \vee, \wedge, \to, \neg)$ the algebra of formulas of language L. We consider the algebra of formulas of TK, $(\mathbf{For}(\mathrm{TK}), \vee, \to, \neg, \blacklozenge, 0, 1)$, such that $\vee$ and $\to$ are binary operators, $\neg$ and $\blacklozenge$ are unary operators and 0 and 1 are constants and $\mathbf{A}\to\mathbf{B} =_{\mathrm{df}} \neg\mathbf{A}\vee\mathbf{B}$. As usual, we define the Lindenbaum algebra of LK.

We define an equivalence relation ~ by:

$$\mathbf{A} \sim \mathbf{B} \Leftrightarrow_{\mathrm{df}} \vdash_{\mathbf{T}} \mathbf{A} \to \mathbf{B} \text{ and } \vdash_{\mathbf{T}} \mathbf{B} \to \mathbf{A}.$$

The relation ~, more than an equivalence, is a congruence, since by rule R$\blacklozenge$:

$$\mathbf{A} \sim \mathbf{B} \Rightarrow \vdash \mathbf{A} \to \mathbf{B} \text{ and } \vdash \mathbf{B} \to \mathbf{A} \Rightarrow \vdash \blacklozenge\mathbf{A} \to \blacklozenge\mathbf{B} \text{ and } \vdash \blacklozenge\mathbf{B} \to \blacklozenge\mathbf{A} \Rightarrow \blacklozenge\mathbf{A} \sim \blacklozenge\mathbf{B}.$$

For each $\mathbf{B} \in \mathbf{For}(\mathrm{TK})$, we denote by $[\mathbf{B}] = \{\mathbf{C} \in \mathbf{For}(\mathrm{TK}) \,/\, \mathbf{C} \sim \mathbf{B}\}$ the class of equivalence of **B** modulo ~.

The (Lindenbaum) a*lgebra* of theory **T**, denoted by $\mathcal{A}(\mathbf{T})$, is the quotient algebra $\mathcal{B}_{\sim}$, defined by:

$$\mathcal{A}(\mathbf{T}) = (\mathbf{For}(\text{TK})|_\sim, \mathbf{0}, \mathbf{1}, \vee_\sim, \neg_\sim, \blacklozenge_\sim),$$

such that:

$[\mathbf{A}] \vee_\sim [\mathbf{B}] = [\mathbf{A}\vee\mathbf{B}]$,
$\neg_\sim [\mathbf{A}] = [\neg\mathbf{A}]$,
$\blacklozenge_\sim [\mathbf{A}] = [\blacklozenge\mathbf{A}]$,
$\mathbf{0} = [\mathbf{A}\wedge\neg\mathbf{A}]$ and
$\mathbf{1} = [\mathbf{A}\vee\neg\mathbf{A}]$.

When **T** = TK, we indicate the algebra of **T** by $\mathcal{A}$(TK). In general, we will not indicate the index ~ of operations.

Proposition 4.1: In $\mathcal{A}(\mathbf{T})$ it is valid $[\mathbf{A}] \leq [\mathbf{B}] \Leftrightarrow \vdash \mathbf{A} \to \mathbf{B}$.

Proof: $[\mathbf{A}] \leq [\mathbf{B}] \Leftrightarrow [\mathbf{A}] \vee [\mathbf{B}] = [\mathbf{B}] \Leftrightarrow [\mathbf{A}\vee\mathbf{B}] = [\mathbf{B}] \Leftrightarrow \vdash \mathbf{A}\vee\mathbf{B} \leftrightarrow \mathbf{B} \Leftrightarrow \vdash \mathbf{A} \to \mathbf{B}$. ■

Proposition 4.2: The algebra $\mathcal{A}(\mathbf{T})$ is a TK-algebra**.**
Proof:
Ax_{TK1} $\mathbf{A} \to \blacklozenge\mathbf{A} \Rightarrow [\mathbf{A}] \leq [\blacklozenge\mathbf{A}] \Rightarrow [\mathbf{A}] \leq \blacklozenge[\mathbf{A}]$
Proposition 3.1: $\vdash \blacklozenge\mathbf{A} \to \blacklozenge(\mathbf{A}\vee\mathbf{B}) \Rightarrow [\blacklozenge\mathbf{A}] \leq [\blacklozenge(\mathbf{A}\vee\mathbf{B})] \Rightarrow \blacklozenge[\mathbf{A}] \leq \blacklozenge[\mathbf{A}\vee\mathbf{B}]$.
Ax_{TK2} $\blacklozenge\blacklozenge\mathbf{A} \to \blacklozenge\mathbf{A} \Rightarrow [\blacklozenge\blacklozenge\mathbf{A}] \leq [\blacklozenge\mathbf{A}] \Rightarrow \blacklozenge\blacklozenge[\mathbf{A}] \leq \blacklozenge[\mathbf{A}]$. ■

The algebra $\mathcal{A}$(TK) is the *canonical model* of TK.

Corollary 4.3: Let **A** be a member of **For**(TK). The formula **A** is a theorem of **T** if [**A**] is the unit **1** of $\mathcal{A}(\mathbf{T})$. The formula **A** is irrefutable in **T** if $[\mathbf{A}] \neq \mathbf{0}$. The theory **T** is consistent if TK-algebra $\mathcal{A}(\mathbf{T})$ is non-degenerate.

Proof: Let $\vdash_{\mathbf{T}} \mathbf{A}$. Since $\mathcal{A}(\mathbf{T})$ always has an identity element **1**, then:

1. $\mathbf{A}$	Hypothesis
2. $\mathbf{A}\to(\mathbf{B}\to\mathbf{A})$	Tautology
3. $\mathbf{A}\to((\mathbf{A}\to\mathbf{A})\to\mathbf{A})$	Substitution in 2
4. $(\mathbf{A}\to\mathbf{A})\to\mathbf{A}$	MP in 1 and 3

Hence: $\mathbf{1} = [\mathbf{A}\to\mathbf{A}] \leq [\mathbf{A}]$, that is, $[\mathbf{A}] = \mathbf{1}$.

On the other hand, consider $[\mathbf{A}] = \mathbf{1}$, so $[\mathbf{A}\to\mathbf{A}] \leq [\mathbf{A}]$, this means that $\vdash_{\mathbf{T}} (\mathbf{A}\to\mathbf{A})\to\mathbf{A}$. Since $\vdash_{\mathbf{T}} \mathbf{A}\to\mathbf{A}$, it follows, by MP, that $\vdash_{\mathbf{T}} \mathbf{A}$.

Now, **A** is irrefutable iff $\nvdash_{\mathbf{T}} \neg\mathbf{A}$ iff $[\neg\mathbf{A}] \neq \mathbf{1}$ iff $\neg_\sim [\mathbf{A}] \neq \mathbf{1}$ iff $[\mathbf{A}] \neq \mathbf{0}$.

Finally, $[\mathbf{A}] = \mathbf{1}$ iff $\vdash_{\mathbf{T}} \mathbf{A}$, then $\mathcal{A}(\mathbf{T})$ has a different element of **1** iff there is $\mathbf{A} \in \mathbf{For}(\text{TK})$ such that $\nvdash_{\mathbf{T}} \mathbf{A}$. ■

It results from preceding propositions that for every formula **A**:

$[\mathbf{A}] = \mathbf{1}$ iff $\vdash_{\mathbf{T}} \mathbf{A}$ and

$[\mathbf{A}] = \mathbf{0}$ iff $\vdash_{\mathbf{T}} \neg\mathbf{A}$,

and, since $\mathcal{A}$(TK) is non-degenerate, then TK is consistent.

Theorem 4.4: (*Soundness*) The TK-algebras are correct models for TK logic.

Proof: Let $\mathcal{A} = (A, 0, 1, \vee, \sim, \bullet)$ be a TK-algebra. It remains to prove that the axioms Ax_{TK1} and Ax_{TK2} are valid and the rule R♦ preserves validity:

Ax_{TK1} - $v(\mathbf{A}\to\blacklozenge\mathbf{A}) = v(\mathbf{A}) \rightarrowtail v(\blacklozenge\mathbf{A}) = \sim v(\mathbf{A}) \vee v(\blacklozenge\mathbf{A}) = \sim v(\mathbf{A}) \vee (v(\mathbf{A}) \vee v(\blacklozenge\mathbf{A})) = (\sim v(\mathbf{A}) \vee v(\mathbf{A})) \vee v(\blacklozenge\mathbf{A}) = 1 \vee v(\blacklozenge\mathbf{A}) = 1$.

Ax_{TK2} - $v(\blacklozenge\blacklozenge\mathbf{A}\to\blacklozenge\mathbf{A}) = \bullet\bullet v(\mathbf{A}) \rightarrowtail \bullet v(\mathbf{A}) = \sim\bullet\bullet v(\mathbf{A}) \vee \bullet v(\mathbf{A}) = \sim\bullet v(\mathbf{A}) \vee \bullet v(\mathbf{A}) = 1$.

R♦ - Using Proposition 2.1: $v(\mathbf{A}\to\mathbf{B}) = 1 \Leftrightarrow v(\mathbf{A}) \leq v(\mathbf{B}) \Rightarrow v(\bullet\mathbf{A}) \leq v(\bullet\mathbf{B}) \Leftrightarrow v(\bullet\mathbf{A}\to\bullet\mathbf{B}) = 1$. ■

Corollary 4.5: Propositional calculus TK is consistent.

Proof: Suppose that TK is not consistent. Then there is $\mathbf{A} \in$ **For**(TK) such that $\vdash \mathbf{A}$ and $\vdash \neg\mathbf{A}$. By Soundness Theorem, **A** e $\neg$**A** are valid. Let v be a valuation in a TK-algebra with two elements $\mathbf{2} = \{0, 1\}$. Since **A** is valid, then $v(\mathbf{A}) = 1$ and therefore $v(\neg\mathbf{A}) = \sim v(\mathbf{A}) = 0$. This contradict the fact of $\neg$**A** is valid. ■

Theorem 4.6: Let **A** be a member of **For**(TK). The following assertions are equivalent:

(i) **A** is derivable in TK;

(ii) **A** is valid;

(iii) **A** is valid in every TK-algebra of closed subsets of a deductive system (S, C);

(iv) $v^*_{\mathcal{A}(TK)}(\mathbf{A}) = 1$, where v^* is the valuation defined at the canonical model.

Proof: (i) ⇒ (ii): it follows of Soundness Theorem.

(ii) ⇒ (iii): it suffices to observe that the algebra of closed subsets of any deductive system is a TK-algebra.

(iii) ⇒ (iv): since every TK-algebra is isomorphic to a sub-algebra of closed subsets of a deductive system (S, C) and $\mathcal{A}$(TK) is a TK-algebra, the result follows.

(iv) ⇒ (i): if $\mathbf{A} \in$ **For**(TK) and it is not derivable in TK, by Corollary 4.3, [**A**] do not coincide with the unity of $\mathcal{A}$(TK) and, thus $v^*_{\mathcal{A}(TK)}(\mathbf{A}) \neq 1$. Therefore **A** is not a valid formula. ■

Corollary 4.7: (*Completeness*) For each $\mathbf{A} \in$ **For**(TK), if **A** is valid, then **A** is derivable in TK. ■

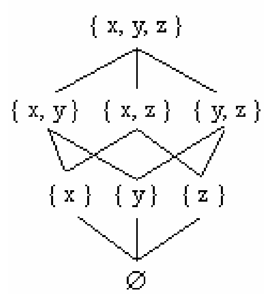
In next proposition it is proved by showing a counter example that the formula $(\mathbf{A} \to \mathbf{B}) \to (\blacklozenge\mathbf{A} \to \blacklozenge\mathbf{B})$ is not TK valid.

Proposition 4.8: $\not\models (\mathbf{A} \to \mathbf{B}) \to (\blacklozenge\mathbf{A} \to \blacklozenge\mathbf{B})$.

Proof: There is a TK-algebra in which does not hold the above formula.

Let E = {x, y, z} and take the Boolean algebra $(\mathcal{P}(E), {}^{C}, \cap, \cup, \varnothing, E)$. Now, define the following consequence operator over $(\mathcal{P}(E), {}^{C}, \cap, \cup, \varnothing, E)$: $\bullet\{x\} = \{x, y\}$, $\bullet\{x, y\} = \{x, y\}$, $\bullet\{x, z\} = \{x, y, z\}$, and $\bullet X = X$, for all the other sets in $\mathcal{P}(E)$. Then $(\mathcal{P}(E), {}^{C}, \cap, \cup, \bullet, \varnothing, E)$ is a TK-algebra, but $(\mathbf{A} \to \mathbf{B}) \to (\blacklozenge\mathbf{A} \to \blacklozenge\mathbf{B})$ is not valid in it.

We will show that $v(\mathbf{A} \to \mathbf{B}) \not\subseteq v(\blacklozenge\mathbf{A} \to \blacklozenge\mathbf{B})$, when **A** is interpreted by {x} and **B** by {z}:



$\{x\} \rightarrowtail \{z\} = \{x\}^{C}\cup\{z\} = \{y, z\}\cup\{z\} = \{y, z\}$ and
$\bullet\{x\} \rightarrowtail \bullet\{z\} = (\bullet\{x\})^{C}\cup\bullet\{z\} = \{z\}\cup\{z\} = \{z\}$. ■

As a consequence of the previous proposition, follows that the Deduction Theorem is not valid for the TK Logic when it was applied the rule R♦ in a deduction.

5. Theories: consistency and models.

A model for a theory **T** is a valuation v: **Var**($\mathcal{L}$) → $\mathcal{B}$, in which $\mathcal{B}$ is a TK-algebra that makes valid all the non-logical axioms of **T**.

Proposition 5.1: Let **T** = (L, C, Δ) be a theory of TK. If **A** is a theorem of **T**, then every model of **T**, in any TK-algebra $\mathcal{B}$, is a model for **A**.
Proof: Let v: **Var**($\mathcal{L}$) → $\mathcal{B}$ be a model to **T** in $\mathcal{B}$. Since $\mathcal{B}$ is a TK-algebra, then v is a model to every logical axiom and for every non-logical axiom, that is, for every formula of Δ. Like in Theorem 4.4, rules of TK preserve validity and $\vdash_{\mathbf{T}}$ **A** then $v_{\mathcal{B}}$(**A**) = 1. ■

Proposition 5.2: Let **T** be a theory. If there is a model v: Var($\mathcal{L}$) → $\mathcal{B}$ in a TK-algebra $\mathcal{B}$, then **T** is consistent.
Proof: If **A** and ¬**A** are theorems of **T**, then $v_{\mathcal{B}}$(**A**) = 1 and $v_{\mathcal{B}}$(¬**A**) = 1. Considering that $v_{\mathcal{B}}$(¬**A**) = 1, it follows that ~$v_{\mathcal{B}}$(**A**) = 1 and, therefore, $v_{\mathcal{B}}$(**A**) = 0, a contradiction. ■

A model v: Var($\mathcal{L}$) → $\mathcal{B}$ is *adequate* for **T** when for every **A** ∈ **For**($\mathcal{L}$), **A** is theorem of **T** iff v is a model to **A**.

Proposition 5.3: Let **T** = (L, C, Δ) be a consistent TK-theory. Then the canonical valuation is an adequate model to **T**.
Proof: Since **T** is consistent, then algebra $\mathcal{A}$(**T**) is not degenerate. Considering the canonical valuation: $v^*_{\mathcal{A}(\mathbf{T})}$: **For**($\mathcal{L}$) → $\mathcal{A}$(**T**), $v^*_{\mathcal{A}(\mathbf{T})}$(**A**) = [**A**], by Corollary 4.3, $v^*_{\mathcal{A}(\mathbf{T})}$(**A**) = 1 iff **A** ∈ C(Δ). Therefore we have that v^* is an adequate model to **T**. ■

Theorem 5.4: (*Adequacy*) For any theory **T** in TK, the following conditions are equivalent:
(i) **T** is consistent;
(ii) there is an adequate model to **T**;
(iii) there is an adequate model to **T** in a TK-algebra $\mathcal{B}$ of all closed subsets of a deductive system **S** = (S, C);
(iv) there is a model to **T**.

Proof: (i) $\Rightarrow$ (ii) It follows of preceding proposition.

(ii) $\Rightarrow$ (iii) Since $\mathcal{A}$(TK) is a TK-algebra and every TK-algebra is isomorphic to a sub-algebra of closed sets of a deductive system (S, C), then the result follows.

(iii) $\Rightarrow$ (iv) It is a immediate consequence.

(vi) $\Rightarrow$ (i) It results directly by Proposition 5.2. ■

Corollary 5.5: For any formula **A** in a consistent theory **T**, the following conditions are equivalent:

(i) **A** is a theorem of **T**;

(ii) every model of **T** is a model to **A**;

(iii) every model of **T** in a TK-algebra of all closed subsets of a deductive system **S** = (S, C) is a model to **A**;

(iv) $v^*_{\mathcal{A}(\mathbf{T})}(\mathbf{A}) = 1$ for every canonical valuation v^*. ■

6. Filters in TK-algebras

Let $\mathcal{A} = (A, 0, 1, \vee, \sim, \bullet)$ be a TK-algebra. A *filter* in $\mathcal{A}$ is a nonempty set $F \subseteq A$ such that for all $x, y \in A$:

(i) $x \in F$ and $y \in F \Rightarrow x \wedge y$ F

(ii) $x \in F$ and $x \leq y \Rightarrow y \in F$.

The filter F is a *TK-filter* when for all $x \in A$ it is valid:

(iii) $\bullet x \in F \Rightarrow x \in F$.

The filter F is a *prime filter* when $F \neq A$ and for all $x, y \in A$ it is valid:

(iv) $x \rightarrowtail y \in F$ or $y \rightarrowtail x \in F$.

Let $\mathcal{A}$ be a TK-algebra and F a prime TK-filter in $\mathcal{A}$. Consider the following equivalence relation $\equiv_F$ in $\mathcal{A}$:

$$x \equiv_F y \text{ if } x \rightarrowtail y \in F \text{ and } y \rightarrowtail x \in F.$$

The relation $\equiv_F$ is a congruence in $\mathcal{A}$ with respect to $\vee$, $\wedge$, $\rightarrow$, that is, $[a]\vee[b] = [a\vee b]$, $[a]\wedge[b] = [a\wedge b]$, $\sim[a] = [\sim a]$, $[a]\rightarrowtail[b] = [a\rightarrow b]$. Also, [1] is the unit, [0] is the zero element of $\mathcal{A} \mid \equiv_F$ and $[a] \leq [b]$ if $a \rightarrowtail b \in F$ (Rasiowa, Sikorski, 1968, p. 63).

Proposition 6.1: Considering $\bullet[x] = [\bullet x]$, the relation $\equiv_F$ is a congruence and the quotient algebra $\mathcal{A} \mid \equiv_F$ is a TK-algebra.

Proof:

(1º step) verify that $\bullet \mid \equiv_F$ is well defined.

$[a] = [b] \Rightarrow a \rightarrowtail b \in F$ and $b \rightarrowtail a \in F \Rightarrow \sim a\vee b \in F$ and $\sim b \vee a \in F$

(Case 1) $\sim a \in F \Rightarrow a \notin F \Rightarrow \bullet a \notin F \Rightarrow \sim\bullet a \in F \Rightarrow \sim\bullet a \vee \bullet b \in F \Rightarrow \bullet a \rightarrowtail \bullet b \in F$.

$\sim a \in F \Rightarrow a \notin F \Rightarrow \sim b \in F \Rightarrow b \notin F \Rightarrow \bullet b \notin F \Rightarrow \sim\bullet b \in F \Rightarrow \sim\bullet b \vee \bullet a \in F \Rightarrow \bullet b \rightarrowtail \bullet a \in F$.

Hence, $[\bullet a] = [\bullet b]$.

(Case 2) $\sim a \notin F \Rightarrow a \in F \Rightarrow \bullet a \in F \Rightarrow \bullet a \vee \sim\bullet b \in F \Rightarrow \bullet b \rightarrowtail \bullet a \in F$.

$\sim a \notin F \Rightarrow b \in F \Rightarrow \bullet b \in F \Rightarrow \sim\bullet a \vee \bullet b \in F \Rightarrow \bullet a \rightarrowtail \bullet b \in F$.

Hence, $[\bullet a] = [\bullet b]$.

Therefore, in any case, $\bullet[a] = \bullet[b]$.

(2º step) verify that $\bullet \mid \equiv_F$ preserves the properties of operator $\bullet$.

$[x] \vee \bullet[x] = [x] \vee [\bullet x] = [x \vee \bullet x] = [\bullet x] = \bullet[x]$.

$\bullet[x] \vee \bullet[x \vee y] = \bullet[x] \vee [\bullet(x \vee y)] = [\bullet x \vee \bullet(x \vee y)] = [\bullet(x \vee y)] = \bullet[(x \vee y)]$.

$\bullet\bullet[x] = [\bullet\bullet x] = [\bullet x] = \bullet[x]$. ■

Proposition 6.2: The quotient algebra $\mathcal{A} \mid \equiv_F$ is linearly ordered.

Proof:

Let F be a prime filter and consider that x, y ∈ A. It follows that $x \rightarrowtail y \in F$ or $y \rightarrowtail x \in F$, that is, $[x] \leq [y]$ or $[y] \leq [x]$ and, therefore $\mathcal{A} \mid \equiv_F$ is linearly ordered. ■

Proposition 6.3: Let $\mathcal{A}$ be a TK-algebra and $1 \neq x \in A$. Then there is a prime TK-filter F in $\mathcal{A}$ that does not contain x.

Proof: This is easily obtained by generalizing the proof of the well-known Ultrafilter Theorem. ■

Theorem 6.4: Each TK-algebra is a sub-algebra of the direct product of a system of linearly ordered TK-algebras.

Proof: Let **S** be the system of prime TK-filters of $\mathcal{A}$. For each $F \in \mathbf{S}$, let $\mathcal{A}_F = \mathcal{A} \mid_{\equiv F}$ and take the set:

$$\mathbf{B} = \Pi_{F \in \mathbf{S}} \mathcal{A}_F.$$

This way, **B** is the direct product of linearly ordered TK-algebras $\{\mathcal{A}_F / F \in \mathbf{S}\}$.

For each $x \in \mathcal{A}$, let $\varphi(x)$ be the element $\{[x]_F\}_{F \in \mathbf{S}}$ of **B**.

The function φ preserves operations.

For each prime filter, like in Proposition 6.1, the operation φ preserves the operations of TK-algebra.

The function φ is injective.

If x, y ∈ A and $x \neq y$, then $x \nleq y$ or $y \nleq x$. Without loss of generality, assume that $x \nleq y$, that is, $x \rightarrowtail y \neq 1$. By preceding proposition, let F be a prime filter in $\mathcal{A}$ that does not contain $x \rightarrowtail y$. It follows that in $\mathcal{A}|_F$, $[x]_F \nleq [y]_F$ and, therefore, $[x]_F \neq [y]_F$, that is, $\varphi(x) \neq \varphi(y)$. ■

Lemma 6.5: If a formula **A** is valid in every linearly ordered TK-algebra, then **A** is valid in every TK-algebra.

Proof: Suppose that **A** is not valid in some TK-algebra $\mathcal{A}$ and let **B** be a direct product of a system of linearly ordered TK-algebras for which $\mathcal{A}$ is a sub-algebra. So, **A** is not valid in some linearly ordered TK-algebra. ■

Theorem 6.6: (*Adequacy*) For each formula **A** of TK the following assertions are equivalent:

(i) **A** is demonstrable in TK;
(ii) **A** is valid in each linearly ordered TK-algebra;
(iii) **A** is valid in each TK-algebra.

Proof: (i) ⇒ (ii) is given by 4.4. (ii) ⇒ (iii) is in 6.5 and (iii) ⇒ (i) if **A** ∈ **For**(TK) and it is not derivable in TK, by Corollary 4.3, [**A**] does not coincide with unity of $\mathcal{A}$(TK) and so, $v^*_{\mathcal{A}(\mathrm{TK})}(\mathbf{A}) \neq 1$. Then **A** is not a valid formula. ■

Acknowledgements:

This work has been sponsored by FAPESP through the Projects 2004/14107-2 and 2005/00408-3.

We wish to express our thanks to the referee for interesting suggestions.

Bibliography:

BELL, J. L., MACHOVER, M. (1977) **A course in mathematical logic**. Amsterdam: North-Holland.

CARNIELLI, W. A., VELOSO, P. A. S. (1997) Ultrafilter logic and generic reasoning. In: **Proceedings of Kurt Gödel Colloquium**, 5, 1997, Berlin. Berlin: Springer-Verlag. p. 34-53.

EBBINGHAUS, H. D., FLUM, J., THOMAS, W. (1984) **Mathematical logic**. New York: Springer-Verlag.

GRÁCIO, M. C. C. (1999) **Lógicas moduladas e raciocínio sob incerteza**. Doctor Thesis (in Portuguese), Institute of Philosophy and Human Sciences, State University of Campinas, Campinas, Brazil. 194 p.

GRÁCIO, M. C. C., FEITOSA, H. A., NASCIMENTO, M. (200_) ***Uma lógica característica dos operadores de conseqüência de Tarski***. (to appear)

HÁJEK, P. (1998). **Metamathematics of fuzzy logic**. Dordrecht: Kluwer.

HAMILTON, A. G. (1978). **Logic for mathematicians**. Cambridge: Cambridge University Press.

MENDELSON, E. (1987) **Introduction to mathematical logic**. 3. ed. Monterey, CA: Wadsworth & Brooks/Cole Advanced Books & Software.

MIRAGLIA, F. (1987). **Cálculo proposicional: uma interação da álgebra e da lógica**. Campinas: UNICAMP/CLE. (Coleção CLE, v. 1)

NASCIMENTO, M. C., FEITOSA, H. A. (2005) As álgebras dos operadores de conseqüência. São Paulo: ***Revista de Matemática e Estatística***, v. 23, n. 1, p. 19-30.

RASIOWA, H. (1974). **An algebraic approach to non-classical logics**. Amsterdam: North-Holland.

RASIOWA, H., SIKORSKI, R. (1968) **The mathematics of metamathematics**. 2. ed. Waszawa: PWN - Polish Scientific Publishers.

SETTE, A. M., CARNIELLI, W. A., VELOSO, P. (1999) An alternative view of default reasoning and its logic. In: HAUESLER, E. H., PEREIRA, L. C. (Eds.) **Pratica: Proofs, types and categories**. Rio de Janeiro: PUC. p. 127-58.

VICKERS, S. (1990) **Topology via logic**. Cambridge: Cambridge University Press.

WÓJCICKI, R. (1988) **Theory of logical calculi:** basic theory of consequence operations. Dordrecht: Kluwer, 1988. (Synthese Library, v. 199)