To distribute or not to distribute?

Jean-Yves. Béziau[1] and Marcelo E. Coniglio[2]

[1]Department of Philosophy
Federal University of Ceará, Fortaleza, Brazil
jyb@ufc.br

[2]Institute of Philosophy and Human Sciences (IFCH) and
Centre for Logic, Epistemology and The History of Science (CLE)
State University of Campinas (UNICAMP), Campinas, SP, Brazil
coniglio@cle.unicamp.br

Abstract

In this paper we address some central problems of combination of logics through the study of a very simple but highly informative case, the combination of the logics of disjunction and conjunction. At first it seems that it would be very easy to combine such logics, but the following problem arises: if we combine these logics in a straightforward way, distributivity holds. On the other hand, distributivity does not arise if we use the usual notion of extension between consequence relations. A detailed discussion about this phenomenon, as well as some elucidation for it, is given.

Contents

1 Introduction

Combination of logics is still a fairly young subject. It arose from the study of some particular cases especially connected with modal logics, through the techniques of *fusion*, *product* and *fibring*. This was a first stage of the development of this new field of research, where the initial techniques (mainly fibring) were generalized to general logics (see, for instance, [7] and [9]). However this theory is still in construction: many concepts still need to be clarified, and many important problems still need to be solved. In fact, combination of logics may look at first sight as an easy subject, but after a closer examination, it looks like a hazardous field of research.

Combination of logics is related to some fundamental phenomena of logic which are still not properly understood, connected to what a logic is and what are the relations between different formulations of a given logic. These questions are the subject matter of universal logic. So a general theory of combination of logics must be developed within universal logic, but on the other hand it is through the study of problems such as the ones appearing in combination of logics that universal logic grows. That is one of the reasons why combination of logics is a very interesting subject.

In this paper, which is an extended and improved version of [5], we are tackling some central problems of combination of logics through the study of a very simple but highly informative case: the combination of the logics of disjunction and conjunction. This is a non-technical paper about combination of logics: no new combining technique is herein introduced, and the use of well-known combination methods will be briefly mentioned without entering in technical details. Indeed, as mentioned above, the main purpose of this paper is presenting some important questions on combining logics, illustrated by the analysis of an specific example. As it is known in mathematics, for instance with Fermat's theorem, it is not because a problem is easy to state and understand, that it is easy to solve. Mathematical problems which are very easy to formulate but very hard to solve are fascinating, because this kind of discrepancies are challenges to our minds, it is like if we were able to see the Eldorado in its full splendor but were unable to reach it.

The problem we are studying here is the combination of the logic of conjunction with the logic of disjunction. Such logics are extremely simple and have barely being studied by themselves. Considered from the outdated concept of logic taken as a set of tautologies they don't even have a sense, since there are not tautologies in such logics. At first it seems that it would be very easy to combine such logics whether it be from a proof theoretical viewpoint (Gentzen rules), semantic viewpoint (truth tables) or a purely consequence viewpoint (consequence operators or consequence relations), but the following problem arises: if we combine these logics in a straightforward way, distributivity holds. We have introduced in a previous work [4] the suggestive terminology "copulation paradox" to describe this phenomenon, because conjunction and disjunction are interacting between each other generating a new law, distributivity.

But distributivity is in fact a paradox and a problem, because the standard view of combinations of logics is that the combined logic is the smallest one defined on the combined language, which extends the two combined logics. One could think that there is no paradox because the logic of conjunction and disjunction is necessary distributive, and as we will see this not completely false, but also someone who knows a bit of lattice theory is aware that non-distributivity is possible. One question here is to study if there is a strict parallel between logic and algebra or not. We already have pointed out the many dangers and incorrectness of a reductionist point of view, according to which logic can properly be treated by algebraic methods [2], it seems we have here again a phenomenon showing that logic does not reduce to algebra, that it is more complex.

2 The logic of conjunction

We begin by recalling the notion of structural Tarskian consequence systems, that will be used throughout this paper.

From now on, For will denote a set (of *formulas*) being the domain of an absolutely free algebra generated by a given set of connectives from a set $\mathcal{P}$ of generators (called *atomic formulas*). A *substitution* over For is a homomorphism $\sigma : \mathsf{For} \to \mathsf{For}$. A structural Tarskian consequence system is a logic given as a pair $\langle \mathsf{For}, \vdash \rangle$ such that $\vdash \subseteq \wp(\mathsf{For}) \times \mathsf{For}$ is a (consequence) relation satisfying the following properties:

(EXT) If $\varphi \in \Gamma$ then $\Gamma \vdash \varphi$

(CUT) If $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$ for every $\psi \in \Gamma$ then $\Delta \vdash \varphi$

(FIN) If $\Gamma \vdash \varphi$ then $\Delta \vdash \varphi$ for some finite $\Delta \subseteq \Gamma$

(STR) If $\Gamma \vdash \varphi$ then $\sigma(\Gamma) \vdash \sigma(\varphi)$ for every substitution σ over For

As it is well known, from (EXT) and (CUT) it follows the usual property of monotonicity:

(MON) If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash \varphi$

By "the logic of conjunction" we informally mean a propositional logic with a single connective $\wedge$ having the properties of a (classical) conjunction. This logic can be defined in several ways; for instance, a simple possibility is to consider a semantical presentation of conjunction. In fact, it is enough to consider bivaluations (that is, valuations on the set $\{0, 1\}$ of truth-values) satisfying the following standard condition:

$$v(\varphi \wedge \psi) = 1 \quad \text{iff} \quad v(\varphi) = 1 \ \text{ and } \ v(\psi) = 1$$

for every formula φ and ψ. The set of mappings (bivaluations) v satisfying the condition above characterize the logic of classical conjunction. This presentation of the logic of conjunction corresponds to the usual truth-table below.

$\wedge$	0	1
0	0	0
1	0	1

In more formal terms, the semantics of the logic of conjunction is the set of homomorphisms between the absolutely free algebra $\mathsf{For}_\wedge$ generated by a binary operator $\wedge$ (called *conjunction*) from the set $\mathcal{P}$ of atomic formulas, and the algebra on $\{0, 1\}$ with a binary operator defined by the truth-table above. Frequently the same symbol "$\wedge$" is used both for the operator in the algebra of language and the operator in the algebra of truth-values.

The algebra $\langle\{0, 1\}; \wedge\rangle$ together with the subset $\{1\}$ of distinguished values constitute the logical matrix $\langle\langle\{0, 1\}; \wedge\rangle; \{1\}\rangle$, from which it can be defined as usual a new structure where the main relation is a consequence relation $\vDash_\wedge$ such that, for every set $\Gamma \cup \{\varphi\}$ of formulas in $\mathsf{For}_\wedge$,

$$\Gamma \vDash_\wedge \varphi \quad \text{iff, for every homomorphism } v, \quad v(\Gamma) \subseteq \{1\} \quad \text{implies that} \quad v(\varphi) = 1.$$

It is easy to show that $\langle \mathsf{For}_\wedge, \vDash_\wedge\rangle$ is a structural Tarskian consequence system.

The logic of conjunction can also be defined using a proof system, for example of Gentzen type. Clearly this is just the subsystem of LK including all the structural rules plus the two logical rules for conjunction. It is worth noting that the logic of conjunction can be alternatively defined by the three following laws, which are closer to the Gentzen rules for conjunction:

($\wedge_1$) If $\Gamma, \varphi \vdash \delta$ then $\Gamma, \varphi \wedge \psi \vdash \delta$

($\wedge_2$) If $\Gamma, \psi \vdash \delta$ then $\Gamma, \varphi \wedge \psi \vdash \delta$

($\wedge_3$) If $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$ then $\Gamma, \Delta \vdash \varphi \wedge \psi$.

It is also possible to prove that these three laws are enough to define the logic of conjunction in the sense that a model of these three laws together with the laws defining a structural Tarskian consequence relation (as above) is the logic of conjunction, and vice-versa. This is the "consequence-relation" approach to logic, promoted especially by Polish logicians, not to be confused with a Gentzen type approach, although sometimes they look very similar.

A different approach to conjunction is also possible. Let $\mathrm{L}_\wedge = \langle \mathsf{For}_\wedge, \vdash_\wedge \rangle$ be the logic such that, for every $\Gamma \cup \{\varphi\} \subseteq \mathsf{For}_\wedge$,

$$\Gamma \vdash_\wedge \varphi \ \text{ iff } \ \Gamma \neq \emptyset \ \text{ and } \ Var(\varphi) \subseteq Var(\Gamma)$$

where $Var(\varphi)$ is the set of atomic formulas occurring in φ, and $Var(\Gamma) = \bigcup_{\psi \in \Gamma} Var(\psi)$. It is easy to see that $\vdash_\wedge$ is a in fact a structural Tarskian consequence relation. This is a purely syntactic approach to conjunction, without using logical rules, logical laws or semantics. It is not a hard task to prove that $\mathrm{L}_\wedge$ corresponds to the logic obtained by a Hilbert-style presentation of the logic of conjunction given by the following inference rules:

$$\frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi} \qquad \frac{\varphi \quad \psi}{\varphi \wedge \psi}$$

So, it is not surprising that the logic $\mathrm{L}_\wedge$ is nothing more than the logic of conjunction defined by the usual truth-tables described above:

Theorem 2.1 *For every* $\Gamma \cup \{\varphi\} \subseteq \mathsf{For}_\wedge$*,*

$$\Gamma \vdash_\wedge \varphi \ \text{ iff } \ \Gamma \vDash_\wedge \varphi.$$

Proof: Suppose that $\Gamma \vdash_\wedge \varphi$; then Γ is nonempty and $Var(\varphi) \subseteq Var(\Gamma)$. Let $v : \mathsf{For}_\wedge \to \{0,1\}$ be a (classical) bivaluation such that $v(\Gamma) \subseteq \{1\}$. By definition of v, $v(Var(\Gamma)) \subseteq \{1\}$ and so $v(Var(\varphi)) \subseteq \{1\}$. From this it follows that $v(\varphi) = 1$ and so $\Gamma \vDash_\wedge \varphi$.

Suppose now that $\Gamma \nvdash_\wedge \varphi$; then, there exist $p \in Var(\varphi) \setminus Var(\Gamma)$. Let v be a classic bivaluation such that $v(Var(\Gamma)) \subseteq \{1\}$ and $v(p) = 0$. Then $v(\Gamma) \subseteq \{1\}$ but $v(\varphi) = 0$, that is, $\Gamma \nvDash_\wedge \varphi$. ■

As an easy consequence of the last theorem it follows that $\mathrm{L}_\wedge$ is the least logic satisfying the laws of conjunction $(\wedge_1)$-$(\wedge_3)$. In more precise terms:

Theorem 2.2 *$L_\wedge$ coincides with LC, the least structural Tarskian consequence relation over $\mathsf{For}_\wedge$ satisfying laws* $(\wedge_1)$-$(\wedge_3)$*.*

From now on, and when there is no risk of ambiguity, by "the logic of conjunction" we will mean the structural and Tarskian consequence system LC. The fact that LC satisfies clauses $(\wedge_1)$-$(\wedge_3)$ will be useful below.

3 The logic of disjunction

The logic of disjunction is, in a sense that can be made precise, the dual of the logic of conjunction. As in the case of conjunction, the classical disjunction can be characterized by bivaluations such that:

$$v(\varphi \vee \psi) = 1 \quad \text{iff} \quad v(\varphi) = 1 \;\; \text{or} \;\; v(\psi) = 1$$

for every formula φ and ψ in $\mathsf{For}_\vee$, the algebra of formulas generated by the binary connective $\vee$. This generates a structural Tarskian consequence relation $\models_\vee$. Of course this definition corresponds to the classical truth-table of disjunction. Instead of recalling the well-known Gentzen rules for disjunction, we just give now the following three laws which can be used to define the logic of disjunction in the framework of consequence relations:

($\vee_1$) If $\Gamma \vdash \varphi$ then $\Gamma \vdash \varphi \vee \psi$

($\vee_2$) If $\Gamma \vdash \psi$ then $\Gamma \vdash \varphi \vee \psi$

($\vee_3$) If $\Gamma, \varphi \vdash \delta$ and $\Delta, \psi \vdash \delta$ then $\Gamma, \Delta, \varphi \vee \psi \vdash \delta$.

As it was done with conjunction, it is possible to define a structural Tarskian consequence system for the logic of disjunction in purely syntactic terms. Thus, let $\mathrm{L}_\vee = \langle \mathsf{For}_\vee, \vdash_\vee \rangle$ such that, for every $\Gamma \cup \{\varphi\} \subseteq \mathsf{For}_\vee$,

$$\Gamma \vdash_\vee \varphi \;\; \text{iff} \;\; Var(\psi) \subseteq Var(\varphi) \;\; \text{for some } \psi \in \Gamma.$$

It can be proven that $\mathrm{L}_\vee$ is the logic defined by the Hilbert system over $\mathsf{For}_\vee$ given by the following inference rules:

$$\frac{\psi}{\varphi} \qquad \text{whenever } Var(\psi) \subseteq Var(\varphi)$$

It is interesting to note that the Hilbert presentation of the logic of disjunction requires infinite inference rules. Another feature of this logic is that there is no interaction between the premises within a deduction.

It can be proven that the logic $\mathrm{L}_\vee$ coincides with the logic of disjunction defined by the classical bivaluations:

Theorem 3.1 *For every* $\Gamma \cup \{\varphi\} \subseteq \mathsf{For}_\vee$,

$$\Gamma \vdash_\vee \varphi \;\; \textit{iff} \;\; \Gamma \models_\vee \varphi.$$

Proof: Suppose that $\Gamma \vdash_\vee \varphi$; then $Var(\psi) \subseteq Var(\varphi)$ for some $\psi \in \Gamma$. Consider a classical bivaluation $v : \mathsf{For}_\vee \to \{0, 1\}$ such that $v(\Gamma) \subseteq \{1\}$; in particular, $v(\psi) = 1$. By definition of v it follows that $v(p) = 1$ for some $p \in Var(\psi)$. But $p \in Var(\varphi)$, by hypothesis, and so $v(\varphi) = 1$. This means that $\Gamma \models_\vee \varphi$.

Suppose now that $\Gamma \nvdash_\vee \varphi$; then, for every $\psi \in \Gamma$ there exist $p \in Var(\psi) \setminus Var(\varphi)$. Let v be a classic bivaluation such that $v(p) = 1$ iff $p \notin Var(\varphi)$. Then $v(\Gamma) \subseteq \{1\}$ but $v(\varphi) = 0$, therefore $\Gamma \not\models_\vee \varphi$. ■

From this, it follows that $\mathrm{L}_\vee$ is the least logic satisfying the laws of disjunction.

Theorem 3.2 *$L_\vee$ coincides with LD, the least structural Tarskian consequence system over $\mathsf{For}_\vee$ satisfying laws* $(\vee_1)$-$(\vee_3)$.

From now on, and when there is no risk of ambiguity, by "the logic of disjunction" we will mean the structural and Tarskian consequence system LD. By the very definition, LD satisfies clauses $(\vee_1)$-$(\vee_3)$.

4 A semantical combination of the logics of conjunction and disjunction

When we talk about "the logic of conjunction and disjunction", there are at least two possible readings for this:

(1) It is the propositional logic which consists exclusively of two connectives: a classical conjunction and a classical disjunction;

(2) It is the propositional logic obtained by the combination (by some suitable method) of the logic of conjunction and the logic of disjunction.

One of the main points of this paper is to show that the first interpretation (1) is by no means a clear concept: it is not obvious if distributivity between both connectives, an interaction law (or a "bridge principle", in the terminology of [8]), should hold. By its turn, the second interpretation (2) brings us to the second point of this paper: how can the logic of conjunction and the logic of disjunction be combined? Distributivity depends upon the method used for combining both logics?

Of course both questions are interdependent. Moreover, an interesting question (that it will not analyzed here) is to determine the exact relationship between (1) and (2). For instance, does (1) follow from (2)?

Now, we will begin for analyzing the second point: how to combine conjunction and disjunction. The semantical approach will be firstly analyzed.

The logic of conjunction LC and the logic of disjunction LD can be combined in a semantical way, by considering the mixed language containing both connectives and by putting together the conditions defining the sets of bivaluations. It is straightforward to prove that the obtained logic, that we will call LCD, satisfies the following laws , from now on called "laws of distributivity":

(D_1) $\quad \varphi \wedge (\psi \vee \delta) \dashv\vdash (\varphi \wedge \psi) \vee (\varphi \wedge \delta)$

(D_2) $\quad \varphi \vee (\psi \wedge \delta) \dashv\vdash (\varphi \vee \psi) \wedge (\varphi \vee \delta)$

Here, as usual, $\varphi \dashv\vdash \psi$ is an abbreviation for "$\varphi \vdash \psi$ and $\psi \vdash \varphi$". The spontaneous arising of the laws of distributivity suggest two conjectures, related to (1) above:

(conjecture 1) We have applied the wrong procedure to combine LC and LD, because distributivity is not intrinsic to the logic of conjunction and disjunction: we have to find the right one, which leads to a a logic of conjunction and disjunction in which distributivity does not hold

(conjecture 2) We have applied the right procedure: combination of LC and LD necessary leads to distributivity, because it is an intrinsic property of the logic of conjunction and disjunction.

In both cases, we are in hazardous zones, for, if the first conjecture is true, we have to understand what went wrong with this combination procedure and find the proper one; and if the second conjecture is true, we have to understand how is produced the interaction between conjunction and disjunction which leads to some new feature, namely distributivity.

One may think that the first conjecture is more plausible, because non-distributivity is a real phenomenon, especially known in quantum logic and lattice theory, and also because when we examine in a closer way how we have combined the semantics of conjunction and disjunction, we see that it seems that there is confusion: we have supposed that the set of values is the same in both cases.

In sections 5 and 7 we will try to justify the first conjecture by means of semantical arguments.

5 Distributivity laws and non-distributive lattices

The logic of conjunction and disjunction has naturally associated the order-theoretic structure of lattice. A lattice can be defined as a partial order structure with two binary operators ("infimum" and "supremum") verifying laws very similar to the ones of conjunction and disjunction, respectively. Moreover, the symbols mostly used in the literature to refer to these operators are $\wedge$ and $\vee$, identical to the symbols frequently used for conjunction and disjunction, respectively. The properties satisfied for such operators are idempotency, associativity and commutativity, as well as the following ones (here, a, b and c are arbitrary elements of a given lattice):

(infimum)

$a \wedge b \leq a$

$a \wedge b \leq b$

If $c \leq a$ and $c \leq b$ then $c \leq a \wedge b$

(supremum)

$a \leq a \vee b$

$b \leq a \vee b$

If $a \leq c$ and $b \leq c$ then $a \vee b \leq c$

The following property holds in any lattice L:

$$a \leq b \quad \text{iff} \quad a \wedge b = a \quad \text{iff} \quad a \vee b = b.$$

Lattices and the logic of conjunction and disjunction are closely related in a precise way: indeed, by using the well-known Lindenbaum-Tarski method, the algebra associated to the logic of conjunction and disjunction (defined by appropriate laws) is precisely the logic of lattices, as we shall prove in Theorem 10.1 below.

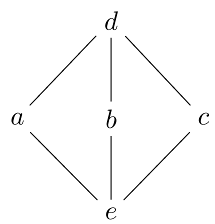
But it is well-known that not every lattice is distributive. That is, there are some lattices in which the following laws do not hold:

$a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$

$(a \vee b) \wedge (a \vee c) \leq a \vee (b \wedge c)$

Since the converse laws are always true in any lattice, the laws above are the algebraic counterpart of the laws of distributivity (D_1)-(D_2) of the previous section.

One possibility to construct a semantics for a non-distributive logic of conjunction and disjunction would be to consider a semantics based on a non-distributive lattice. The idea behind this strategy is that in case of classical logic, its Lindenbaum-Tarksi algebra is a boolean algebra and its semantics is a matrix whose algebra is the boolean algebra on $\{0, 1\}$. Moreover, intuitively, it seems reasonable that a matrix semantics based on a non-distributive lattice could falsify the distributivity law. So the idea is to consider one of the two simplest non-distributive lattices ([6] pp. 75), knowing in particular that a lattice which is not distributive contains one of them as a sublattice. Consider, for instance, the diamond lattice M_3:



Then

$$a \wedge (b \vee c) = a \wedge d = a \nleq e = e \vee e = (a \wedge b) \vee (a \wedge c)$$

and

$$(a \vee b) \wedge (a \vee c) = d \wedge d = d \nleq a = a \vee e = a \vee (b \wedge c).$$

Building a matrix semantics with these non-distributive lattices means having to choose the set of distinguished elements. After toying a bit with these

5 elements lattice one sees that the task is not so easy. Either we are able to falsify distributivity but not verifying the laws for conjunction and disjunction, or we satisfy these laws but distributivity also holds. In fact it is possible to prove that there is no way out, as we shall see in Section 7.

6 The logic of conjunction and disjunction is distributive

Consider now standard systems of sequents, with many formulas on the left side of the sequent, and with any number of formulas on the right (possibly reduced to zero or one as in the case of intuitionistic sequent systems). Then, it is easy to prove distributivity in such sequent systems with the standard rules for conjunction and disjunction and the usual structural rules. Instead of giving a proof of this result, an analogous version of it will be shown below. Such result, which corresponds to a meta-theorem in the theory of consequence relations, shows that the logic of conjunction, taken as a consequence relation, combined with the logic of disjunction, taken also as a consequence relation, is necessary distributive.

From now on, For will denote the set of formulas generated by conjunction $\wedge$ and disjunction $\vee$.

Let $\mathrm{L}_{\wedge\vee} = \langle \mathsf{For}, \vdash \rangle$ be the least structural Tarskian consequence relation over For satisfying laws $(\wedge_1)$-$(\wedge_3)$ and $(\vee_1)$-$(\vee_3)$ from sections 2 and 3. The logic $\mathrm{L}_{\wedge\vee}$ can be considered a possible definition of the logic of conjunction and disjunction, taken as a consequence relation. Thus, meta-derivations in this logic use exclusively laws $(\wedge_1)$-$(\wedge_3)$, $(\vee_1)$-$(\vee_3)$, (EXT) and (CUT).

Theorem 6.1 *The logic $L_{\wedge\vee}$ is distributive.*

Proof: Consider the following meta-derivations in $\mathrm{L}_{\wedge\vee}$:

(1) $\varphi, \psi \vdash (\varphi \wedge \psi)$ by (EXT) and $(\wedge_3)$
(2) $\varphi, \psi \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \delta)$ from (1) using $(\vee_1)$
(3) $\varphi, \delta \vdash (\varphi \wedge \delta)$ by (EXT) and $(\wedge_3)$
(4) $\varphi, \delta \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \delta)$ from (3) using $(\vee_2)$
(5) $\varphi, (\psi \vee \delta) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \delta)$ from (2), (4) using $(\vee_3)$
(6) $\varphi \wedge (\psi \vee \delta) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \delta)$ from (5) using $(\wedge_1)$, $(\wedge_2)$.

Consider now the following sequence of meta-derivations in $\mathrm{L}_{\wedge\vee}$:

(1) $\varphi \vdash \varphi$ by (EXT)
(2) $\psi \vdash \psi$ by (EXT)
(3) $\psi \vdash \psi \vee \delta$ from (2) using $(\vee_1)$
(4) $\varphi, \psi \vdash \varphi \wedge (\psi \vee \delta)$ from (1), (3) using $(\wedge_3)$
(5) $\varphi \wedge \psi \vdash \varphi \wedge (\psi \vee \delta)$ from (4) using $(\wedge_1)$, $(\wedge_2)$
(6) $\delta \vdash \delta$ by (EXT)

(7) $\delta \vdash \psi \vee \delta$ from (6) using $(\vee_2)$
(8) $\varphi, \delta \vdash \varphi \wedge (\psi \vee \delta)$ from (1), (7) using $(\wedge_3)$
(9) $\varphi \wedge \delta \vdash \varphi \wedge (\psi \vee \delta)$ from (8) using $(\wedge_1)$, $(\wedge_2)$
(10) $(\varphi \wedge \psi) \vee (\varphi \wedge \delta) \vdash \varphi \wedge (\psi \vee \delta)$ from (5), (9) using $(\vee_3)$.

This shows that the distributivity law (D_1) holds in $\mathrm{L}_{\wedge\vee}$. The proof of (D_2) is analogous, and is left to the reader. ■

This result could be considered as the syntactical counterpart of that of Section 4. In fact, the logic of conjunction and disjunction considered here is a kind of syntactical combination between the logic LC, characterized by the "syntactical" laws $(\wedge_1)$-$(\wedge_3)$, and the logic LD, given by the "syntactical" laws $(\vee_1)$-$(\vee_3)$ (and using the fact that (C_1)-(C_3) are derived laws). Thus, the combination is defined by putting together the laws of both logics. In formal terms, it is a fibring, that is, a coproduct in an appropriate category, as it will be analyzed in Section 9 below.

7 No semantics for non-distributive logics

As a direct consequence of Theorem 6.1 we will now prove a very general result about the impossibility to find a semantics for a non-distributive logic of conjunction and disjunction taken as a consequence relation. To prove this result we use a very general definition of semantics.

A *semantic* is any set T of values divided into two subsets, of distinguished and non-distinguished values, as well as a set of mappings $v : \mathsf{For} \to T$ called *valuations*. This data is then used to define a consequence relation following the standard Tarskian notion of semantical consequence: $\Gamma \vDash \varphi$ iff, for every valuation v, if $v(\psi)$ is distinguished for every $\psi \in \Gamma$ then $v(\varphi)$ is distinguished.

This general definition of semantics encompasses any matrix semantics finite or not and also any truth-functional semantics. Moreover, we have shown (see [1]) that, if we are working within the framework of Tarskian semantical consequence, any semantics can be reduced to such a concept of semantics. The following result really shows that there are no semantics for non-distributive logics within the Tarskian paradigm.

Theorem 7.1 *The laws of distributivity hold in every semantics validating the laws* $(\wedge_1)$-$(\wedge_3)$ *and* $(\vee_1)$-$(\vee_3)$.

Proof: Consider a semantics S which validates the laws $(\wedge_1)$-$(\wedge_3)$ and $(\vee_1)$-$(\vee_3)$. By induction of the length of a meta-derivation in the theory of consequence relations, it follows that every meta-theorem in the logic of conjunction and disjunction $\mathrm{L}_{\wedge\vee}$ is valid in the semantics S. In particular, by Theorem 6.1, the laws of distributivity hold in S. ■

8 The combination of the logic of conjunction and disjunction is not necessarily distributive

In the preceding sections, we have considered the logic of conjunction and disjunction to be the logic $L_{\wedge\vee}$ verifying the laws $(\wedge_1)$-$(\wedge_3)$ and $(\vee_1)$-$(\vee_3)$ (observing that (C_1)-(C_3) are derived laws). It could be argued that such a logic is not the genuine combination of the logic of conjunction and the logic of disjunction. Usually, as in the case of fibring (see for instance [9]), the combination of two logics is defined as being the smallest logic on the combined language which is an extension of both logics. If we take here "extension" in the usual sense of inclusion of consequence relations, we will show below that there exists a logic which is an extension of both (the logic of conjunction and the logic of disjunction) where $(\vee_3)$ does not hold and where also distributivity does not hold.

In order to prove this, let $\vdash_{Lat}$ be the relation defined as follows: for any $\Gamma \cup \{\varphi\} \subseteq \mathsf{For}$, $\Gamma \vdash_{Lat} \varphi$ iff there exists a nonempty finite set $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ such that $v(\gamma_1) \wedge \ldots \wedge v(\gamma_n) \leq v(\varphi)$ holds in L, for every lattice L and every homomorphism $v : \mathsf{For} \to L$.

It can be straightforwardly proven that the relation $\vdash_{Lat}$ is a structural Tarskian consequence relation over the set For of formulas (recall Section 2). This defines the logic of lattices $L_{Lat} = \langle \mathsf{For}, \vdash_{Lat} \rangle$.

Now, let CD be the combination of the logics LC (of conjunction) and LD (of disjunction) defined as being the smallest structural Tarskian consequence system over the combined language For which is an extension of both logics. In more precise terms, let $CONS_{\wedge\vee}$ be the set of structural Tarskian consequence relations defined over For. The set $CONS_{\wedge\vee}$ ordered by inclusion is a complete lattice (cf. [11]). Then, CD is the logic defined over For by the consequence relation

$$\vdash_{CD} = \bigwedge \{\vdash \in CONS_{\wedge\vee} \ : \ \vdash_\wedge \subseteq \vdash \ \text{ and } \ \vdash_\vee \subseteq \vdash\}.$$

Of course the infimum $\bigwedge$ is taken in the complete lattice $CONS_{\wedge\vee}$. Obviously $\vdash_{CD}$ is the supremum in $CONS_{\wedge\vee}$ of $\vdash_\wedge$ and $\vdash_\vee$, and CD is the least structural Tarskian system which simultaneously extends LC and LD. Then we obtain the following result:

Theorem 8.1 *The logic CD obtained by the combination of the logic of conjunction and the logic of disjunction is non-distributive.*

Proof: Let $\mathcal{R} = \{\vdash \in CONS_{\wedge\vee} \ : \ \vdash_\wedge \subseteq \vdash \ \text{ and } \ \vdash_\vee \subseteq \vdash\}$. It is clear that the relation $\vdash_{Lat}$ defined above belongs to $CONS_{\wedge\vee}$. The following step is to show that $\vdash_{Lat} \in \mathcal{R}$.

Let $\Gamma \cup \{\varphi\} \subseteq \mathsf{For}_\wedge$ such that $\Gamma \vdash_\wedge \varphi$. Then $Var(\varphi) \subseteq Var(\Gamma)$, by Theorem 2.2. But $Var(\varphi)$ is finite, then there is a finite set $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ contained in Γ such that $Var(\varphi) \subseteq Var(\Delta)$. Since $\Delta \cup \{\varphi\} \subseteq \mathsf{For}_\wedge$, it is easy to see that, for every lattice L and every homomorphism $v : \mathsf{For} \to L$, $v(\gamma_1) \wedge \ldots \wedge v(\gamma_n) \leq v(\varphi)$ holds in L. That is, $\Gamma \vdash_{Lat} \varphi$ and so $\vdash_\wedge \subseteq \vdash_{Lat}$.

Now, let $\Gamma \cup \{\varphi\} \subseteq \mathsf{For}_\vee$ such that $\Gamma \vdash_\vee \varphi$. Then, $Var(\psi) \subseteq Var(\varphi)$ for some $\psi \in \Gamma$, by Theorem 3.2. Since both ψ and φ are in $\mathsf{For}_\vee$ then $v(\psi) \leq v(\varphi)$ for every homomorphism $v : \mathsf{For} \to L$ and every lattice L. But then $\Gamma \vdash_{Lat} \varphi$ and so $\vdash_\vee \subseteq \vdash_{Lat}$.

This shows that $\vdash_{Lat}$ belongs to $\mathcal{R}$ and then $\vdash_{CD} \subseteq \vdash_{Lat}$. But $\vdash_{Lat}$ does not validate the distributivity laws, since there exist non-distributive lattices, for instance M_3 (recall Section 5). Therefore the logic CD is non-distributive. ■

The trick is that if we consider the logic of disjunction just as a consequence relation, the law $(\vee_3)$, which appears as a kind of meta-property, does not necessarily hold in a logic which extend LD. The question here is to know whether a logic in which $(\vee_3)$ does not hold can properly be called a logic of disjunction. We will return to the analysis of $(\vee_3)$ in the next section.

A possible explication for the loss of $(\vee_3)$ could be found in the very definition of combination of logics. Thus, combination of logics should not be defined as the "smallest logic on the combined language which is an extension of both in the sense of inclusion of consequence relation". But then how to proceed? It is not possible to proceed in general as we have done here by putting together some laws that define the two logics, because we want to have a definition of combination of logics even when the logics are not given by some laws. One solution is to consider that a logical structure is not completely given by the consequence relation but must be further be specified by some meta-properties, but then how to define the set of relevant meta-properties?

At this point, it is worth noting that the operation of algebraic fibring (see [9]) is defined as a coproduct in the category of logic systems under consideration. In the case of the category of logics defined by structural Tarskian consequence relations and translations between logics as morphisms, the coproduct coincides with the supremum of the consequence relations. Thus, in terms of combination of logics, the logic CD is the algebraic fibring of LC and LD, that is, the coproduct of LC and LD in this category.

Under this perspective, the situation obtained in Theorem 8.1 is not so surprising in the field of combination of logics. Basically, it is a typical case of non-preservation of completeness by fibring. In fact, the logic LC (considered as a consequence system) is sound and complete with respect to the semantic of bivaluations or, in other words, $\vdash_\wedge$ coincides with $\vDash_\wedge$. Analogously, the logic LD is sound and complete with respect to the semantic of bivaluations, that is, $\vdash_\vee = \vDash_\vee$. But the fibring (in the category of consequence systems) of LC and LD produces the non-distributive logic CD whereas the fibring of $\vDash_\wedge$ and $\vDash_\vee$ (in the category of consequence systems given by semantic of bivaluations) produces the distributive logic LCD, as shown in Section 4. That is, the fibring CD of LC and LD is no longer complete with respect to the fibring LCD of the respective semantic of bivaluations.

9 A problem with translations?

The failure of completeness preservation by fibring LC and LD or, in other words, the loss of distributivity in CD, is not intrinsic to fibring, and admits a solution.

Indeed, as we proved in a previous work [10], a stronger logic can be obtained through fibring provided that a stronger notion of translation between logics is considered as morphisms. Recall that a translation between a logic L_1 (defined over the set For_1 of formulas) and a logic L_2 (defined over the set For_2 of formulas) is a mapping $f : \mathsf{For}_1 \to \mathsf{For}_2$ such that $\Gamma \vdash_{L_1} \varphi$ implies that $f(\Gamma) \vdash_{L_2} f(\varphi)$. It is possible to substitute this notion of translation by a stronger one, viz. a mapping $f : \mathsf{For}_1 \to \mathsf{For}_2$ satisfying the following: if the logic L_1 (defined over the set For_1 of formulas) satisfies a meta-property of the form

$$\texttt{If } \Gamma_1 \vdash_{L_1} \varphi_1 \texttt{ and } \ldots \texttt{ and } \Gamma_n \vdash_{L_1} \varphi_n \texttt{ then } \Gamma \vdash_{L_1} \varphi$$

then the logic L_2 (defined over the set For_2 of formulas) must satisfy the meta-property

$$\texttt{If } f(\Gamma_1) \vdash_{L_2} f(\varphi_1) \texttt{ and } \ldots \texttt{ and } f(\Gamma_n) \vdash_{L_2} f(\varphi_n) \texttt{ then } f(\Gamma) \vdash_{L_2} f(\varphi).$$

If this kind of meta-translations are considered, then the coproduct of LC and LD in this category (that is, the smallest logic given by a structural Tarskian consequence relation over the set For which extend both LC and LD by means of meta-translations) is exactly the logic LCD obtained in Theorem 6.1, which is distributive. In [10] we defined a formal framework for combining consequence relations by means of meta-translations. This combination process, called *meta-fibring*, is essentially a fibring (coproduct) in the category if sequent calculi and morphisms which preserve sequent rules. In [3] we already pointed out the importance of meta-properties (such as those considered above) in the analysis of logics.

There is still another solution in the case of the logic of disjunction, which is to consider logics as multiple-conclusion consequence relations. In this case we can state the law ($\vee_3$) without going at the meta-level. But then we can reformulate the result of Section 6 and show that the combination of the logics of disjunction and conjunction is necessarily distributive.

10 A problem with a law?

Theorem 8.1 showed that the logic of lattices is not characterized by the usual laws for disjunction and disjunction: while the logic $\mathrm{L}_{\wedge\vee}$ governed by the usual laws is distributive (cf. Theorem 6.1), the logic L_{Lat} of lattices is not distributive.

In this section we will obtain the syntactical consequence-relation counterpart of the logic L_{Lat} of lattices. Thus, we will prove that L_{Lat} can be seen as the logic of conjunction and disjunction provided that an appropriate set of laws is considered. The problem with the laws for conjunction and disjunction given in sections 2 and 3 resides exclusively in law $(\vee_3)$, which presupposes (or induces) distributivity. In fact, this law states that from $\Gamma, \varphi \vdash \delta$ and $\Delta, \psi \vdash \delta$ it follows $\Gamma, \Delta, \varphi \vee \psi \vdash \delta$. In particular:

$(*)$ If $\gamma, \varphi \vdash \delta$ and $\gamma, \psi \vdash \delta$ then $\gamma, \varphi \vee \psi \vdash \delta$.

As long as just disjunction is present in the formulas above, the meta-property $(*)$ is valid even in the logic of lattices. Indeed, since $\{0, 1\}$ with the classical bivaluations for $\wedge$ and $\vee$ is a lattice then the validity of the premises of $(*)$ in this lattice in particular implies that either $Var(\gamma) \subseteq Var(\delta)$ or $Var(\varphi \vee \psi) = Var(\varphi) \cup Var(\psi) \subseteq Var(\delta)$. Therefore $\gamma, \varphi \vee \psi \vdash_{Lat} \delta$, because just disjunction occurs in these formulas.

But if conjunction is allowed in the formulas of $(\vee_3)$ (and, in particular, in the formulas of $(*)$) then the validity of $(*)$ implies that the lattices should be distributive. Indeed, consider and instance of $(*)$ such that $\delta = (\gamma \wedge \varphi) \vee (\gamma \wedge \psi)$. Then the premises of $(*)$ hold in any lattice, but the consequence holds iff

$$\gamma \wedge (\varphi \vee \psi) \leq (\gamma \wedge \varphi) \vee (\gamma \wedge \psi).$$

Analogously, by using $(*)$ two times it can be proven that

$$(\gamma \vee \varphi) \wedge (\gamma \vee \psi) \leq \gamma \vee (\varphi \wedge \psi)$$

and so the validity of $(*)$ implies distributivity.

A solution to this dilemma is to consider a weaker version of $(\vee_3)$, namely

$(\vee_{3w})$ If $\varphi \vdash \delta$ and $\psi \vdash \delta$ then $\varphi \vee \psi \vdash \delta$.

It is important to note that, as long as just disjunction is involved, the expressive power of $(\vee_3)$ and $(\vee_{3w})$ is the same. In other words, the logic over $\mathsf{For}_\vee$ governed by laws $(\vee_1)$-$(\vee_3)$ is the same as the logic over $\mathsf{For}_\vee$ governed by laws $(\vee_1)$-$(\vee_{3w})$. But when another connectives are introduced (in the case under study, a conjunction) then things change.

In order to see this, let $\mathrm{L}^{\wedge\vee} = \langle \mathsf{For}, \vdash^{\wedge\vee} \rangle$ be the least structural Tarskian consequence relation over For satisfying laws $(\wedge_1)$-$(\wedge_3)$, $(\vee_1)$, $(\vee_2)$ and $(\vee_{3w})$. Then $\mathrm{L}^{\wedge\vee}$ captures the logic of lattices, being so a non-distributive logic of conjunction and disjunction.

Theorem 10.1 *The logic of conjunction and disjunction $L^{\wedge\vee}$ is precisely the logic of lattices L_{Lat}, being therefore non-distributive.*

Proof: By using the usual Lindenbaum-Tarski method. Consider the following relation $\sim$ on For:

$$\varphi \sim \psi \ \text{ iff } \ \varphi \vdash^{\wedge\vee} \psi \ \text{ and } \ \psi \vdash^{\wedge\vee} \varphi.$$

Then $\sim$ is an equivalence relation. We denote by $[\varphi]$ the equivalence class of φ. In the quotient set $\mathsf{For}/_{\sim}$ define the following:

$[\varphi] \leq [\psi]$ iff $\varphi \vdash^{\wedge\vee} \psi$;

$[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$;

$[\varphi] \vee [\psi] = [\varphi \vee \psi]$.

It is easy to show that the relation and the operations above are well-defined, and $\mathsf{For}/_{\sim}$ together with $\leq$, $\wedge$ and $\vee$ is a lattice. Moreover, the canonical mapping $v_c : \mathsf{For} \to \mathsf{For}/_{\sim}$ given by $v_c(\varphi) = [\varphi]$ is a homomorphism.

Now, if $\Gamma \cup \{\varphi\} \subseteq \mathsf{For}$ is such that $\Gamma \vdash^{\wedge\vee} \varphi$ then $\Gamma \vdash_{Lat} \varphi$, because all the laws of $\mathrm{L}^{\wedge\vee}$ are valid in any lattice. Suppose now that $\Gamma \nvdash^{\wedge\vee} \varphi$. Then $\gamma_1, \ldots, \gamma_n \nvdash^{\wedge\vee} \varphi$ for every finite nonempty $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$. That is, $[\gamma_1] \wedge \ldots \wedge [\gamma_n] \nleq [\varphi]$ for every finite nonempty $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$. Thus, for every $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ there exists a lattice $\mathsf{For}/_{\sim}$ and a homomorphism $v_c : \mathsf{For} \to \mathsf{For}/_{\sim}$ such that $v_c(\gamma_1) \wedge \ldots \wedge v_c(\gamma_n) \nleq v_c(\varphi)$. Therefore $\Gamma \nvdash_{Lat} \varphi$, concluding the proof. ■

This means that $\mathrm{L}^{\wedge\vee}$ is sound and complete for lattice semantics. The result above can be seen from another perspective, connected with completeness preservation by fibring consequence systems as mentioned in Section 9. Recall that a meet-semilattice is a poset $\langle L, \leq \rangle$ having a binary operator $\wedge$ for infimum satisfying the laws described in Section 5. Dually, a join-semilattice is a poset with a binary operator $\vee$ for supremum satisfying the properties defined in Section 5.

The logic $\mathrm{L}_{Meet} = \langle \mathsf{For}_{\wedge}, \vdash_{Meet} \rangle$ of meet-semilattices is defined in an analogous way to the logic L_{Lat} of lattices defined in Section 8, but now using meet-semilattices instead of lattices. On the other hand, the logic $\mathrm{L}_{Join} = \langle \mathsf{For}_{\vee}, \vdash_{Join} \rangle$ of join-semilattices is defined as follows: for any $\Gamma \cup \{\varphi\} \subseteq \mathsf{For}_{\vee}$, $\Gamma \vdash_{Join} \varphi$ iff there exists $\psi \in \Gamma$ such that $v(\psi) \leq v(\varphi)$ holds in L, for every join-semilattice L and every homomorphism $v : \mathsf{For}_{\vee} \to L$.

Recall that LC is defined to be the least structural Tarskian consequence system over $\mathsf{For}_{\wedge}$ satisfying laws $(\wedge_1)$-$(\wedge_3)$. On the other hand, let LD_w be the least structural Tarskian consequence system over $\mathsf{For}_{\vee}$ satisfying laws $(\vee_1)$, $(\vee_2)$ and $(\vee_{3w})$. As one would expect, the logic LC is the logic L_{Meet} of meet-semilattices, whereas LD_w is the logic L_{Join} of join-semilattices. The proof of this facts is analogous to that of Theorem 10.1. In other words, LC is sound and complete for meet-semilattices semantics, as well as LD_w is sound and complete for join-semilattices semantics.

As mentioned in Section 9, the (meta)fibring of LC and LD (seen as sequent calculi) is LCD, the distributive logic of conjunction and disjunction. By its turn, it can be easily proven that the non-distributive $\mathrm{L}^{\wedge\vee}$ is the (meta)fibring of LC and LD_w (seen as sequent calculi), whereas L_{Lat} is the fibring of L_{Join} and L_{Meet}. Therefore, Theorem 10.1 states the completeness preservation of the combination of LC and LD_w by fibring.

This example clearly shows that different presentations of the same logic produce different logics when the given logic is combined with other. An extreme case of this phenomenon is the syntactical presentation of disjunction given by the Hilbert calculus presented in Section 3 formed by the following inference rules:

$$\frac{\psi}{\varphi} \qquad \text{whenever } Var(\psi) \subseteq Var(\varphi).$$

It is easy to see that these inference rules are not even sound when combined with the rules of other connectives, for instance the inference rules for conjunction given in Section 2. For example, the deduction

$$\frac{\varphi}{\varphi \wedge \psi}$$

is allowed in the fibring of both Hilbert calculi, because $Var(\varphi) \subseteq Var(\varphi \wedge \psi)$. But then ψ follows from $\varphi \wedge \psi$, by using a (sound) rule for conjunction, and so ψ is a logical consequence of φ, for every φ and ψ. That is, the combination of this particular presentation of the logics of conjunction and disjunction produces the trivial logic, since we arrive to Prior's Tonk problem.

The dependency on the representation of the logics partly explains why the distributivity laws appear in some combinations of conjunction and disjunction and do not arise in others.

11 Another bridge principles

We analyzed the spontaneous creation of interaction laws between conjunction and disjunction when these connectives are combined, namely the distributivity laws. As observed in [8], this kind of interaction (or bridge principles, following the terminology of that paper) is not a consequence of the application of the rules of each logic separately, but it arises when rules from both logics are combined. Thus, the interaction laws

$$\varphi \vdash \varphi \vee (\varphi \wedge \psi)$$

$$\varphi \wedge (\varphi \vee \psi) \vdash \varphi$$

are not bridge principles (or unexpected, or paradoxical) since they are instances of the following properties of conjunction and disjunction, respectively:

$$\varphi \vdash \varphi \vee \psi \quad \text{and} \quad \varphi \wedge \psi \vdash \varphi.$$

The latter hold in CD, the minimal logic extending LC and LD, and so the interaction laws mentioned above are also valid in CD. In particular, they are valid in the logic of lattices.

But now consider the converse laws:

(A_1) $\varphi \vee (\varphi \wedge \psi) \vdash \varphi$

(A_2) $\varphi \vdash \varphi \wedge (\varphi \vee \psi)$

The latter laws will be called *absorption laws*, because its lattice counterpart constitute the well-known absorption laws

$$a = a \vee (a \wedge b) \quad \text{and} \quad a = a \wedge (a \vee b).$$

It could be expected that the minimal logic of conjunction and disjunction CD would satisfy the absorption laws, because they are valid in the logic of lattices. The next result shows that the intuition about considering the logic of lattices as being the minimal logic of conjunction and disjunction is wrong:

Theorem 11.1 *The logic CD of conjunction and disjunction does not satisfy the absorption laws.*

Proof: It is enough to exhibit a structural Tarskian consequence system over For which simultaneously extends LC and LD such that the absorption laws fail.

Let $[0,1]$ be the real interval with its usual product $\cdot$ and the truncated sum $\oplus$ such that

$$a \oplus b = \begin{cases} a+b & \text{if } a+b \leq 1 \\ 1 & \text{otherwise} \end{cases}.$$

Consider homomorphisms $v : \mathsf{For} \to [0,1]$ such that $v(\varphi \wedge \psi) = v(\varphi) \cdot v(\psi)$ and $v(\varphi \vee \psi) = v(\varphi) \oplus v(\psi)$. Define a relation $\vdash_{[0,1]}$ as follows: given $\Gamma \cup \{\varphi\} \subseteq \mathsf{For}$, $\Gamma \vdash_{[0,1]} \varphi$ iff there exists a finite nonempty set $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ such that, for every homomorphism $v : \mathsf{For} \to [0,1]$, $v(\gamma_1) \cdot \ldots \cdot v(\gamma_n) \leq v(\varphi)$.

It is easy to show that $\langle \mathsf{For}, \vdash_{[0,1]} \rangle$ is a structural Tarskian consequence system. In particular, structurality is a direct consequence of the following fact: if v is a homomorphism and σ is a substitution over For, then the homomorphism v' given by $v'(p) = v(\sigma(p))$ for every generator $p \in \mathcal{P}$ is such that $v'(\varphi) = v(\sigma(\varphi))$ for every $\varphi \in \mathsf{For}$. Thus, if $\Gamma \vdash_{[0,1]} \varphi$ and $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ is such that $v(\gamma_1) \cdot \ldots \cdot v(\gamma_n) \leq v(\varphi)$ for every homomorphism v, let σ be a substitution and v a homomorphism. Define a homomorphism v' as above; then

$$v(\sigma(\gamma_1)) \cdot \ldots \cdot v(\sigma(\gamma_n)) = v'(\gamma_1) \cdot \ldots \cdot v'(\gamma_n) \leq v'(\varphi) = v(\sigma(\varphi))$$

and so $\sigma(\Gamma) \vdash_{[0,1]} \sigma(\varphi)$.

Now, let $\Gamma \cup \{\varphi\} \subseteq \mathsf{For}_\wedge$ such that $\Gamma \vdash_\wedge \varphi$. Then $Var(\varphi) \subseteq Var(\{\gamma_1, \ldots, \gamma_n\})$ for some $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$. Therefore $v(\gamma_1) \cdot \ldots \cdot v(\gamma_n) \leq v(\varphi)$ for every homomorphism v, because $\{\gamma_1, \ldots, \gamma_n, \varphi\} \subseteq \mathsf{For}_\wedge$ (observe that $a \cdot b \leq a$ for every $a, b \in [0,1]$). Thus $\Gamma \vdash_{[0,1]} \varphi$ and so $\vdash_\wedge \subseteq \vdash_{[0,1]}$.

Finally, let $\Gamma \cup \{\varphi\} \subseteq \mathsf{For}_{\vee}$ such that $\Gamma \vdash_{\vee} \varphi$. Then $Var(\psi) \subseteq Var(\varphi)$ for some $\psi \in \Gamma$. Thus $v(\psi) \leq v(\varphi)$ for every homomorphism v, because $\psi, \varphi \in \mathsf{For}_{\vee}$ and $a \leq a \oplus b$ for every $a, b \in [0,1]$. Therefore $\Gamma \vdash_{[0,1]} \varphi$ and so $\vdash_{\vee} \subseteq \vdash_{[0,1]}$.

This means that $\vdash_{[0,1]}$ belongs to $\mathcal{R}$ (recall the proof of Theorem 8.1) and then $\vdash_{CD} \subseteq \vdash_{[0,1]}$. But $\vdash_{[0,1]}$ does not validate the absorption laws: for instance,

$$\frac{1}{2} \not\leq \frac{1}{2} \cdot (\frac{1}{2} \oplus \frac{1}{4}).$$

Therefore the logic CD does not satisfy the absorption laws. ■

12 Are there non-distributive logics? and what is the right method to combine logics?

Following the result of Section 6, one may want to conclude that there are no non-distributive logics, that the combination of the logic of disjunction and disjunction is the standard fragment of classical propositional logic, whose semantics is given by the usual truth-tables for conjunction and disjunction and whose proof theory is provided by the usual Gentzen rules.

From this point of view, there are here no problems related to combination of logics, and one may conjecture that from a semantical viewpoint, to combine bivalent semantics, truth-functional or not, we just have to put together the set of conditions in the same way as what we combine proof systems by putting the rules together.

However it is not that simple. We can argue that there really are non-distributive logics in a strong sense where $(\vee_3)$ does not hold, as it was shown in sections 8 and 10. The idea of non-distributivity makes sense in lattice theory and in algebra in general. Clearly, Theorem 10.1 supports this point of view. On the other hand, by comparing theorems 6.1 and 8.1, it seems that the underlying notion of morphisms between logics adopted for the process of combining logics is crucial. By its turn, the discussion at the end of Section 10 points out the relevance that the presentation of the logics has for the combining methods.

The framework of Tarskian consequence relations has been over the last decades challenged in different ways, for example a alternative to it is the notion of multiple-conclusion consequence relations. However this option it is not a solution for our problem, since Theorem 6.1 still hold for this framework, as we have observed. A possible solution is going in the opposite direction, by restricting the set of formulas on both sides of the consequence relation to the cardinality one. If we do that we find our way to a non-distributive logic of conjunction and disjunction which can be generated by a sequent system having an analogous feature. It is easy to prove that in a sequent system with only one formula on both sides of the sequent (generating a "monomonoconsequence relation"), distributivity cannot be derived. The proof is analogous to that of Theorem 10.1. But the analysis done in Section 10 suggests that the full use of monomonoconsequence relations is not necessary: it is enough to consider a monomonoconsequence version of law ($\vee_3$), namely ($\vee_{3w}$), and keep the other laws unchanged. After all, despite the logic of disjunction is characterized by having no interaction between the premises within a deduction, an intrinsic feature of the logic of conjunction is the interaction between premises: from premises φ and ψ (among other premises) it follows $\varphi \wedge \psi$.

One may say that the problem we face with non-distributivity is connected to substructurality. We prove distributivity in the standard sequent system for conjunction and disjunction by using contraction. As we have seen, one way to block this phenomenon is to reduce the cardinality of the sets of the formulas to one, which already leads to sequent systems that can be called substructural. A monomonoconsequence relation could by analogy also be called a substructural logic. Another possibility in order to block distributivity is to throw away or to restrict contraction. But then several problems arise: we have to see how we will define logics, i.e. consequence relations, without contraction rules. This problem is in general not faced by people working in linear logic, who rather considered logic as set of tautologies. A solution to this problem is to consider multiset or alternatively further operators between formulas which properly form a substructure. Then we have to study the techniques to combine such substructural logics and related proof systems and semantics for such logics.

Of course all the considerations for distributivity mentioned above apply to the other interaction laws generated spontaneously by some combining methods, the absorption laws. This reinforces our claim that an apparently trivial problem such as the combination of classical conjunction and disjunction hidden interesting and non-trivial questions.

Acknowledgements: We thank to the anonymous referees for their useful comments and suggestions. The first author was supported by a grant of CNPq-DCR-FUNCAMP, Brazil. The second author was supported by The State of São Paulo Research Foundation (FAPESP), Brazil, Thematic Project number 2004/14107-2 ("ConsRel"), and by an individual research grant from The National Council for Scientific and Technological Development (CNPq), Brazil.

References

[1] J.-Y. Béziau. *Sur la vérité logique*, PhD thesis, Department of Philosophy, University of São Paulo, Brazil, 1996.

[2] J.-Y. Béziau. Logic may be simple, *Logic and Logical Philosophy*, 5:129–147, 1997.

[3] J.-Y. Béziau. Rules, derived rules, permissible rules and the various types of systems of deduction. In L. C. Pereira and E. H. Hauesler, editors, *Proof, Types and Categories*, PUC, Rio de Janeiro, pages 159–184, 1999.

[4] J.-Y. Béziau. A paradox in the combination of logics. In W. A. Carnielli, F. M. Dionísio, and P. Mateus, editors, *Proceedings of Comblog'04*, IST, Lisbon, pages 87–92, 2004.

[5] J.-Y. Béziau and M. E. Coniglio. Combining Conjunction with Disjunction. In B. Prasad, editor, *Proceedings of the 2nd Indian International Conference on Artificial Intelligence (IICAI 2005)*, Pune, India, pages 1648–1658, 2005.

[6] G. Birkhoff. *Lattice theory*, American Mathematical Society, New York, 1940.

[7] W. A. Carnielli and M. E. Coniglio. Combining Logics. In E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*, 2007.
URL = `http://www.plato.stanford.edu/entries/logic-combining/`

[8] W. A. Carnielli and M. E. Coniglio. Bridge principles and combined reasoning. In T. Müller and A. Newen, editors, *Logik, Begriffe, Prinzipien des Handelns (Logic, Concepts, Principles of Action)*, Mentis Verlag, Paderborn, pages 32–48, 2007.

[9] W. A. Carnielli, M. E. Coniglio, D. Gabbay, P. Gouveia and C. Sernadas. *Analysis and Synthesis of Logics - How To Cut And Paste Reasoning Systems*, vol. 35 of *Applied Logic Series*, Springer, Amsterdam, 2008.

[10] M. E. Coniglio. Recovering a logic from its fragments by meta-fibring. *Logica Universalis*, 1(2):377–416, 2007. Preprint available as :"The Meta-Fibring environment: Preservation of meta-properties by fibring", *CLE e-Prints*, v. 5., n. 4, 2005.
URL = `http://www.cle.unicamp.br/e-prints/vol_5,n_4,2005.html`

[11] R. Wójcicki. *Theory of Logical Calculi*, Synthese Library, Kluwer Academic Publishers, 1988.