On the Way to a Wider Model Theory: Completeness Theorems for First-Order Logics of Formal Inconsistency

Walter Carnielli, Marcelo E. Coniglio,
Rodrigo Podiacki and Tarcísio Rodrígues
Centre for Logic, Epistemology and the History of Science – CLE
and
Department of Philosophy – UNICAMP
P.O. Box 6133, 13083-970
Campinas, SP, Brazil
walter.carnielli@cle.unicamp.br, coniglio@cle.unicamp.br
podiacki@gmail.com, tarcisio.genaro@gmail.com

Abstract

This paper investigates the question of characterizing first-order **LFI**s (logics of formal inconsistency) by means of two-valued semantics. **LFI**s are powerful paraconsistent logics that encode classical logic and permit a finer distinction between contradictions and inconsistencies, with a deep involvement in philosophical and foundational questions. Although focused on just one particular case, namely, the quantified logic **QmbC**, the method proposed here is completely general for this kind of logics, and can be easily extended to a large family of quantified paraconsistent logics, supplying a sound and complete semantical interpretation for such logics. However, certain subtleties involving term substitution and replacement, that are hidden in classical structures, have to be taken into account when one ventures into the realm of non-classical reasoning. This paper shows how such difficulties can be overcome, and offers detailed proofs showing that a smooth treatement of semantical characterization can be given to all such logics. Although the paper is well-endowed in technical details and results, it has a significant philosophical aside: it shows how slight extensions of classical methods can be used to construct the basic model theory of logics that are weaker than traditional logic due to the absence of certain rules present in classical logic. Several such logics, however, as in the case of the **LFI**s treated here, are notorious for their wealth of models precisely because they do not make indiscriminate use of certain rules; these models thus require new methods. In the case of this paper, by just appealing to a refined version of the Principle of Explosion, or Pseudo-Scotus, some new constructions and crafty solutions to certain non-obvious subtleties are proposed. The result is that a richer extension of model theory can be inaugurated, with interest not only for paraconsistency, but hopefully to other enlargements of traditional logic.

Introduction

Logics of Formal Inconsistency — from now on **LFIs** — are logics able to internalize, in a precise sense, the notions of "consistency" and "inconsistency" at the object-language level, be it by introducing primitive unary connectives, or by means of appropriate definitions using the familiar propositional connectives. Such logics are paraconsistent in the following sense: given a contradiction of the form $(\varphi \wedge \neg\varphi)$, it is not possible in general to deduce an arbitrary formula ψ from the contradiction. That is, such logics do not fall into deductive triviality when exposed to a contradiction. This means that the Principle of Explosion, or Pseudo-Scotus, is not valid for such logics in general. However, **LFIs** may "explode" if, besides φ being contradictory, there is an additional stipulation, namely, that φ is consistent, or that φ behaves classically. **LFIs** are therefore submitted to a more restricted principle of explosion, called in [7] the *Gentle Principle of Explosion*: an **LFI** explodes if φ, $\neg\varphi$ and $\circ\varphi$ occur simultaneusly, for some arbitrary φ, such that '$\circ\varphi$' expresses the fact that φ is consistent. This constitutes a wide generalization of the well-known C-systems introduced by da Costa through the hierarchy of systems C_n, for $n \geq 1$ (see [12]). In C_1, for instance, consistency (or well-behavior, in da Costa's words) is defined by the formula $\circ\varphi = \neg(\varphi \wedge \neg\varphi)$.

In their beginnings, paraconsistent logics were mainly developed syntactically, that is, presented through Hilbert calculi, without committing to a semantical interpretation. The first semantics for propositional paraconsistent logics (that is, for the calculi C_n of da Costa) had to wait until the 70s, and were known as *valuation semantics* (see [14]). Nevertheless, the problem concerning a convenient interpretation for first-order paraconsistent logic persisted. In 1984, Alves proposed a method which can be called *pre-structural semantics* (see [1]).

The present paper proposes an axiomatization and first-order semantics for **QmbC**, the first-order extension of **mbC**, which is the simplest logic in the hierarchy of **LFI**s proposed in [8, 7]. The semantics, adapted from [25], differs slightly from the one adopted in [23]. This improved formulation is preferred, as it can be better and more naturally extended to other cases.

The structure of the paper is as follows: in Section 1 the basic system **QmbC** is introduced, and some useful theorems are proved. Section 2 establishes some metatheorems of **QmbC**. The (Tarskian) 2-valued semantics for **QmbC** is introduced in Section 3. The soundness and completeness theorems of **QmbC** with respect to the proposed semantics are obtained in sections 4 and 5, respectively. In Section 6 some fundamental theorems of Model Theory for **QmbC** are given: Compactness and the Lowenhëim-Skolem Theorems. In Section 7, the system **QmbC** is expanded with the predicate for standard equality, as is usually done in Model Theory. Section 8 presents the axiomatic extension of **QmbC** (possibly expanded with equality) to other **LFI**'s studied in [8, 7]. In Section 8 a brief survey of previous approaches to first-order **LFI**'s proposed in the literature is presented. The final section discusses what was done from a conceptual point of view.

1 The logic QmbC

This section introduces the logic **QmbC**, the first-order **LFI** to be investigated in detail in this paper.

Definition 1.1. *Assume the set of connectives* $\{\neg, \circ, \wedge, \vee, \rightarrow\}$ *for negation, consistency, conjunction, disjunction and implication, as well as the symbols* $\forall$ *(universal quantifier) and* $\exists$ *(existential quantifier), and punctuation marks (commas and parentheses). Let* $Var = \{v_1, v_2, \ldots\}$ *be a denumerable set of individual variables. A first-order signature* Σ *for* ***LFI****s is composed of the following elements:*

- *a set* C *of individual constants;*
- *for each* $n \geq 1$*, a set of function symbols of arity* n*;*
- *for each* $n \geq 1$*, a set of predicate symbols of arity* n*.*

As usual, given a signature Σ, it is assumed that it has at least one predicate symbol. The set of terms and of formulas of Σ (which are defined recursively, as usual) are denoted by T_Σ and L_Σ, respectively. Also, the notions of subformula, scope of an occurrence of a quantifier in a formula, free and bound occurrence of a variable in a formula, and of a term free for a variable in a formula, are the usual ones (the reader is referred to [26] and [21]).

The set of atomic formulas and of sentences (i.e., formulas without free variables) of Σ are denoted by At_Σ and S_{L_Σ}, respectively.

The notation $\varphi[x/t]$ will stand for the formula obtained from φ by substituting every free occurrence of variable x by the term t.

Definition 1.2. *Let* φ *and* ψ *be formulas. If* φ *can be obtained from* ψ *by means of addition or deletion of void quantifiers, or by renaming bound variables (keeping the same free variables in the same places), then* φ *and* ψ *are said to be variants of each other.*

The logic **mbC** was introduced in [7] as a basic **LFI**, meaning that its axioms embody a minimum proof power sufficient to preserve the positive theorems of classical propositional logic, while at the same time being capable of avoiding trivialization in the presence of contradictions. The extension of **mbC** to first-order logic is called **QmbC**, and is defined as follows:

Definition 1.3. *Let* Σ *be a first-order signature. The logic* **QmbC** *(over* Σ*) is defined by the following Hilbert calculus:*

Axiom Schemas

(Ax1) $\alpha \rightarrow (\beta \rightarrow \alpha)$

(Ax2) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma))$

(Ax3) $\alpha \to (\beta \to (\alpha \wedge \beta))$

(Ax4) $(\alpha \wedge \beta) \to \alpha$

(Ax5) $(\alpha \wedge \beta) \to \beta$

(Ax6) $\alpha \to (\alpha \vee \beta)$

(Ax7) $\beta \to (\alpha \vee \beta)$

(Ax8) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

(Ax9) $\alpha \vee (\alpha \to \beta)$

(Ax10) $\alpha \vee \neg\alpha$

(Ax11) $\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$

(Ax12) $\varphi[x/t] \to \exists x\varphi$, *if* t *is a term free for* x *in* φ

(Ax13) $\forall x\varphi \to \varphi[x/t]$, *if* t *is a term free for* x *in* φ

(Ax14) $\forall x(\alpha \to \beta) \to (\alpha \to \forall x\beta)$, *if* x *is not free in* α

(Ax15) $\alpha \to \beta$, *whenever* α *is a variant of* β

Inference Rules

MP: $\alpha, \ \alpha \to \beta / \beta$

$\forall$-***In***: $\alpha \to \beta / \alpha \to \forall x\beta$, *if* x *is not free in* α

$\exists$-***In***: $\alpha \to \beta / \exists x\alpha \to \beta$, *if* x *is not free in* β

The consequence relation of **QmbC** will be denoted by $\vdash$. Thus, if $\Gamma \cup \{\varphi\} \subseteq L_\Sigma$ then $\Gamma \vdash \varphi$ will denote that there exists a derivation in **QmbC** of φ from Γ.

It is worth noting that (Ax1)-(Ax11) plus **MP** (considered in a propositional language) is a Hilbert calculus for the propositional logic **mbC**, while (Ax1)-(Ax9) plus **MP** is a Hilbert calculus for positive propositional classical logic (see [7]).

As it was proved in [7], the logic **mbC** can be characterized in terms of valuations over $\{0, 1\}$, or *bivaluations*:

Definition 1.4. *Let* $\mathcal{L}$ *be the algebra of formulas of* **mbC**. *A function* $v : \mathcal{L} \to \{0, 1\}$ *is a* valuation for **mbC** *if it satisfies the following clauses:*

(*vAnd*) $v(\alpha \wedge \beta) = 1 \iff v(\alpha) = 1$ *and* $v(\beta) = 1$

(*vOr*) $v(\alpha \vee \beta) = 1 \iff v(\alpha) = 1$ *or* $v(\beta) = 1$

(*vImp*) $v(\alpha \to \beta) = 1 \iff v(\alpha) = 0$ *or* $v(\beta) = 1$

(*vNeg*) $v(\alpha) = 0 \implies v(\neg\alpha) = 1$

(*vCon*) $v(\circ\alpha) = 1 \implies v(\alpha) = 0$ *or* $v(\neg\alpha) = 0$.

The semantical consequence relation w.r.t. valuations for **mbC** is defined as expected: $\Gamma \vDash_{\mathbf{mbC}} \varphi$ iff, for every valuation v for **mbC**, if $v(\gamma) = 1$ for every $\gamma \in \Gamma$ then $v(\varphi) = 1$.

Theorem 1.5. *([7]) For every set of formulas* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$*:*

$$\Gamma \vdash_{\mathbf{mbC}} \varphi \ \textit{ if and only if } \ \Gamma \vDash_{\mathbf{mbC}} \varphi\,.$$

2 Some useful (meta)theorems of QmbC

In this section, some useful theorems and meta-theorems of **QmbC** will be established. Throughout this section, a fixed first-order signature Σ will be assumed.

Theorem 2.1.

1. For every formula α*:* $\vdash \alpha \to \alpha$.

2. If $\Gamma \vdash \alpha \to \gamma$ *and* $\Gamma \vdash \beta \to \gamma$ *then* $\Gamma, \alpha \vee \beta \vdash \gamma$.

3. If $\Gamma \vdash \neg\alpha \to \gamma$ *and* $\Gamma \vdash \alpha \to \gamma$ *then* $\Gamma \vdash \gamma$.

4. If $\Gamma \vdash \phi$ *then* $\Gamma \vdash \forall x\, \phi$.

5. If x *is not free in* β *then* $\forall x(\alpha \to \beta) \vdash \exists x\alpha \to \beta$.

6. If x *is not free in* α *then* $\forall x(\alpha \to \beta) \vdash \alpha \to \forall x\beta$.

7. For every formulas α, β, γ*:* $\vdash (\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$.

8. For every formulas α, β, γ*:* $\vdash (\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$.

9. If $\Gamma \vdash \alpha \to \beta$ *then* $\Gamma, \alpha \vdash \beta$.

10. $\alpha \to \beta, \beta \to \gamma \vdash \alpha \to \gamma$.

11. If $\vdash \alpha \to \beta$ *then* $\vdash (\gamma \to \alpha) \to (\gamma \to \beta)$*, for every formula* γ.

Proof. Item 1: It is well known that any Hilbert calculus containing axiom schemas (Ax1), (Ax2) and (**MP**) derives the schema $\alpha \to \alpha$. The following derivation was taken from [21]:

1. $(\alpha \to ((\alpha \to \alpha) \to \alpha)) \to ((\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha))$ (Ax2)
2. $\alpha \to ((\alpha \to \alpha) \to \alpha)$ (Ax1)
3. $(\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha)$ (**MP** 1,2)

4. $\alpha \to (\alpha \to \alpha)$ (Ax1)

5. $\alpha \to \alpha$ (**MP** 3,4)

Item 2: By hypothesis, $\Gamma, \alpha \vee \beta \vdash \alpha \to \gamma$ and $\Gamma, \alpha \vee \beta \vdash \beta \to \gamma$ hold. But $\vdash (\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$, by (Ax8), and so $\Gamma, \alpha \vee \beta \vdash (\alpha \vee \beta) \to \gamma$, by using **MP** twice. By applying **MP** once again it follows that $\Gamma, \alpha \vee \beta \vdash \gamma$.

Item 3: It is a consequence of Item 2, by observing that $\vdash \alpha \vee \neg\alpha$, by (Ax10).

Item 4: Consider the (meta)derivation below.

$\Gamma \vdash \phi$	Hypothesis
$\Gamma \vdash \phi \to (\neg\forall x\, \phi \to \phi)$	Ax1
$\Gamma \vdash \neg\forall x\, \phi \to \phi$	**MP**
$\Gamma \vdash \neg\forall x\, \phi \to \forall x\, \phi$	**∀-In**
$\Gamma \vdash \forall x\, \phi \to \forall x\, \phi$	Item 1
$\Gamma \vdash \forall x\, \phi$	Item 2

Item 5: Consider the derivation in **QmbC** below.

1. $\forall x(\alpha \to \beta)$ (premise)

2. $\forall x(\alpha \to \beta) \to (\alpha \to \beta)$ (Ax13)

3. $\alpha \to \beta$ (**MP** 1,2)

4. $\exists x\alpha \to \beta$ (**∃-In** 3)

Item 6: It follows by (Ax14) and **MP**.

Item 7: By considering the semantics of bivaluations for **mbC** given above, it is easy to see that $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$ is a valid formula. By completeness of **mbC** w.r.t. bivaluations, that formula is derivable in **mbC**, for every α, β, γ. Since **QmbC** extends **mbC**, it follows that the schema $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$ is derivable in **QmbC**.

Item 8: The proof is identical to that of Item 7.

Item 9: Consider the (meta)derivation below.

$\Gamma \vdash \alpha \to \beta$	Hypothesis
$\Gamma, \alpha \vdash \alpha \to \beta$	Monotonicity
$\Gamma, \alpha \vdash \beta$	**MP**

Item 10: It follows from Item 7 and Item 9 (used two times).

Item 11: From items 7 and 8 it follows that $\vdash (\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$. Then, by hypothesis and **MP**, $\vdash (\gamma \to \alpha) \to (\gamma \to \beta)$. □

Recall from [7] that, in **mbC**, a strong (classical) negation can be defined as $\sim_\beta \alpha = \alpha \rightarrow \bot_\beta$, where $\bot_\beta = (\beta \wedge (\neg\beta \wedge \circ\beta))$ is a bottom formula,[1] for any formula β. In the case of first-order languages, the formula β must be a sentence. For simplicity, a privileged one will be chosen, and the subscript β will be omitted in $\bot_\beta$ and $\sim_\beta$ from now on.

Proposition 2.2 (Strong Negation)**.** *The strong negation $\sim$ satisfies the following properties in* **mbC** *(and, therefore, also in* **QmbC***):*

(i) $\vdash \sim\alpha \rightarrow (\alpha \rightarrow \psi)$ *for every α and ψ;*

(ii) $\vdash \alpha \vee \sim\alpha$

(iii) $\vdash \alpha \rightarrow \sim\sim\alpha$ *and* $\vdash \sim\sim\alpha \rightarrow \alpha$

(iv) If $(\Gamma \vdash \alpha \rightarrow \gamma)$ *and* $(\Delta, \vdash \sim\alpha \rightarrow \gamma)$ *then* $(\Gamma, \Delta \vdash \gamma)$

(v) $\vdash (\alpha \rightarrow \beta) \rightarrow (\sim\beta \rightarrow \sim\alpha)$ *and so* $\alpha \rightarrow \beta \vdash \sim\beta \rightarrow \sim\alpha$

(vi) $\vdash (\sim\alpha \rightarrow \sim\beta) \rightarrow (\beta \rightarrow \alpha)$ *and so* $\sim\alpha \rightarrow \sim\beta \vdash \beta \rightarrow \alpha$

(vii) $\sim\alpha \rightarrow \beta \vdash \sim\beta \rightarrow \alpha$

(viii) $\vdash \sim(\alpha \rightarrow \beta) \rightarrow (\alpha \wedge \sim\beta)$

(ix) $\vdash \bot \rightarrow \alpha$

(x) $\vdash \forall x \sim\alpha \rightarrow \sim\exists x\alpha$

(xi) If $\vdash \alpha \rightarrow \beta$ *then* $\vdash \forall x\alpha \rightarrow \forall x\beta$

(xii) $\vdash \forall x(\alpha \rightarrow \beta) \rightarrow (\exists x\alpha \rightarrow \beta)$*, if x does not occur free in β.*

Proof. Items (i)-(iii) and (v)-(ix): It follows from the completeness of **mbC** with respect to bivaluations (see Theorem 1.5), by observing that $v(\bot) = 0$, and $v(\sim\alpha) = 1$ iff $v(\alpha) = 0$, for every formula α and every valuation v for **mbC**. An argument similar to the proof of Theorem 2.1 Item 7 can then be used.

Item (iv): It follows from Item (ii) and from Item 2 of Theorem 2.1.

Item (x): From (Ax13) it holds that $\vdash \forall x \sim\alpha \rightarrow \sim\alpha$ and so $\vdash \sim\sim\alpha \rightarrow \sim\forall x \sim\alpha$, by Item (v). Using Item (iii) and Theorem 2.1 Item 10, $\vdash \alpha \rightarrow \sim\forall x \sim\alpha$. By rule (**∃-In**), $\vdash \exists x\alpha \rightarrow \sim\forall x \sim\alpha$ whence $\vdash \sim\sim\forall x \sim\alpha \rightarrow \sim\exists x\alpha$, using again Item (v). Finally, by Item (iii) and Theorem 2.1 Item 10, $\vdash \forall x \sim\alpha \rightarrow \sim\exists x\alpha$.

Item (xi): From (Ax13), $\vdash \forall x\alpha \rightarrow \alpha$ and then, by hypothesis and Theorem 2.1 Item 10, $\vdash \forall x\alpha \rightarrow \beta$. By rule (**∀-In**), $\vdash \forall x\alpha \rightarrow \forall x\beta$.

Item (xii): By items (v) and (xi) it follows that $\vdash \forall x(\alpha \rightarrow \beta) \rightarrow \forall x(\sim\beta \rightarrow \sim\alpha)$. But $\vdash \forall x(\sim\beta \rightarrow \sim\alpha) \rightarrow (\sim\beta \rightarrow \forall x \sim\alpha)$, by (Ax14), and then $\vdash \forall x(\alpha \rightarrow \beta) \rightarrow (\sim\beta \rightarrow \forall x \sim\alpha)$, by Theorem 2.1 Item 10. By Item (x) and Theorem 2.1 Item 11, $\vdash (\sim\beta \rightarrow \forall x \sim\alpha) \rightarrow (\sim\beta \rightarrow \sim\exists x\alpha)$. From this, $\vdash \forall x(\alpha \rightarrow \beta) \rightarrow (\sim\beta \rightarrow \sim\exists x\alpha)$,

[1]That is: $\bot_\beta \vdash \psi$ for every ψ.

by Theorem 2.1 Item 10. By Item (vi), $\vdash (\sim\beta \rightarrow \sim\exists x\alpha) \rightarrow (\exists x\alpha \rightarrow \beta)$. Finally, $\vdash \forall x(\alpha \rightarrow \beta) \rightarrow (\exists x\alpha \rightarrow \beta)$, using again Theorem 2.1 Item 10. □

The next step is to establish a particularly important meta-theorem of **QmbC**: the *Deduction Meta-Theorem* (DMT). As in the case of classical first-order logic, it does not hold in the general case, but does hold under certain assumptions concerning free variables of the formula being discharged from the assumptions. The proof will be done by adapting the case for classical logic presented in [21].

Definition 2.3. *Let $d = \varphi_1, \ldots, \varphi_n$ be a derivation in* **QmbC** *from a set of hypothesis Γ, and let $\varphi \in \Gamma$. Then φ_i is said* to depend upon *φ in d if:*

- *$\varphi_i = \varphi$; or*
- *φ_i is obtained from φ_j and φ_k (with $j, k < i$) by (**MP**), where φ_j or φ_k depend upon φ in d; or*
- *φ_i is obtained from φ_j (with $j < i$) by (**∃-In**), where φ_j depends upon φ in d; or*
- *φ_i is obtained from φ_j (with $j < i$) by (**∀-In**), where φ_j depends upon φ in d.*

Of course the notion above can be adjusted to any Hilbert calculus. The next result holds in any Hilbert calculus (see [21]).

Lemma 2.4. *If ψ does not depend upon φ in the derivation of ψ from $\Gamma \cup \{\varphi\}$, then $\Gamma \vdash \psi$.*

Theorem 2.5 (Deduction Meta-Theorem (DMT) for **QmbC**)**.** *Suppose that there exists in* **QmbC** *a derivation of ψ from $\Gamma \cup \{\varphi\}$, such that no application of the rules (**∃-In**) and (**∀-In**) to formulas that depend upon φ have as their quantified variables free variables of φ. Then $\Gamma \vdash \varphi \rightarrow \psi$.*

Proof. Let $d = \varphi_1, \ldots, \varphi_n$ be a derivation in **QmbC** of ψ from $\Gamma \cup \{\varphi\}$, satisfying the conditions of the hypothesis of the theorem; then $\varphi_n = \psi$. It will be proven by induction on n that $\Gamma \vdash \varphi \rightarrow \varphi_i$ for every $1 \leq i \leq n$. From this it follows that $\Gamma \vdash \varphi \rightarrow \psi$, as required.

The proof is identical to that presented in [21] for first-order classical logic, with the exception of the rules for quantification (which are different from the rules of **QmbC**), and so this case is the only one to be treated here (observe that the part of the proof in [21] omitted here uses that $\alpha \rightarrow \alpha$ is a theorem, as proved here in Item 1 of Theorem 2.1).

Thus, suppose that $\Gamma \vdash \varphi \rightarrow \varphi_j$ for every $1 \leq j < i$, with $i \geq 2$. By the considerations above, just two cases need to be analyzed:

1) There exists $j < i$ such that $\varphi_j = \alpha \rightarrow \beta$ and $\varphi_i = \exists x\alpha \rightarrow \beta$ (with x not free in β) is obtained from φ_j by (**∃-In**). By induction hypothesis, $\Gamma \vdash \varphi \rightarrow \varphi_j$ and, by the hypothesis on d, either φ_j does not depend upon φ or x does not occur free in φ. There are two subcases to be analyzed:
1.1) φ_j does not depend upon φ. By Lemma 2.4, $\Gamma \vdash \varphi_j$, that is, $\Gamma \vdash \alpha \rightarrow \beta$. By applying rule (**∃-In**) it follows that $\Gamma \vdash \exists x\alpha \rightarrow \beta$, that is, $\Gamma \vdash \varphi_i$. From this, $\Gamma \vdash \varphi \rightarrow \varphi_i$.

1.2) x does not occur free in φ. As $\Gamma \vdash \varphi \to \varphi_j$, that is, $\Gamma \vdash \varphi \to (\alpha \to \beta)$, then $\Gamma \vdash \varphi \to \forall x(\alpha \to \beta)$, by applying rule (**∀-In**). By Proposition 2.2 Item (xii) it follows that $\vdash \forall x(\alpha \to \beta) \to (\exists x\alpha \to \beta)$ and then $\vdash (\varphi \to \forall x(\alpha \to \beta)) \to (\varphi \to (\exists x\alpha \to \beta))$, by Theorem 2.1 Item 11. From this, $\Gamma \vdash \varphi \to (\exists x\alpha \to \beta)$, by **MP**. That is, $\Gamma \vdash \varphi \to \varphi_i$.

2) There exists $j < i$ such that $\varphi_j = \alpha \to \beta$ and $\varphi_i = \alpha \to \forall x\beta$ (with x not free in α) is obtained from φ_j by (**∀-In**). The proof of this case is quite similar to that of Case 1 and so it will be omitted (it is worth noting that, in the second subcase, axiom (Ax14) is used instead of Proposition 2.2 Item (xii)). □

The version of (DMT) presented above is very general, but for this reason it can be a bit complicated to determine the conditions under which it can be applied.A particular case is given below, which is simpler than the general case but enough for most applications.

Corollary 2.6 ((DMT), simplified version)**.** *Suppose that there exists in* **QmbC** *a derivation of* ψ *from* $\Gamma \cup \{\varphi\}$*, such that no application of the rules* (**∃-In**) *and* (**∀-In**) *have as their quantified variables free variables of* φ *(in particular, this holds when* φ *is a sentence). Then* $\Gamma \vdash \varphi \to \psi$*.*

The Deduction Meta-Theorem simplifies considerably the derivations in **QmbC**. The proof of the following result, which will be used in the sequel, takes profit of DMT:

Theorem 2.7. *If* $\alpha \vdash \beta$ *then* $\gamma \to \alpha \vdash \gamma \to \beta$ *and* $\beta \to \gamma \vdash \alpha \to \gamma$*, for every* γ*.*

Proof. Consider the following (meta) derivation in **QmbC**:

$\gamma \to \alpha, \gamma \vdash \alpha$	**MP**
$\alpha \vdash \beta$	Hypothesis
$\gamma \to \alpha, \gamma \vdash \beta$	Transitivity

Therefore, by (DMT), $\gamma \to \alpha \vdash \gamma \to \beta$.

Finally, the following (meta)derivation above can be considered:

$\alpha \vdash \beta$	Hypothesis
$\beta \to \gamma, \alpha \vdash \beta$	Monotonicity
$\beta \to \gamma, \alpha \vdash \beta \to \gamma$	Reflexivity
$\beta \to \gamma, \alpha \vdash \gamma$	**MP**

Therefore, by (DMT), $\beta \to \gamma \vdash \alpha \to \gamma$. □

Some variations of the inference rules **∃-In** and **∀-In** will now be discussed. These results are essential in order to prove the Completeness theorem for **QmbC**, more precisely when proving that non-trivial theories can be conservatively extended to non-trivial Henkin theories (Theorem 5.3). First, however, some technical results must be obtained, recalling that $\sim$ denotes the strong negation.

Lemma 2.8. *In* **QmbC** *the following hold:*

(i) $\vdash (\alpha \to \beta)$ *implies* $\vdash (\exists x\alpha \to \exists x\beta)$

(ii) $\vdash \exists x{\sim}{\sim}\alpha \to \exists x\alpha$

(iii) $\vdash {\sim}\forall x\, \alpha \to \exists x\, {\sim}\alpha$

(iv) $\vdash {\sim}\forall x{\sim}\alpha \to \exists x\alpha$

(v) $\vdash (\forall x\alpha \to \beta) \to \exists x(\alpha \to \beta)$ *if x does not occur free in β*

(vi) $\vdash (\alpha \to \exists x\, \beta) \to \exists x(\alpha \to \beta)$ *if x does not occur free in α.*

Proof. (i) Suppose that $\vdash (\alpha \to \beta)$. Then $\vdash (\beta \to \exists x\beta)$, by (**∃-Ax**), and so $\vdash (\alpha \to \exists x\beta)$, by transitivity of $\to$. The result follows by applying rule (**∃-In**).

(ii) As $\vdash {\sim}{\sim}\alpha \to \alpha$, by Proposition 2.2(iii), the result follows from item (i).

(iii) By (Ax12), $\vdash {\sim}\alpha \to \exists x{\sim}\alpha$. By Proposition 2.2(vii), $\vdash {\sim}\exists x{\sim}\alpha \to \alpha$. By **∀-In**, $\vdash {\sim}\exists x{\sim}\alpha \to \forall x\alpha$. Finally, by Proposition 2.2(vii), $\vdash {\sim}\forall x\alpha \to \exists x{\sim}\alpha$.

(iv) By item (iii), $\vdash {\sim}\forall x{\sim}\alpha \to \exists x{\sim}{\sim}\alpha$. The result follows by item (ii) and by transitivity of $\to$.

(v) By **MP**, $(\forall x\alpha \to \beta), \forall x\alpha \vdash \beta$. But $\vdash \beta \to (\alpha \to \beta)$ and $\vdash (\alpha \to \beta) \to \exists x(\alpha \to \beta)$, by (Ax1) and (Ax12), respectively. Then, $(\forall x\alpha \to \beta), \forall x\alpha \vdash \exists x(\alpha \to \beta)$. On the other hand, $\vdash {\sim}\alpha \to (\alpha \to \beta)$, by Proposition 2.2(i), and so $\vdash \exists x{\sim}\alpha \to \exists x(\alpha \to \beta)$, by item (i). But $\vdash {\sim}\forall x\alpha \to \exists x{\sim}\alpha$, by item (iii) , therefore $\vdash {\sim}\forall x\alpha \to \exists x(\alpha \to \beta)$, by transitivity of $\to$. From this, $(\forall x\alpha \to \beta), {\sim}\forall x\alpha \vdash \exists x(\alpha \to \beta)$. Thus, by (DMT) and Proposition 2.2(iv), $(\forall x\alpha \to \beta) \vdash \exists x(\alpha \to \beta)$. The result follows by (DMT), as x does not occur free in β.

(vi) From $(\alpha \to \exists x\, \beta), \forall x{\sim}(\alpha \to \beta)$ it follows that $(\alpha \to \exists x\, \beta), {\sim}(\alpha \to \beta)$, by (Ax13), and from this $(\alpha \to \exists x\, \beta), \alpha, {\sim}\beta$, by Proposition 2.2(viii), (Ax4) and (Ax5). From this $\exists x\, \beta, {\sim}\beta$ is obtained by **MP**. But ${\sim}\beta = \beta \to \bot$ and $\beta \to \bot \vdash \exists x\beta \to \bot$, by (**∃-In**). That is, ${\sim}\beta \vdash {\sim}\exists x\, \beta$. Combining this with the inference above, from $(\alpha \to \exists x\, \beta), \forall x{\sim}(\alpha \to \beta)$, it follows that $\exists x\, \beta, {\sim}\exists x\, \beta$ and from this one obtains $\bot$. Therefore, $(\alpha \to \exists x\, \beta) \vdash {\sim}\forall x{\sim}(\alpha \to \beta)$, by (DMT) and the definition of ${\sim}$. By item (iv), $(\alpha \to \exists x\, \beta) \vdash \exists x(\alpha \to \beta)$. As x does not occur free in α, the result follows again by (DMT). □

Lemma 2.9. *If x does not occur free in φ and ψ, the following holds in* **QmbC***:*

1. If $\Gamma \vdash (\phi \to \varphi) \to \psi$ *then* $\Gamma \vdash (\forall x\, \phi \to \varphi) \to \psi$

2. If $\Gamma \vdash (\phi \to \varphi)$ *then* $\Gamma \vdash (\forall x\, \phi \to \varphi)$

3. If $\Gamma \vdash (\varphi \to \phi) \to \psi$ *then* $\Gamma \vdash (\varphi \to \exists x\, \phi) \to \psi$

4. If $\Gamma \vdash (\varphi \to \phi)$ *then* $\Gamma \vdash (\varphi \to \exists x\, \phi)$.

Proof. 1. By Lemma 2.8(v), $\vdash (\forall x\phi \to \varphi) \to \exists x(\phi \to \varphi)$. Then, by Theorem 2.7, $\vdash (\exists x(\phi \to \varphi) \to \psi) \to ((\forall x\phi \to \varphi) \to \psi)$.

Thus, suppose that $\Gamma \vdash (\phi \to \varphi) \to \psi$. Then $\Gamma \vdash \exists x(\phi \to \varphi) \to \psi$, by (**∃-In**). By the observation above, $\Gamma \vdash (\forall x\phi \to \varphi) \to \psi$.

2. Consider the following derivation in **QmbC**:

1. $\phi \to \varphi$ (premise)
2. $\forall x\phi$ (premise)
3. $\forall x\phi \to \phi$ (Ax13)
4. ϕ (**MP** 2,3)
5. φ (**MP** 1,4)

Thus, $\phi \to \varphi, \forall x\phi \vdash \varphi$, and so, by (DMT), it follows that $\phi \to \varphi \vdash \forall x\phi \to \varphi$. The result follows by transitivity of derivations.

3. By Lemma 2.8(vi), $\vdash (\varphi \to \exists x\, \phi) \to \exists x(\varphi \to \phi)$. By Theorem 2.7, it follows that $\vdash (\exists x(\varphi \to \phi) \to \psi) \to ((\varphi \to \exists x\, \phi) \to \psi)$.

Now, suppose that $\Gamma \vdash (\varphi \to \phi) \to \psi$. By (**∃-In**) it follows that $\Gamma \vdash \exists x(\varphi \to \phi) \to \psi$. Then $\Gamma \vdash (\varphi \to \exists x\, \phi) \to \psi$, by the observation above.

4. Consider the following derivation in **QmbC**:

1. $\varphi \to \phi$ (premise)
2. φ (premise)
3. ϕ (**MP** 1,2)
4. $\phi \to \exists x\phi$ (Ax12)
5. $\exists x\phi$ (**MP** 3,4)

Then, $\varphi \to \phi, \varphi \vdash \exists x\phi$. By (DMT), $\varphi \to \phi \vdash \varphi \to \exists x\phi$. Therefore, if $\Gamma \vdash \varphi \to \phi$ then $\Gamma \vdash \varphi \to \exists x\phi$, by transitivity of derivations. □

This quick treatment of the syntax is already sufficient for the enterprise of semantics, and for clarifying certain subtleties therein.

3 First-order paraconsistent structures

This section introduces the semantic interpretation for **QmbC**, which consists of the (usual) Tarskian structures endowed with paraconsistent valuations. In the following sections the soundness and completeness of **QmbC** with respect to such interpretations will be proved.

Definition 3.1 (Structures). *Let Σ be a first-order signature (see Definition 1.1). A* structure *over Σ is pair $\mathfrak{A} = \langle A, I_{\mathfrak{A}}\rangle$ such that A is a non-empty set (the domain of the structure) and $I_{\mathfrak{A}}$ is an interpretation mapping assigning, to each individual constant $c \in C$, an element $I_{\mathfrak{A}}(c)$ of A; to each function symbol f of arity n, a function $I_{\mathfrak{A}}(f) : A^n \to A$; and to each predicate symbol P of arity n, a relation $I_{\mathfrak{A}}(P) \subseteq A^n$.*

A structure $\mathfrak{A}$ over Σ defines an interpretation mapping $(\cdot)^{\mathfrak{A}} : CT_\Sigma \to A$ from the set CT_Σ of closed terms (that is, without variables) of Σ to the set A. This mapping is defined recursively as follows:

- $c^{\mathfrak{A}} = I_{\mathfrak{A}}(c)$ if c is an individual constant;
- $f(t_1, \ldots, t_n)^{\mathfrak{A}} = I_{\mathfrak{A}}(f)(t_1^{\mathfrak{A}}, \ldots, t_n^{\mathfrak{A}})$ if f is a function symbol of arity n and $t_1, \ldots, t_n \in CT_\Sigma$.

From now on, the notaton $f^{\mathfrak{A}}$ and $P^{\mathfrak{A}}$ will be used, instead of $I_{\mathfrak{A}}(f)$ and $I_{\mathfrak{A}}(P)$, for a function symbol f and a predicate symbol P, respectively.

In order to interpret the quantifiers in a given structure $\mathfrak{A}$, it is useful to give a formal name to each element of the domain A of $\mathfrak{A}$ by means of new individual constants.

Definition 3.2 (Diagram languages and extended structures)**.** *Consider a structure* $\mathfrak{A}$ *over* Σ*. The* diagram language *of* $\mathfrak{A}$*, denoted by* $L_\Sigma(\mathfrak{A})$*, or simply* $L(\mathfrak{A})$*, is defined over the signature* Σ_A *obtained from* Σ *by adding a new individual constant* $\bar{a}$ *for each element* a *of the domain* A *of* $\mathfrak{A}$*. The notation* $T_\Sigma(\mathfrak{A})$*, or simply* $T(\mathfrak{A})$*, will be used to denote the set of terms of the diagram language of* $\mathfrak{A}$*.*

The structure $\mathfrak{A}$ *can be naturally extended to a structure* $\widehat{\mathfrak{A}} = \langle A, I_{\widehat{\mathfrak{A}}} \rangle$ *over* Σ_A *by defining* $I_{\widehat{\mathfrak{A}}}(\bar{a}) = a$ *for every* $a \in A$*.*

In order to define **QmbC**-valuations able to interpret formulas in a given structure, it will be necessary for technical reasons to deal with some notions introduced in the following definition:

Definition 3.3 (Multiple substitution)**.** *Let* $\mathfrak{A}$ *be a structure for a signature* Σ*, and* $\vec{x} = x_1, \ldots, x_n$ *a sequence of different variables. The set of formulas of* $L(\mathfrak{A})$ *whose free variables occur in the sequence* $\vec{x}$ *is denoted by* $L(\mathfrak{A})_{\vec{x}}$*, and* $\vec{x}$ *is said to be a* context *for the formulas in* $L(\mathfrak{A})_{\vec{x}}$*. The set* $L_{\Sigma\vec{x}}$ *of all the formulas of* L_Σ *with context* $\vec{x}$ *is defined analogously. Given a sequence* $\vec{a} = a_1, \ldots, a_n$ *of elements in* A *and* $\varphi \in L(\mathfrak{A})_{\vec{x}}$*, the notation* $\varphi[\vec{x}/\vec{a}]$ *denotes the sentence of* $S_{L(\mathfrak{A})}$*, obtained from* φ *by substituting simultaneously every free occurrence of variable* x_i *by the constant* $\bar{a}_i$*, for* $1 \leq i \leq n$*. In the same way, if* t *is a term over the signature* Σ_A *whose variables occur in the sequence* $\vec{x}$*, then* $t[\vec{x}/\vec{a}]$ *is the closed term obtained from* t *by substituting simultaneously every occurrence of variable* x_i *by the constant* $\bar{a}_i$*, for* $1 \leq i \leq n$*. The set of all the terms of the signature* Σ_A *of* $L(\mathfrak{A})$ *with context* $\vec{x}$ *will be denotd by* $T(\mathfrak{A})_{\vec{x}}$*.*

Remark 3.4. *Observe that when* $n = 1$*, the notation introduced in Definition 3.3 is different to that introduced in Section 1 concerning substitutions of terms for variables. In fact, according to the latter,* $\varphi[x/\bar{a}]$ *denotes the substitution of constant* $\bar{a}$ *for variable* x*. But, according to Definition 3.3, the same formula can be denoted by* $\varphi[x/a]$ *(when considering* x *as a context). As it will be convenient to identify (informally) an element* b *of* A *with the constant* $\bar{b}$ *of* Σ_A*, this duality is not problematic. (Notice that this duality already appears in the following definition.)*

Definition 3.5 (**QmbC**-valuations)**.** *Let* $\mathfrak{A}$ *be a structure over* Σ *with domain* A*. A mapping* $v : S_{L(\mathfrak{A})} \to \{0, 1\}$ *is a* **QmbC***-valuation over* $\mathfrak{A}$ *if it satisfies the following clauses:*

(vPred) $v(P(t_1, \ldots, t_n)) = 1 \iff \langle\, t_1^{\widehat{\mathfrak{A}}}, \ \ldots\ , t_n^{\widehat{\mathfrak{A}}} \,\rangle \in I_{\mathfrak{A}}(P)$, *for* $P(t_1, \ldots, t_n) \in At_{\Sigma_A}$

(vOr) $v(\alpha \vee \beta) = 1 \iff v(\alpha) = 1$ *or* $v(\beta) = 1$

(vAnd) $v(\alpha \wedge \beta) = 1 \iff v(\alpha) = 1$ *and* $v(\beta) = 1$

(vImp) $v(\alpha \to \beta) = 1 \iff v(\alpha) = 0$ *or* $v(\beta) = 1$

(vNeg) $v(\alpha) = 0 \implies v(\neg\alpha) = 1$

(vCon) $v(\circ\alpha) = 1 \implies v(\alpha) = 0$ *or* $v(\neg\alpha) = 0$

(vVar) $v(\phi) = v(\psi)$ *whenever* ϕ *is a variant of* ψ

(vEx) $v(\exists x\phi) = 1 \iff v(\phi[x/\bar{a}]) = 1$ *for some* $a \in A$

(vUni) $v(\forall x\phi) = 1 \iff v(\phi[x/\bar{a}]) = 1$ *for every* $a \in A$

(sNeg) *For every context* $(\vec{x}; z)$ *and* $(\vec{x}; \vec{y})$*, for every sequence* $(\vec{a}; \vec{b})$ *in A interpreting* $(\vec{x}; \vec{y})$*, for every* $\varphi \in L(\mathfrak{A})_{\vec{x};z}$ *and every* $t \in T(\mathfrak{A})_{\vec{x};\vec{y}}$ *such that t is free for z in* φ*, if* $\varphi[z/t] \in L(\mathfrak{A})_{\vec{x};\vec{y}}$ *and* $b = (t[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^{\widehat{\mathfrak{A}}}$ *then:*

$v((\varphi[z/t])[\vec{x}; \vec{y}/\vec{a}; \vec{b}]) = v(\varphi[\vec{x}; z/\vec{a}; b]) \implies v((\neg\varphi[z/t])[\vec{x}; \vec{y}/\vec{a}; \vec{b}]) = v(\neg\varphi[\vec{x}; z/\vec{a}; b])$

(sCon) *For every context* $(\vec{x}; z)$ *and* $(\vec{x}; \vec{y})$*, for every sequence* $(\vec{a}; \vec{b})$ *in A interpreting* $(\vec{x}; \vec{y})$*, for every* $\varphi \in L(\mathfrak{A})_{\vec{x};z}$ *and every* $t \in T(\mathfrak{A})_{\vec{x};\vec{y}}$ *such that t is free for z in* φ*, if* $\varphi[z/t] \in L(\mathfrak{A})_{\vec{x};\vec{y}}$ *and* $b = (t[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^{\widehat{\mathfrak{A}}}$ *then:*

$v((\varphi[z/t])[\vec{x}; \vec{y}/\vec{a}; \vec{b}]) = v(\varphi[\vec{x}; z/\vec{a}; b]) \implies v((\circ\varphi[z/t])[\vec{x}; \vec{y}/\vec{a}; \vec{b}]) = v(\circ\varphi[\vec{x}; z/\vec{a}; b])$.

Notice that, in particular,

$$v(P(\bar{a}_1, \ldots, \bar{a}_n)) = 1 \iff \langle a_1, \ \ldots \ , a_n \rangle \in P^{\mathfrak{A}}$$

for every n-ary predicate symbol P and every $a_1, \ldots, a_n \in A$. On the other hand,

$$v(\sim\alpha) = 1 \iff v(\alpha) = 0$$

for every formula α.

Definition 3.6 (Interpretations)**.** *An interpretation for* **QmbC** *(over signature* Σ*) is a pair* $\langle\mathfrak{A}, v\rangle$*, such that* $\mathfrak{A}$ *is a structure over* Σ *and* $v : S_{L(\mathfrak{A})} \to \{0, 1\}$ *is a* **QmbC***-valuation.*

As would be expected from the previous definition, the notions of satisfiability (and thus the semantical consequence relation) are defined for sentences of the extended language.

Definition 3.7 (Semantical consequence relation)**.** *An interpretation* $\langle\mathfrak{A}, v\rangle$ *for* **QmbC** *over* Σ *is said to satisfy a sentence* $\varphi \in S_{L(\mathfrak{A})}$*, denoted by* $\mathfrak{A}, v \vDash \varphi$*, if* $v(\varphi) = 1$*. If* $\Gamma \subseteq S_{L(\mathfrak{A})}$*, then* $\mathfrak{A}, v \vDash \Gamma$ *means that* $\mathfrak{A}, v \vDash \gamma$ *for every* $\gamma \in \Gamma$*. Finally, given* $\Gamma \cup \{\varphi\} \subseteq S_{L(\mathfrak{A})}$*, we say that* φ *is a semantical consequence of* Γ*, denoted by* $\Gamma \vDash \varphi$*, if* $\mathfrak{A}, v \vDash \Gamma$ *implies that* $\mathfrak{A}, v \vDash \varphi$*, for every interpretation* $\langle\mathfrak{A}, v\rangle$*.*

Remark 3.8. *Clauses (sNeg) and (sCon) are of a purely technical character, and they establish that if two related formulas in the diagram language involving substitutions* *get the same truth value, then this equality must be preserved through the non-truth-functional connectives ¬ and ∘. For instance, let P be a symbol for a unary predicate and f a symbol for a unary function. Let $\vec{x}$ be the empty context (and so $\vec{a}$ is also empty); $\vec{y} = x$ (a single variable), $\vec{b} = a$ (a single element of A), $\varphi = P(z)$ (where z is a variable) and $t = f(x)$. Let $b = (t[x/a])^{\widehat{\mathfrak{A}}} = f(\bar{a})^{\widehat{\mathfrak{A}}} = f^{\mathfrak{A}}(a)$. Then,*

$$v((P(z)[z/t])[x/a]) = v(P(t)[x/a]) = v(P(f(x))[x/a]) = v(P(f(\bar{a})))$$

while

$$v(P(z)[z/b]) = v(P(\bar{b})) = v\left(P\left(\overline{f^{\mathfrak{A}}(a)}\right)\right).$$

By (vPred), $v(P(f(\bar{a})))$ and $v\left(P\left(\overline{f^{\mathfrak{A}}(a)}\right)\right)$ coincide. However, only clauses (sNeg) and (sCon) can guarantee that $v(\#P(f(\bar{a}))) = v\left(\#P\left(\overline{f^{\mathfrak{A}}(a)}\right)\right)$ for $\# \in \{\neg, \circ\}$, as expected. This feature will be fundamental in order to prove the Substitution Lemma (Theorem 3.13) which, in turn, is crucial in the proof of the soundness of **QmbC** *with respect to interpretations.*

It is important to note that in the absence of (sNeg) and (sCon), it is possible to find interpretations falsifying axioms (Ax12) and (Ax13) (see Remark 4.2).

The semantical notions introduced above can be extended to general formulas, that is, to formulas having free variables, by using some concepts from Definition 3.3.

Definition 3.9 (Extended valuation)**.** *Let $\mathfrak{A}$ be a structure over Σ, $\vec{x}$ a context and $\vec{a}$ a sequence of elements in A interpreting $\vec{x}$. If $v : S_{L(\mathfrak{A})} \to \{0, 1\}$ is a* **QmbC***-valuation over $\mathfrak{A}$, its extension $v^{\vec{a}}_{\vec{x}} : L(\mathfrak{A})_{\vec{x}} \to \{0, 1\}$ is defined as follows: $v^{\vec{a}}_{\vec{x}}(\varphi) = v(\varphi[\vec{x}/\vec{a}])$, for every $\varphi \in L(\mathfrak{A})_{\vec{x}}$.*

Remarks 3.10.

(1) Clearly, if $\varphi \in L(\mathfrak{A})_{\vec{x}}$ and $\vec{y} = (\vec{x}; \vec{z})$ with $\vec{z} = z_1, \dots, z_m$ then $v^{\vec{a};\vec{b}}_{\vec{y}}(\varphi) = v^{\vec{a}}_{\vec{x}}(\varphi)$ for every sequence $\vec{b} = b_1, \dots, b_m$ in A. In particular, $v(\varphi) = v^{\vec{a}}_{\vec{x}}(\varphi)$ for every $\vec{x}$ and $\vec{a}$, whenever $\varphi \in S_{L(\mathfrak{A})}$.

(2) The clauses for **QmbC***-valuations (see Definition 3.5) can be reintroduced in terms of extended valuations. The clauses for connectives and quantifiers are essentially the same: it is enough to carry on the context $\vec{x}$ and the sequence $\vec{a}$ interpreting it. For instance, clause (vUni) changes to*

(vUni)' $v^{\vec{a}}_{\vec{x}}(\forall x\phi) = 1 \iff v^{\vec{a}}_{\vec{x}}(\phi[x/\bar{a}]) = 1$ *for every $a \in A$,*

observing that whether x occurs or not in $\vec{x}$ is irrelevant. In order to see this, note that $v^{\vec{a}}_{\vec{x}}(\forall x\phi) = v((\forall x\phi)[\vec{x}/\vec{a}]) = v((\forall y(\phi[x/y]))[\vec{x}/\vec{a}])$, where y is a variable that does not occur either in $\vec{x}$ or in ϕ, by clause (vVar) and the definition of substitution. But $(\forall y(\phi[x/y]))[\vec{x}/\vec{a}] = \forall y(\phi[x/y][\vec{x}/\vec{a}])$, and so $v^{\vec{a}}_{\vec{x}}(\forall x\phi) = v(\forall y(\phi[x/y][\vec{x}/\vec{a}])) = 1$ iff $v(\phi[x/y][\vec{x}/\vec{a}][y/a]) = v(\phi[x/a][\vec{x}/\vec{a}]) = 1$, for every $a \in A$. But the latter is equivalent to saying that $v^{\vec{a}}_{\vec{x}}(\phi[x/\bar{a}]) = 1$, for every $a \in A$.

On the other hand, the two clauses concerning substitution can be presented in a simplified way. Thus, under the same notation and assumptions as for (sNeg) and *(sCon), the corresponding clauses for extended valuations are the following:*

(*sNeg*)' $v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\varphi[z/t]) = v^{\vec{a};b}_{\vec{x};z}(\varphi) \implies v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\neg\varphi[z/t]) = v^{\vec{a};b}_{\vec{x};z}(\neg\varphi)$

(*sCon*)' $v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\varphi[z/t]) = v^{\vec{a};b}_{\vec{x};z}(\varphi) \implies v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\circ\varphi[z/t]) = v^{\vec{a};b}_{\vec{x};z}(\circ\varphi).$

Definition 3.11 (Extended semantical consequence relation)**.** *An interpretation $\langle\mathfrak{A}, v\rangle$ is said to satisfy a formula $\varphi \in L(\mathfrak{A})_{\vec{x}}$, denoted by $\mathfrak{A}, v \vDash_{\vec{x}} \varphi$, if $v^{\vec{a}}_{\vec{x}}(\varphi) = 1$ for every sequence $\vec{a}$ in A. If $\Gamma \subseteq L(\mathfrak{A})_{\vec{x}}$, then $\mathfrak{A}, v \vDash_{\vec{x}} \Gamma$ means that $\mathfrak{A}, v \vDash_{\vec{x}} \gamma$, for every $\gamma \in \Gamma$. Finally, given $\Gamma \cup \{\varphi\} \subseteq L(\mathfrak{A})_{\vec{x}}$, φ is said to be a semantical consequence of Γ in context $\vec{x}$, denoted by $\Gamma \vDash_{\vec{x}} \varphi$, if $\mathfrak{A}, v \vDash_{\vec{x}} \Gamma$ implies that $\mathfrak{A}, v \vDash_{\vec{x}} \varphi$, for every interpretation $\langle\mathfrak{A}, v\rangle$.*

Remark 3.12. *Observe that when $\Gamma \cup \{\varphi\} \subseteq S_{L(\mathfrak{A})}$, the notions $\vDash_{\vec{x}}$ and $\vDash$ coincide. Moreover,*

$$\Gamma \vDash_{\vec{x}} \varphi \iff (\forall)\Gamma \vDash (\forall)\varphi$$

where $(\forall)\varphi = \forall x_1 \cdots \forall x_n \varphi$ and $(\forall)\Gamma = \{(\forall)\gamma \ : \ \gamma \in \Gamma\}$.

This section concludes with the proof of a technical result which is fundamental in order to state the soundness of **QmbC** with respect to the proposed semantics. In order to lighten notation, and without loss of generality, extended valuations will be used.

Theorem 3.13 (Substitution Lemma)**.** *Let t be a term free for the variable z in the formula φ. Suppose that $(\vec{x}; z)$ and $(\vec{x}; \vec{y})$ are contexts for φ and $\varphi[z/t]$, respectively. Let $\langle\mathfrak{A}, v\rangle$ be an interpretation for* **QmbC**. *If $b = (t[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^{\widehat{\mathfrak{A}}}$ then:*

$$v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\varphi[z/t]) = v^{\vec{a};b}_{\vec{x};z}(\varphi)\ .$$

Proof. The proof is identical with that for classical logic, by induction on the complexity of $\varphi \in L(\mathfrak{A})_{\vec{x};z}$.
(a) $\varphi = P(t_1, \ldots, t_k)$, with P a symbol for predicate and $t_1, \ldots, t_k$ terms in $T(\mathfrak{A})_{\vec{x};z}$. Then, $\varphi[z/t] = P(t_1[z/t], \ldots, t_k[z/t])$. By the definition of extended valuation it follows that

$$v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\varphi[z/t]) = 1 \ \text{ iff } \ \langle\ ((t_1[z/t])[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^{\widehat{\mathfrak{A}}},\ \ldots\ ,((t_k[z/t])[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^{\widehat{\mathfrak{A}}}\ \rangle \in I_{\mathfrak{A}}(P).$$

By induction on the complexity of the term $u \in T(\mathfrak{A})_{\vec{x};z}$, it is easy to prove that

$$((u[z/t])[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^{\widehat{\mathfrak{A}}} = (u[\vec{x}; z/\vec{a}; b])^{\widehat{\mathfrak{A}}}$$

for $b = (t[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^{\widehat{\mathfrak{A}}}$. From this,

$$\langle\ ((t_1[z/t])[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^{\widehat{\mathfrak{A}}},\ \ldots\ ,((t_k[z/t])[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^{\widehat{\mathfrak{A}}}\ \rangle \in I_{\mathfrak{A}}(P)$$

if and only if

$$\langle\ (t_1[\vec{x}; z/\vec{a}; b])^{\widehat{\mathfrak{A}}},\ \ldots\ ,(t_k[\vec{x}; z/\vec{a}; b])^{\widehat{\mathfrak{A}}}\ \rangle \in I_{\mathfrak{A}}(P)\ .$$

As

$$\langle\ (t_1[\vec{x}; z/\vec{a}; b])^{\widehat{\mathfrak{A}}},\ \ldots\ ,(t_k[\vec{x}; z/\vec{a}; b])^{\widehat{\mathfrak{A}}}\ \rangle \in I_{\mathfrak{A}}(P) \ \text{ iff } \ v^{\vec{a};b}_{\vec{x};z}(\varphi) = 1$$

it follows that

$$v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\varphi[z/t]) = 1 \text{ iff } v^{\vec{a};b}_{\vec{x};z}(\varphi) = 1\ .$$

That is, $v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\varphi[z/t]) = v^{\vec{a};b}_{\vec{x};z}(\varphi)$.

(b) $\varphi = (\alpha\#\beta)$, with $\# \in \{\vee, \wedge, \rightarrow\}$. Assuming that α and β satisfy the property (by induction hypothesis), then φ also satisfies the property, as v is truth-functional for these connectives.

(c) $\varphi = \forall x\psi$. If z does not occur free in φ, the result is obviously true. If z occurs free in φ then, as t is free for z in φ, it follows that x does not occur in t. Thus, $\varphi[z/t] = (\forall x\psi)[z/t] = \forall x(\psi[z/t])$. By definition of extended valuation and Remark 3.10(2), if y is a variable that does not occur in either $\vec{x}; \vec{y}; z$ or ψ, $v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\varphi[z/t]) = v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\forall x(\psi[z/t])) = v((\forall x(\psi[z/t]))[\vec{x}; \vec{y}/\vec{a}; \vec{b}]) = v(\forall y(\psi[z/t][x/y][\vec{x}; \vec{y}/\vec{a}; \vec{b}]))$. Then, by ($vUni$) and the equations above,

$$v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\varphi[z/t]) = v(\forall y(\psi[z/t][x/y][\vec{x}; \vec{y}/\vec{a}; \vec{b}])) = 1$$

if and only if

$$v(\psi[z/t][x/y][\vec{x}; \vec{y}/\vec{a}; \vec{b}][y/a]) = 1 \text{ for every } a \in A.$$

But $\psi[z/t][x/y][\vec{x}; \vec{y}/\vec{a}; \vec{b}][y/a] = (\psi[x/y])[z/t][\vec{x}; \vec{y}; y/\vec{a}; \vec{b}; a]$, as individual constants are being substituted for variables, and so the simultaneous substitution coincides with the iterative substitution. Thus, $v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\varphi[z/t]) = 1$ iff $v((\psi[x/y])[z/t][\vec{x}; \vec{y}; y/\vec{a}; \vec{b}; a]) = v^{\vec{a};\vec{b};a}_{\vec{x};\vec{y};y}((\psi[x/y])[z/t]) = 1$, for every $a \in A$.

By the induction hypothesis applied to $\psi[x/y]$, and given that $b = (t[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^{\widehat{\mathfrak{A}}} = (t[\vec{x}; \vec{y}; y/\vec{a}; \vec{b}; a])^{\widehat{\mathfrak{A}}}$ (as y is new),

$$v^{\vec{a};\vec{b};a}_{\vec{x};\vec{y};y}((\psi[x/y])[z/t]) = 1 \text{ iff } v^{\vec{a};b;a}_{\vec{x};z;y}(\psi[x/y]) = 1\ .$$

On the other hand,

$$v^{\vec{a};b;a}_{\vec{x};z;y}(\psi[x/y]) = 1 \text{ for every } a \text{ iff } v^{\vec{a};b}_{\vec{x};z}(\forall y\psi[x/y]) = v^{\vec{a};b}_{\vec{x};z}(\forall x\psi) = v^{\vec{a};b}_{\vec{x};z}(\varphi) = 1$$

and so

$$v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\varphi[z/t]) = 1 \text{ iff } v^{\vec{a};b}_{\vec{x};z}(\varphi) = 1\ .$$

That is, $v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\varphi[z/t]) = v^{\vec{a};b}_{\vec{x};z}(\varphi)$.

(d) $\varphi = \exists x\psi$. This is a consequence of the fact that $v(\exists x\delta) = v(\sim\forall x\sim\delta)$.

(e) $\varphi = \#\psi$, with $\# \in \{\neg, \circ\}$. By induction hypothesis,

$$v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\psi[z/t]) = v^{\vec{a};b}_{\vec{x};z}(\psi)$$

and then, by clauses ($vNeg$)' and ($vCon$)',

$$v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\varphi[z/t]) = v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\#\psi[z/t]) = v^{\vec{a};b}_{\vec{x};z}(\#\psi) = v^{\vec{a};b}_{\vec{x};z}(\varphi)\ .$$

□

The importance of clauses ($vNeg$) and ($vCon$) is clear, therefore, from the proof of the above theorem.

4 Soundness of QmbC

The next step is to prove that the semantics of paraconsistent interpretations is adequate for the logic **QmbC**, presented as a Hilbert calculus. For simplicity, the result will be proved just for sentences (i.e., for formulas without free variables). That is, if $\Delta \cup \{\varphi\}$ is a set of sentences, then

$$\Delta \vdash \varphi \qquad \Longleftrightarrow \qquad \Delta \vDash \varphi \,.$$

It should be observed that, despite the fact that the premises (the set Δ) and the conclusion (the formula φ) are sentences, a given derivation of φ from Δ can involve formulas with free variables, and so the use of extended valuations will be required. On the other hand, there is no loss of generality by proving soundness and completeness just for sentences, by virtue of Remark 3.12 (which obviously also holds in the Hilbert calculus **QmbC**).

In this section, the soundness of **QmbC** is stated:

Theorem 4.1 (Soundness of **QmbC** with respect to interpretations)**.** *For every set of sentences $\Delta \cup \{\varphi\}$: if $\Delta \vdash \varphi$ then $\Delta \vDash \varphi$.*

Proof. By induction on the length n of a derivation $\varphi_1, \ldots, \varphi_n$ of φ from Δ in **QmbC**, it will be proved that given a structure $\mathfrak{A}$, each **QmbC** valuation v over $\mathfrak{A}$ such that $\mathfrak{A}, v \vDash \Delta$, satisfies the following condition: $v^{\vec{a}}_{\vec{x}}(\varphi_i) = 1$ for every sequence $\vec{a}$ in A and every $i \leq i \leq n$, where $\vec{x}$ is a context for every φ_i $(1 \leq i \leq n)$. In particular, it will be proved that $v(\varphi) = 1$, as desired.

It is clear that, in order to get the desired result, it is enough to prove the following:

(i) $v^{\vec{a}}_{\vec{x}}(\psi) = 1$ for every $\vec{a}$ and every instance ψ of an axiom schema of **QmbC**

(ii) if $v^{\vec{a}}_{\vec{x}}(\psi_1) = 1$ and $v^{\vec{a}}_{\vec{x}}(\psi_1 \to \psi_2) = 1$ for every $\vec{a}$ then $v^{\vec{a}}_{\vec{x}}(\psi_2) = 1$ for every $\vec{a}$

(iii) if $v^{\vec{a};b}_{\vec{x};y}(\psi_1 \to \psi_2) = 1$ for every $(\vec{a}; b)$, and if the variable y does not occur free in ψ_1, then $v^{\vec{a}}_{\vec{x}}(\psi_1 \to \forall y \psi_2) = 1$ for every $\vec{a}$

(iv) if $v^{\vec{a};b}_{\vec{x};y}(\psi_1 \to \psi_2) = 1$ for every $(\vec{a}; b)$, and if the variable y does not occur free in ψ_1, then $v^{\vec{a}}_{\vec{x}}(\exists y \psi_1 \to \psi_2) = 1$ for every $\vec{a}$.

In order to prove (i) it is enough to analyze the axioms involving quantifiers; the others are true because of the soundness theorem of **mbC** for bivaluations (see Theorem 1.5). The same holds for item (ii) (concerning **MP**), which is obviously true. Thus consider the following cases for item (i):

(i.1) $\psi = \forall z \alpha \to \alpha[z/t]$ where t is free for z in α. Let $\vec{x}$ be a context formed by all the variables occurring free in $\forall z \alpha$ and let $(\vec{x}; \vec{y})$ be a context formed by the variables occurring free in $\alpha[z/t]$. Consider a sequence $(\vec{a}; \vec{b})$ in A interpreting $(\vec{x}; \vec{y})$. If $v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\forall z \alpha) = 0$ then $v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\psi) = 1$. If, on the other hand, $v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\forall z \alpha) = v^{\vec{a}}_{\vec{x}}(\forall z \alpha) = 1$ then $v^{\vec{a};b}_{\vec{x};z}(\alpha) = 1$ for every $b \in A$ (by the considerations above concerning simultaneous and iterated substitutions). In particular, $v^{\vec{a};b}_{\vec{x};z}(\alpha) = 1$ for $b = (t[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^{\widehat{\mathfrak{A}}}$. By the Substitution Lemma

(Theorem 3.13), $v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\alpha[z/t]) = v^{\vec{a};b}_{\vec{x};z}(\alpha)$, as t is free for z in α. From this $v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\alpha[z/t]) = 1$, as required.

(i.2) $\psi = \alpha[z/t] \to \exists z\alpha$ where t is free for z in α. The proof is analogous to that of item (i.1).

(i.3) $\psi = \forall x(\alpha \to \beta) \to (\alpha \to \forall x\beta)$, where x does not occur free in α. Suppose that $v^{\vec{a}}_{\vec{x}}(\forall x(\alpha \to \beta)) = v^{\vec{a}}_{\vec{x}}(\alpha) = 1$. If $a \in A$ then $v^{\vec{a}}_{\vec{x}}((\alpha \to \beta)[x/\bar{a}]) = 1$ and $v^{\vec{a}}_{\vec{x}}(\alpha[x/\bar{a}]) = 1$, since x does not occur free in α. From this, $v^{\vec{a}}_{\vec{x}}(\beta[x/\bar{a}]) = 1$ and so $v^{\vec{a}}_{\vec{x}}(\forall x\beta) = 1$. This shows that $v^{\vec{a}}_{\vec{x}}(\psi) = 1$ for every v and $\vec{a}$.

(i.4) $\psi = \alpha \to \beta$, where α is a variant of β. This is an obvious consequence of clause ($vVar$) and the fact that $\alpha[\vec{x}/\vec{a}]$ is a variant of $\beta[\vec{x}/\vec{a}]$ whenever α is a variant of β.

Now, in order to prove (iii), suppose that $v^{\vec{a};b}_{\vec{x};y}(\psi_1 \to \psi_2) = 1$ for every $(\vec{a}; b)$, where the variable y does not occur free in ψ_1. Fix the sequence $\vec{a}$. If $v^{\vec{a}}_{\vec{x}}(\psi_1) = 0$ then $v^{\vec{a}}_{\vec{x}}(\psi_1 \to \forall y\psi_2) = 1$. On the other hand, if $v^{\vec{a}}_{\vec{x}}(\psi_1) = v^{\vec{a};b}_{\vec{x};y}(\psi_1) = 1$ then, by hypothesis, $v^{\vec{a};b}_{\vec{x};y}(\psi_2) = v^{\vec{a}}_{\vec{x}}(\psi_2[y/\bar{b}]) = 1$, for every $b \in A$. From this, $v^{\vec{a}}_{\vec{x}}(\forall y\psi_2) = 1$.

Item (iv) is proved in a similar way. □

Remark 4.2. *As observed in Remark 3.8, clauses (sNeg) and (sCon) are crucial in order to prove the soundness theorem above.*

Consider, for instance, $\alpha = \neg P(z)$ and $t = f(x, y)$, with P a symbol denoting a unary predicate. Suppose that $v^{a;b}_{x;y}(\forall z\alpha) = v(\forall z\neg P(z)) = 1$. Then,

$$v(\neg P(\bar{e})) = 1 \textit{ for every } e \in A. \qquad (1)$$

In particular,

$$v\left(\neg P\left(\overline{f^{\mathfrak{A}}(a, b)}\right)\right) = 1\,. \qquad (2)$$

On the other hand,

$$v^{a;b}_{x;y}(\alpha[z/t]) = v^{a;b}_{x;y}(\neg P(f(x, y))) = v(\neg P(f(\bar{a}, \bar{b}))). \qquad (3)$$

In order to guarantee that $v(\neg P(f(\bar{a}, \bar{b}))) = 1$, one has to ensure that

$$v\left(\neg P\left(\overline{f^{\mathfrak{A}}(a, b)}\right)\right) = v(\neg P(f(\bar{a}, \bar{b}))). \qquad (*)$$

But the latter is only obtained from the Substitution Lemma or, in this specific case, by clause (sNeg). In other words, without (sNeg) it would be possible to find a valuation v over a structure $\mathfrak{A}$ such that $v^{a;b}_{x;y}(\forall z\neg P(z)) = 1$ but $v^{a;b}_{x;y}(\neg P(f(x, y))) = 0$. That is, it would be possible to falsify the instance

$$\forall z\, \neg P(z) \to \neg P(f(x, y))$$

of axiom schema (Ax13). By a similar argument, it would be possible to falsify the instance

$$\forall z \circ P(z) \to \circ P(f(x, y))$$

of axiom schema (Ax13) without the presence of clause (sCon).

5 Completeness of QmbC

Given a first-order signature Σ, any set of sentences in L_Σ will be called a *theory*.

This section is dedicated to prove the completeness of **QmbC** with respect to interpretations. The proof will be analogous to that for classical logic: given a theory Γ which does not deduce a given sentence φ (being, therefore, non-trivial), a canonical interpretation will be constructed which satisfies Γ but does not satisfy φ. Therefore, it will be proved that:

$$\Gamma \nvdash \varphi \qquad \Longrightarrow \qquad \Gamma \nvDash \varphi\,.$$

In order to do this, the original theory Γ will be conservatively extended to a Henkin theory Δ in an extended signature, that is, to a theory containing a *witness* for each existential sentence. Since Δ is a conservative extension of Γ, it does not derive φ. Thus, by using a classical and general result by Lindenbaum-Łos, Δ will be extended to a maximal theory $\bar{\Delta}$ which does not derive φ and is still a Henkin theory. Using a canonical structure generated from $\bar{\Delta}$, the characteristic map of $\bar{\Delta}$ will constitute a **QmbC**-valuation which, as required, satisfies Γ but does not satisfy φ.

5.1 Henkin theories

A Henkin theory is a theory designed to comply with the inference rules for quantifiers.

Definition 5.1 (Henkin theory)**.** *Given a theory $\Delta \subseteq S_L$ and a non-empty set C of constants of the signature Σ of L, Δ is called a C-*Henkin theory *in* **QmbC** *if it satisfies the following: for every sentence of the form $\exists x\phi$ in S_L, there exists a constant c in C such that if $\Delta \vdash \exists x\phi$ then $\Delta \vdash \phi[x/c]$.*

The set C is called a set of *witnesses* of Δ. The next step is to prove that any theory can be conservatively extended to a C-Henkin theory, for some C.

Theorem 5.2 (Theorem of Constants)**.** *Let $\Delta \subseteq S_L$ be a theory in* **QmbC** *over a signature Σ, and let $\vdash_C$ be the consequence relation of* **QmbC** *over the signature Σ_C, which is obtained from Σ by adding a set C of new individual constants. Then, for every $\varphi \in S_L$,*

$$\Delta \vdash \varphi \;\text{ iff }\; \Delta \vdash_C \varphi.$$

That is, **QmbC** *(over Σ_C) is a conservative extension of* **QmbC** *(over Σ).*

Proof. The proof is analogous to that for classical first-order logic: given a derivation π of φ from Γ in **QmbC** over Σ_C, the constants of C occurring in π are replaced uniformly by new variables, obtaining a finite sequence π' of formulas over Σ. But the instances over Σ_C of axioms of **QmbC** occurring in π become instances over Σ of axioms of **QmbC**, and the same holds for the instances of inference rules. Then, π' is in fact a derivation of φ from Γ in **QmbC** over Σ. The converse is obvious. □

Theorem 5.3. *Every theory $\Delta \subseteq S_L$ in* **QmbC** *over a signature Σ can be conservatively extended to a C-Henkin theory Δ^H in* **QmbC** *over a signature Σ_C, as in Theorem 5.2. That is, $\Delta \subseteq \Delta^H$ and if $\varphi \in S_L$ then $\Delta \vdash \varphi$ iff $\Delta^H \vdash_C \varphi$. Additionally, any extension of Δ^H by sentences in the signature Σ_C is also a C-Henkin theory.*

Proof. Let us define an increasing denumerable sequence of signatures $\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots$ such that each Σ_{n+1} is obtained from Σ_n by adding new individual constants. The language L_{Σ_n} generated by Σ_n will be denoted by L_n, and so $L_n \subseteq L_{n+1}$.

The definition of the signatures is as follows:

(i) $\Sigma_0 = \Sigma$; then, $L_0 = L_{\Sigma_0} = L_\Sigma$.

(ii) Σ_1 is obtained from Σ_0 by adding the set of new individual constants

$$C_1 = \{\, c_{\exists x\alpha} \ : \ \exists x\alpha \text{ is a sentence of } L_0 \,\};$$

(iii) For $n \geq 1$, $\Sigma_{n+1}$is obtained from Σ_n by adding the set of new individual constants

$$C_{n+1} = \{\, c_{\exists x\alpha} \ : \ \exists x\alpha \text{ is a sentence of } L_n \setminus L_{n-1} \,\}\,.$$

Let $C = \bigcup_{n\geq 1} C_n$ be the set of new individual constants, let $\Sigma_C = \bigcup_{n\geq 0} \Sigma_n$ be the signature obtained by adding the new constants, and let $L_C = L_{\Sigma_C}$.

Consider now the following sequence of sets of non-logical axioms over Σ_C:

$$\begin{aligned} AX_0 &= \emptyset \\ AX_{n+1} &= \{\exists x\phi \to \phi[x/c_{\exists x\phi}] \ : \ \exists x\phi \in S_{L_n}\} \quad (\text{for } n \geq 0). \end{aligned}$$

Finally, let $\Delta^H = \Delta \cup \bigcup_{n\geq 1} AX_n$. Observe that $\Delta^H \subseteq S_{L_C}$, and that it extends Δ. It will be proved now that Δ^H is a conservative extension of Δ. Thus, let $\phi \in S_L$ such that $\Delta^H \vdash_C \phi$, and let π be a derivation of ϕ from Δ^H in **QmbC** over Σ_C. As π is finite, there exists a finite set $\Delta^H_0 \subseteq \Delta^H$ such that $\Delta, \Delta^H_0 \vdash_C \phi$. Let $\exists x\psi \to \psi[x/c_{\exists x\psi}]$ in Δ^H_0, and let $\Delta^H_1 = \Delta^H_0 \setminus \{\exists x\psi \to \psi[x/c_{\exists x\psi}]\}$. Given that Δ^H_0 is a set of sentences, (DMT) can be applied in order to obtain $\Delta, \Delta^H_1 \vdash_C (\exists x\psi \to \psi[x/c_{\exists x\psi}]) \to \phi$. Observe that the constant $c_{\exists x\psi}$ only appears in the conclusion and so, by using the same technique employed in the proof of Theorem 5.2, that constant can be substituted by a new variable, namely y. This means that $\Delta, \Delta^H_1 \vdash_C (\exists x\psi \to \psi[x/y]) \to \phi$. By Lemma 2.9(3), $\Delta, \Delta^H_1 \vdash_C (\exists x\psi \to \exists y(\psi[x/y])) \to \phi$. On the other hand, $\vdash_C \exists x\, \psi \to \exists y(\psi[x/y])$, by axiom (Ax15). From this, $\Delta, \Delta^H_1 \vdash_C \phi$.

By repeating this process, every element of Δ^H_1 can be eliminated in a finite number of steps, proving that $\Delta \vdash_C \phi$. By Theorem 5.2, it is finally obtained that $\Delta \vdash \phi$. This shows that Δ^H is in fact a conservative extension of Δ.

To finish the proof, consider an extension $\Delta^{H'}$ of Δ^H (in particular, one can chosse $\Delta^{H'} = \Delta^H$), formed by sentences of L_C. Suppose that for some sentence $\exists x\varphi \in L_C$, the following condition holds: $\Delta^{H'} \vdash_C \exists x\varphi$. As $\Delta^{H'}$ extends Δ^H, then $\Delta^{H'} \vdash_C \exists x\varphi \to \varphi[x/c_{\exists x\varphi}]$ and so $\Delta^{H'} \vdash_C \varphi[x/c_{\exists x\varphi}]$. This means that $\Delta^{H'}$ (and, in particular, Δ^H) is a C-Henkin theory in **QmbC** over Σ_C. □

5.2 Maximal extensions: the Lindenbaum-Łos theorem

The next step towards the proof of completeness requires the notion of maximal theories with respect to a sentence. In particular, a classical and very useful result due to Lindenbaum and Łos will be crucial. Some well-known concepts, essential ingredients for the proofs, are here recalled.

Definition 5.4 (Tarskian Logic[2]). *Let $\mathcal{L}$ be a logic defined over a language L and with a consequence relation $\vdash$. Then $\mathcal{L}$ is said to be* Tarskian *if it satisfies the following, for every $\Gamma \cup \Delta \cup \{\alpha\} \subseteq L$:*

(1) if $\alpha \in \Gamma$ then $\Gamma \vdash \alpha$;

(2) if $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash \alpha$;

(3) if $\Delta \vdash \alpha$ and $\Gamma \vdash \beta$ for every $\beta \in \Delta$ then $\Gamma \vdash \alpha$.

A Tarskian logic is finitary *if it satisfies, additionally, the following:*

(4) if $\Gamma \vdash \alpha$ then there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \vdash \alpha$.

■

Definition 5.5. *Let $\mathcal{L}$ be a Tarskian logic over the language L, and let $\Gamma \cup \{\varphi\} \subseteq L$. The set Γ is said to be* maximally non-trivial with relation to φ *in $\mathcal{L}$ if $\Gamma \nvdash \varphi$ but $\Gamma, \psi \vdash \varphi$ for any $\psi \in L \setminus \Gamma$.*

■

A proof of the following classical result can be found in [27], Theorem 22.2.

Theorem 5.6 (Lindenbaum-Łos). *Let $\mathcal{L}$ be a Tarskian and finitary logic over the language L. Let $\Gamma \cup \{\varphi\} \subseteq L$ such that $\Gamma \nvdash \varphi$. Then there exists a set Δ, such that $\Gamma \subseteq \Delta \subseteq L$ with Δ maximally non-trivial with relation to φ in $\mathcal{L}$.*

Clearly, any logic defined by means of a Hilbert calculus where the inference rules are finitary is Tarskian and finitary, and so the theorem above holds for it. In particular, it holds for **QmbC** *restricted to sentences*: it is easy to see that the consequence relation of **QmbC**, when restricted to sentences, is Tarskian and finitary. Then, if the set L of Definition 5.5 is also restricted to sentences (that is, to S_L), the following holds:

Corollary 5.7. *Let $\Gamma \cup \{\varphi\} \subseteq S_L$ be a set of sentences such that $\Gamma \nvdash \varphi$ in* **QmbC**. *Then, there exists a set of sentences $\Delta \subseteq S_L$ extending Γ which is maximally non-trivial with relation to φ in* **QmbC** *(by restricting $\vdash$ to sentences).*

Theorem 5.8 (Canonical interpretation). *Let $\Delta \subseteq S_L$ be a set of sentences over a signature Σ containing at least one individual constant. Assume that Δ is a C-Henkin theory in* **QmbC** *for a non-empty set C of individual constants of Σ, and that Δ is also maximally non-trivial with relation to φ in* **QmbC**, *for some sentence φ. Then, Δ induces a canonical structure $\mathfrak{A}$ and a canonical* **QmbC**-*valuation $v : S_{L(\mathfrak{A})} \to \{0, 1\}$ over $\mathfrak{A}$ such that, for every sentence $\psi \in S_L$:*

$$\mathfrak{A}, v \vDash \psi \qquad \Longleftrightarrow \qquad \Delta \vdash \psi\,.$$

Proof. Let $A = CT_\Sigma$ be the set of closed terms (that is, without variables) over the signature Σ. Then a structure $\mathfrak{A} = \langle A, I_\mathfrak{A} \rangle$ over Σ can be defined as follows: $I_\mathfrak{A}(c) = c$, if c is an individual constant; if f is a function symbol, then $I_\mathfrak{A}(f) : A^n \to A$ is such

[2]See, for instance, [27].

that $I_{\mathfrak{A}}(f)(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$ (and so $t^{\mathfrak{A}} = t$ for every $t \in CT_\Sigma$). Finally, define the interpretation of the predicate symbols as follows:

$$\langle t_1, \ldots, t_n \rangle \in I_{\mathfrak{A}}(P) \qquad \Longleftrightarrow \qquad \Delta \vdash P(t_1, \ldots, t_n) \, .$$

Recall from Definition 3.2 the diagram language $L(\mathfrak{A})$ of $\mathfrak{A}$, its set $T(\mathfrak{A})$ of terms and the extended structure $\widehat{\mathfrak{A}}$ such that $I_{\widehat{\mathfrak{A}}}(\bar{t}) = t$. Here, $\bar{t}$ is a new constant introduced for each closed term $t \in CT_\Sigma$ (which is as an element of the domain of $\mathfrak{D}$). Let $CT(\mathfrak{A})$ be the set of closed terms of the language $L(\mathfrak{A})$, and consider a mapping $* : CT(\mathfrak{A}) \to CT_\Sigma$ defined recursively as follows: $(\bar{t})^* = t$ if $t \in CT_\Sigma$; $c^* = c$ if c is an individual constant of Σ; and $(f(t_1, \ldots, t_n))^* = f(t_1^*, \ldots, t_n^*)$ if f is a function symbol of Σ. It is clear that $t^* = t^{\widehat{\mathfrak{A}}}$ for every $t \in CT(\mathfrak{A})$. This mapping can be naturally extended to sentences: let $* : S_{L(\mathfrak{A})} \to S_L$ be defined recursively as follows: $(P(t_1, \ldots, t_n))^* = P(t_1^*, \ldots, t_n^*)$ if $P(t_1, \ldots, t_n)$ is atomic; $(\#\psi)^* = \#(\psi^*)$ if $\# \in \{\neg, \circ\}$; $(\varphi\#\psi)^* = (\varphi^*\#\psi^*)$ if $\# \in \{\wedge, \vee, \to\}$; and $(Qx\psi)^* = Qx(\psi^*)$ if $Q \in \{\forall, \exists\}$. Clearly, φ^* is the sentence of L_Σ obtained from φ by substituting every occurrence of a constant $\bar{t}$ by the term t itself.

Finally, the mapping $v : S_{L(\mathfrak{A})} \to \{0, 1\}$ can be defined as follows:

$$v(\varphi) = 1 \qquad \Longleftrightarrow \qquad \Delta \vdash \varphi^* \, .$$

By construction of v, it is clear that for every sentence $\varphi \in S_L$:

$$\mathfrak{A}, v \vDash \varphi \qquad \Longleftrightarrow \qquad \Delta \vdash \varphi \, .$$

The proof will be completed by showing that v is in fact a **QmbC**-valuation (recall Definition 3.5).

In order to prove (*vPred*), if $P(t_1, \ldots, t_n)$ is an atomic sentence of $L(\mathfrak{A})$, then

$$v(P(t_1, \ldots, t_n)) = 1 \qquad \Longleftrightarrow \qquad \Delta \vdash P(t_1^*, \ldots, t_n^*) \, .$$

But this happens iff $\langle t_1^*, \ldots, t_n^* \rangle \in I_{\mathfrak{A}}(P)$, by definition of $I_{\mathfrak{A}}(P)$. Given that $t^* = t^{\widehat{\mathfrak{A}}}$ for every $t \in CT(\mathfrak{A})$, then

$$v(P(t_1, \ldots, t_n)) = 1 \qquad \Longleftrightarrow \qquad \langle t_1^{\widehat{\mathfrak{A}}}, \ldots, t_n^{\widehat{\mathfrak{A}}} \rangle \in I_{\mathfrak{A}}(P) \, .$$

In order to see that v satisfies clauses (*vOr*), (*vAnd*), (*vImp*), (*vNeg*) and (*vCon*), the reader can consult [7], where the corresponding proof is provided for **mbC**, and so it holds *mutatis mutandis* for **QmbC** (by using the definition of $*$).

The satisfaction of clause (*vVar*) follows from axiom (Ax15) and the definition of $*$.

In order to prove (*vEx*), firstly observe that, if $\exists x\phi \in S_L$ then $\Delta \vdash \exists x\phi$ implies that $\Delta \vdash \phi[x/c]$ for some constant c of C (which is an element of CT_Σ), as Δ is a C-Henkin theory in **QmbC**. On the other hand, if $\Delta \vdash \phi[x/t]$ for some closed term t in CT_Σ then $\Delta \vdash \exists x\phi$, in virtue of (Ax12) and **MP**. Consider now a sentence in $L(\mathfrak{A})$ of the form $\exists x\phi$. Then $v(\exists x\phi) = 1$ iff $\Delta \vdash (\exists x\phi)^*$ iff $\Delta \vdash_{\mathfrak{L}} \exists x(\phi)^*$, by definition of $*$. From this, and by the observation above, one infers that $v(\exists x\phi) = 1$ iff $\Delta \vdash ((\phi)^*)[x/t]$ for some closed term t in CT_Σ. On the other hand, it is easy to prove by induction on the complexity of

ϕ that $((\phi)^*)[x/t] = (\phi[x/\bar{t}])^*$, for every $t \in CT_\Sigma$. Thus, $v(\exists x\phi) = 1$ iff $\Delta \vdash (\phi[x/\bar{t}])^*$ for some t of CT_Σ. From this it follows that $v(\exists x\phi) = 1$ iff $v(\phi[x/\bar{t}]) = 1$ for some element t of CT_Σ.

Concerning (*vUni*), as v satisfies the clauses for the propositional connectives then $v({\sim}\varphi) = 1$ iff $v(\varphi) = 0$. On the other hand, $v(\forall x\varphi) = v({\sim}\exists x{\sim}\varphi)$, because of the theorems $\vdash \forall x\varphi \to {\sim}\exists x{\sim}\varphi$ and $\vdash {\sim}\exists x{\sim}\varphi \to \forall x\varphi$ of **QmbC** (and by the Soundness Theorem). From this, and using clause (*vEx*), it can be immediately seen that v satisfies clause (*vUni*).

Finally, it will be proved that the pair $\langle \mathfrak{A}, v\rangle$ satisfies the Substitution Lemma (see Theorem 3.13) and so the mapping v satisfies the clauses (*sNeg*) and (*sCon*).

Facts: Let t be a term free for a variable z in a formula φ. Suppose that $(\vec{x}; z)$ and $(\vec{x}; \vec{y})$ are contexts for φ and $\varphi[z/t]$, respectively, and let $b = (t[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^*$. Then:
(i) $((u[z/t])[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^* = (u[\vec{x}; z/\vec{a}; b])^*$, for every term $u \in T(\mathfrak{A})_{\vec{x};z}$.
(ii) $((\varphi[z/t])[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^* = (\varphi[\vec{x}; z/\vec{a}; b])^*$.

Item (i) can be easily proved by induction on the complexity of u. (Notice that this fact was already used in item (a) of the proof of Theorem 3.13, given that $\widehat{u^{\mathfrak{A}}} = u^*$ for every term u).

Item (ii) is proved by induction on the complexity of φ. If φ is atomic, the result follows immediately by item (i). The propagation of the induction hypothesis through the connectives $\wedge$, $\vee$, $\to$, $\neg$ and $\circ$ is obvious. The propagation of the induction hypothesis through the quantifiers is a consequence of the fact that t is free for z in φ. Therefore, x does not occur in t when $\varphi = Qx\psi$, with $Q \in \{\forall, \exists\}$. From this, $((Qx\psi)[z/t])[\vec{x}; \vec{y}/\vec{a}; \vec{b}] = Qx((\psi[z/t])[\vec{x}; \vec{y}/\vec{a}; \vec{b}])$, and the result follows by induction hypothesis and the definition of $*$. This concludes the proof of the **Facts**.

Now, let φ be a formula of $L(\mathfrak{A})$, and t a term free for the variable z in φ such that $(\vec{x}; z)$ and $(\vec{x}; \vec{y})$ are contexts for φ and $\varphi[z/t]$, respectively. Then

$$v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\varphi[z/t]) = v((\varphi[z/t])[\vec{x}; \vec{y}/\vec{a}; \vec{b}])$$

and so

$$v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\varphi[z/t]) = 1 \qquad \Longleftrightarrow \qquad \Delta \vdash ((\varphi[z/t])[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^* \;.$$

On the other hand, $v^{\vec{a};b}_{\vec{x};z}(\varphi) = v(\varphi[\vec{x}; z/\vec{a}; b])$ therefore

$$v^{\vec{a};b}_{\vec{x};z}(\varphi) = 1 \qquad \Longleftrightarrow \qquad \Delta \vdash (\varphi[\vec{x}; z/\vec{a}; b])^* \;.$$

Finally, by taking $b = (t[\vec{x}; \vec{y}/\vec{a}; \vec{b}])^*$, it follows by **Facts**(ii) that

$$v^{\vec{a};\vec{b}}_{\vec{x};\vec{y}}(\varphi[z/t]) = v^{\vec{a};b}_{\vec{x};z}(\varphi)$$

as desired. From this, the mapping v satisfies clauses (*sNeg*) and (*sCon*).

This proves that the pair $\langle \mathfrak{A}, v\rangle$ is an interpretation with the required properties. □

Theorem 5.9 (Completeness of **QmbC** with respect to interpretations)**.** *For every set of sentences $\Delta \cup \{\varphi\}$ over a signature Σ, if $\Delta \vDash \varphi$ then $\Delta \vdash \varphi$.*

Proof. Suppose that $\Delta \cup \{\varphi\} \subseteq S_L$ such that $\Delta \nvDash \varphi$. By Theorem 5.3, there exists a C-Henkin theory Δ^H defined over a signature Σ_C which conservatively extends Δ, that is: for every sentence $\psi \in S_L$, $\Delta \vdash \psi$ iff $\Delta^H \vdash_C \psi$, recalling that $\vdash_C$ denotes the consequence relation of the Hilbert calculus **QmbC** over signature Σ_C. By Corollary 5.7, there exists a set of sentences $\overline{\Delta^H}$ over the signature Σ_C which extends Δ and is maximally non-trivial with relation to φ in **QmbC** (as a calculus defined over Σ_C). By Theorem 5.3, $\overline{\Delta^H}$ is also a C-Henkin theory. By Theorem 5.8, a canonical interpretation $\langle \mathfrak{A}, v\rangle$ over Σ_C can be defined such that, for every sentence ψ over Σ_C,

$$\mathfrak{A}, v \vDash \psi \qquad \Longleftrightarrow \qquad \overline{\Delta^H} \vdash_C \psi\ .$$

In particular, $\mathfrak{A}, v \vDash \Delta$ (as $\Delta \subseteq \overline{\Delta^H}$) and $\mathfrak{A}, v \nvDash \varphi$ (as $\overline{\Delta^H} \nvdash_C \varphi$). Finally, let $\overline{\mathfrak{A}}$ be the reduct of $\mathfrak{A}$ to the signature Σ.[3] That is, $I_{\overline{\mathfrak{A}}}$ coincides with $I_{\mathfrak{A}}$ over Σ (and so just 'forgets' the interpretation of the individual constants in C). Let $\bar{v}$ be the restriction of v to the set of sentences $S_{L(\overline{\mathfrak{A}})}$ of the diagram language of $\overline{\mathfrak{A}}$. Clearly, $\overline{\mathfrak{A}}, \bar{v} \vDash \psi$ iff $\mathfrak{A}, v \vDash \psi$, for every sentence $\psi \in S_{L(\overline{\mathfrak{A}})}$. Therefore, $\langle \overline{\mathfrak{A}}, \bar{v}\rangle$ is an interpretation for **QmbC** over Σ such that $\overline{\mathfrak{A}}, \bar{v} \vDash \Delta$ but $\overline{\mathfrak{A}}, \bar{v} \nvDash \varphi$. This shows that $\Delta \nvDash \varphi$, as required. □

6 Compactness and Lowenhëim-Skolem Theorems

This section is devoted to establishing some fundamental theorems of Model Theory for **QmbC**, namely: Compactness and the Lowenhëim-Skolem Theorems. Here are some basic definitions and results to make the arguments more clear and self contained:

Definition 6.1. *Let Σ be a first-order signature for **LFI**s. Consider the sets*

$$C = \{c \ :\ c \ \textit{ is an individual constant of } \Sigma\}$$

$$\mathcal{F} = \{f \ :\ f \ \textit{ is a function symbol of arity } n \textit{ of } \Sigma, \textit{ for some } n \geq 1\}$$

$$\mathcal{P} = \{P \ :\ P \ \textit{ is a predicate symbol of arity } n \textit{ of } \Sigma, \textit{ for some } n \geq 1\}.$$

The cardinal *of Σ, denoted by $||\Sigma||$, is the cardinal of the set*

$$\omega \cup C \cup \mathcal{F} \cup \mathcal{P}$$

where ω denotes the set of natural numbers.

Definition 6.2. *Let $\mathfrak{A} = \langle A, I_{\mathfrak{A}}\rangle$ be a structure. The* cardinal *of $\mathfrak{A}$, denoted by $||\mathfrak{A}||$, is the cardinal of the set A. Given an interpretation $\langle \mathfrak{A}, v\rangle$, its* cardinal *is, by definition, the cardinal of the structure $\mathfrak{A}$.*

Definition 6.3. *Let Γ be a set of sentences over a signature Σ. We say that Γ is* satisfiable *if there exists an interpretation $\langle \mathfrak{A}, v\rangle$ such that $\mathfrak{A}, v \vDash \Gamma$ (recall Definition 3.7). Otherwise, Γ is said to be* unsatisfiable*. If $\mathfrak{A}, v \vDash \Gamma$ then $\langle \mathfrak{A}, v\rangle$ is called a* model *of Γ.*

[3] As usual, if Σ is a subsignature of another signature Σ', then any structure $\mathfrak{A}$ over Σ' can be considered as a structure over Σ, by 'forgetting' the interpretation of the symbols in Σ' that do not belong to Σ. Such a structure over Σ is called the *reduct* of $\mathfrak{A}$ to Σ (see [10]).

By recalling that a strong negation $\sim$ can be defined in **QmbC**, it is immediate to prove the following:

Proposition 6.4. *Let* $\Gamma \cup \{\varphi\}$ *be a set of sentences over a signature* Σ*. Then,* $\Gamma \vDash \varphi$ *if and only if* $\Gamma \cup \{\sim\varphi\}$ *is unsatisfiable.*

Proof. The proof is entirely analogous to that for classical logic, taking into account the definition of the consequence relation $\vDash$ (see Definition 3.7) and the fact that $\mathfrak{A}, v \vDash \varphi$ if and only if $\mathfrak{A}, v \nvDash \sim\varphi$. □

Profiting from the previous result, the following can easily be proved:

Proposition 6.5. *Let* Σ *be a signature. Then, the following statements are equivalent in* **QmbC***:*

(i) For every set of sentences Γ *over* Σ *the following holds: every finite subset of* Γ *is satisfiable if and only if* Γ *is satisfiable;*

(ii) For every set of sentences $\Gamma \cup \{\varphi\}$ *over* Σ *the following holds:* $\Gamma \vDash \varphi$ *if and only if* $\Gamma_0 \vDash \varphi$ *for some finite subset* Γ_0 *of* Γ*.*

Proof. $(i) \Rightarrow (ii)$: Assuming (i), suppose that $\Gamma \vDash \varphi$. Then $\Gamma \cup \{\sim\varphi\}$ is unsatisfiable, by Proposition 6.4. By (i), there exists a finite subset Δ_0 of $\Gamma \cup \{\sim\varphi\}$ such that Δ_0 is unsatisfiable. Let $\Gamma_0 = \Delta_0 \setminus \{\sim\varphi\}$. Then, $\Gamma_0 \cup \{\sim\varphi\}$ is also unsatisfiable (since it contains Δ_0). Using again Proposition 6.4, it follows that $\Gamma_0 \vDash \varphi$, where Γ_0 is a finite subset of Γ. The converse is obvious.

$(ii) \Rightarrow (i)$: Assuming (i), suppose that Γ is a set of sentences over Σ which is unsatisfiable. Since clearly Γ is non-empty, there exists some sentence, say φ, belonging to Γ. Given that $\mathfrak{A}, v \vDash \varphi$ if and only if $\mathfrak{A}, v \vDash \sim\sim\varphi$, for every interpretation $\langle \mathfrak{A}, v \rangle$, it follows that $\Gamma \cup \{\sim\sim\varphi\}$ is unsatisfiable and so $\Gamma \vDash \sim\varphi$, by Proposition 6.4. Using (ii) one infers that $\Gamma_0 \vDash \sim\varphi$ for some finite subset Γ_0 of Γ. Being so, and using Proposition 6.4 again, it follows that $\Gamma_0 \cup \{\sim\sim\varphi\}$ is unsatisfiable. But then $\Gamma_0 \cup \{\varphi\}$ is unsatisfiable, by the observation above, where $\Gamma_0 \cup \{\varphi\}$ is a finite subset of Γ. The converse is immediate. □

Because of the last result, the compactness of **QmbC** can be stated as follows:

Definition 6.6. *The logic* **QmbC** *is (semantically) compact if it satisfies statement* (i) *(or, equivalently, statement* (ii)*) of Proposition 6.5.*

Theorem 6.7 (Compactness of **QmbC**)**.** *The logic* **QmbC** *is (semantically) compact.*

Proof. It will be proved that **QmbC** satisfies statement (ii) of Proposition 6.5. Suppose that $\Gamma \cup \{\varphi\}$ is a set of sentences such that $\Gamma \vDash \varphi$. By the Completeness Theorem 5.9 it follows that $\Gamma \vdash \varphi$. Hence, since the syntactical consequence relation $\vdash$ is finitary, there is some finite subset Γ_0 of Γ such that $\Gamma_0 \vdash \varphi$ and so $\Gamma_0 \vDash \varphi$, by the Soundness Theorem 4.1. The converse is immediate and so **QmbC** satisfies statement (ii) of Proposition 6.5, being therefore compact. □

Definition 6.8. *Given a theory* Γ *over* Σ*, that is, a set of sentences in* L_Σ*,* Γ *is said to be* non-trivial *if* $\Gamma \nvDash \varphi$ *for some sentence* φ *over* Σ*.*

Theorem 6.9 (Downward Lowenhëim-Skolem Theorem for **QmbC**). *Let Σ be a signature. Every non-trivial theory Γ over Σ has a model of cardinal equal to $||\Sigma||$.*

Proof. Suppose that Γ is a non-trivial theory over Σ. Then, there is some sentence φ over Σ such that $\Gamma \nvDash \varphi$. By the proof of the Completeness Theorem 5.9, there exists an interpretation $\langle\overline{\mathfrak{A}}, \bar{v}\rangle$ for **QmbC** over Σ which is a model for Γ, such that the domain A of the structure $\overline{\mathfrak{A}}$ is the set of closed terms over the signature Σ_C. It is routine to prove that the cardinal of A is $||\Sigma||$. □

Lemma 6.10. *Let Σ be a signature, and let Γ be a non-trivial theory over Σ. If Γ has a model of cardinal κ, then it has a model of cardinal κ', for every cardinal κ' greater or equal than κ.*

Proof. Let $\langle\mathfrak{A}, v\rangle$ be a model of Γ of cardinal κ such that $\mathfrak{A} = \langle A, I_{\mathfrak{A}}\rangle$. Let κ' be a cardinal strictly greater than κ, and let A' be a set of cardinality κ' such that $A \subset A'$. Fix an element a of A, and define a structure $\mathfrak{A}' = \langle A', I_{\mathfrak{A}'}\rangle$ over Σ as follows. If c is an individual constant of Σ then $I_{\mathfrak{A}'}(c) = I_{\mathfrak{A}}(c)$. If P is a predicate symbol of Σ of arity n and $\langle a'_1, \ldots, a'_n\rangle \in (A')^n$ then $\langle a'_1, \ldots, a'_n\rangle \in I_{\mathfrak{A}'}(P)$ if and only if $\langle a_1, \ldots, a_n\rangle \in I_{\mathfrak{A}}(P)$, where a_i is a'_i, if $a'_i \in A$, or a_i is a otherwise, for every $1 \leq i \leq n$. If f is a function symbol of Σ of arity n and $\langle a'_1, \ldots, a'_n\rangle \in (A')^n$ then $I_{\mathfrak{A}'}(f)(a'_1, \ldots, a'_n) = I_{\mathfrak{A}}(f)(a_1, \ldots, a_n)$, where each a_i is defined as above. Finally, consider a **QmbC**-valuation v' over $\mathfrak{A}'$ which extends v, such that $v'(\varphi[\vec{x}/\vec{a}']) = v(\varphi[\vec{x}/\vec{a}])$ for every φ with context $\vec{x}$, every $\vec{a}' = a'_1, \ldots, a'_n$ and every $\vec{a} = a_1, \ldots, a_n$ such that each a_i is defined from a'_i as above (it is easy to see that it is always possible to define such a valuation from v). Then, $\langle\mathfrak{A}', v'\rangle$ is a model of Γ of cardinal κ'. □

Theorem 6.11 (Upward Lowenhëim-Skolem Theorem for **QmbC**). *Let Σ be a signature. Every non-trivial theory Γ over Σ has a model of cardinal κ, for every cardinal κ greater or equal than $||\Sigma||$.*

Proof. It is a direct consequence of Theorem 6.9 and Lemma 6.10. □

7 QmbC with equality

In order to develop higher-level applications of the quantified version **QmbC** of **mbC**, such as paraconsistent model theory or paraconsistent set theory, it is necessary to consider a binary predicate $\approx$ for the equality relation satisfying the usual axioms, which should be interpreted as the identity relation. As such, the predicate $\approx$ will be considered as a logical symbol (like the connectives and the quantifiers), not belonging to the signatures. By writing, as usual, $(t \approx t')$ instead of $\approx(t, t')$ (where t and t' are terms of the language), the following definitions are in order:

Definition 7.1. *Let Σ be a first-order signature for **LFI**s (recall Definition 1.1). The set of formulas with equality $\approx$ over Σ, denoted by $L^{\approx}_{\Sigma}$, is defined as usual, but now expressions of the form $(t \approx t')$ (where t and t' are terms of the language) are also atomic formulas. In other words, a new symbol $\approx$ for a binary predicate denoting equality is added to the set of given logical symbols (connectives, quantifiers and punctuation* *marks). The set of sentences of $L^{\approx}_{\Sigma}$ will be denoted by $S^{\approx}_{\Sigma}$. On the other hand, the diagram language of $\mathfrak{A}$ and the corresponding sets of sentences, when including the equality symbol $\approx$, will be denoted by $L^{\approx}(\mathfrak{A})$ and $S_{L^{\approx}(\mathfrak{A})}$ respectively.*

If α is a formula and y is a variable free for the variable x in α, then $\alpha[x \wr y]$ denotes any formula obtained from α by replacing some, but not necessarily all, free occurrences of x by y.

Definition 7.2. *Let Σ be a first-order signature for **LFI**s. The logic* $\mathbf{QmbC}_{\approx}$ *(over Σ) is the extension of* **QmbC** *over $L^{\approx}_{\Sigma}$ obtained by adding to* **QmbC***, besides all the new instances of axioms and inference rules involving the equality predicate $\approx$, the following axiom schemas:*

(AxEq1) $\forall x(x \approx x)$

(AxEq2) $(x \approx y) \to (\alpha \to \alpha[x \wr y])$*, if y is a variable free for x in α*

It is worth noting that axiom (AxEq2) depends on each α and each specific $\alpha[x \wr y]$.

Definition 7.3. *The semantics for* $\mathbf{QmbC}_{\approx}$ *is given by interpretations $\langle \mathfrak{A}, v\rangle$ (recall Definition 3.6) such that the* **QmbC***-valuation $v : S_{L^{\approx}(\mathfrak{A})} \to \{0,1\}$ satisfies, additionally, the following clauses:*

(*vEq*1) $v(t_1 \approx t_2) = 1 \iff \widehat{t_1}^{\mathfrak{A}} = \widehat{t_2}^{\mathfrak{A}}$*, for every $t_1, t_2 \in CT(\mathfrak{A})$ (the set of closed terms of the language $L(\mathfrak{A})$)*

(*vEq*2) $v(\bar{a} \approx \bar{b}) = 1 \implies v(\alpha[x, y/\bar{a}, \bar{b}]) = v((\alpha[x \wr y])[x, y/\bar{a}, \bar{b}])$ *for every $a, b \in A$, if y is a variable free for x in α.*

Since $v(\bar{a} \approx \bar{a}) = 1$ for every $a \in A$, by (*vEq*1), then $v(\forall x(x \approx x)) = 1$, by (*vUni*). However, it is possible to have $v(\neg(t \approx t)) = 1$, that is, $\mathfrak{A}, v \vDash \neg(t \approx t)$, for some interpretation $\langle \mathfrak{A}, v\rangle$ and some term t. In other words, it is not required that $v(\circ(t \approx t)) = 1$ is always the case.

From the clauses (*vEq*1) and (*vEq*2), it is clear that the Substitution Lemma (Theorem 3.13) can be extended to $\mathbf{QmbC}_{\approx}$, as it clearly holds for atomic formulas of the form $(t \approx t')$ (and the proof is done by induction on the complexity of formulas).

Remark 7.4. *At first sight, it would seem that the clause for valuations corresponding to (AxEq1) should be simply $v(\forall x(x \approx x)) = 1$ or, equivalently, $v(\bar{a} \approx \bar{a}) = 1$ for every $a \in A$. However, in order to ensure the validity of the Substitution Lemma, the stronger condition (vEq1) must be required. In fact, recall from Remark 4.2 that the validity of the Substitution Lemma is necessary to guarantee the soundness of (Ax13). Consider again the terms $t_1 = \overline{f^{\mathfrak{A}}(a, b)}$ and $t_2 = f(\bar{a}, \bar{b})$ of Remark 4.2. If one simply requires for the* $\mathbf{QmbC}_{\approx}$*-valuations the condition $v(\bar{a} \approx \bar{a}) = 1$ for every $a \in A$, there is no guarantee that $v(t_1 \approx t_2) = 1$ despite $\widehat{t_1}^{\mathfrak{A}} = \widehat{t_2}^{\mathfrak{A}}$. This situation would violate the Substitution Lemma, and consequently also the Soundness theorem of* $\mathbf{QmbC}_{\approx}$ *with respect to interpretations, as observed above.*

Now, it is easy to extend the previous results in order to prove the following soundness and completeness theorem for $\mathbf{QmbC}_{\approx}$. Thus, by denoting by $\vdash^{\approx}$ the relation consequence of the Hilbert calculus $\mathbf{QmbC}_{\approx}$ and by $\vDash^{\approx}$ the semantical consequence relation with respect to interpretations (see Definition 7.3), the following holds:

Theorem 7.5 (Soundness and Completeness of $\mathbf{QmbC}_{\approx}$ with respect to interpretations)**.** *For every set of sentences $\Delta \cup \{\varphi\} \subseteq S^{\approx}_{\Sigma}$ in a language with equality over a signature Σ:*

$$\Delta \vdash^{\approx} \varphi \quad \Longleftrightarrow \quad \Delta \vDash^{\approx} \varphi\,.$$

Proof. Soundness can be easily established from the considerations above. For completeness, the proof of Theorem 5.9 is adapted as follows: by assuming that $\Delta \cup \{\varphi\} \subseteq S^{\approx}_{L}$ is a set of sentences with equality over a signature Σ such that $\Delta \nvdash^{\approx} \varphi$, let $\overline{\Delta^H}$ be a set of sentences with equality over the signature Σ_C which extends Δ and is maximal non-trivial with relation to φ in **QmbC** (as a calculus defined over Σ_C) and is also a C-Henkin theory. A canonical interpretation $\langle \mathfrak{A}, v\rangle$ over Σ_C will be defined now such that, for every sentence ψ over Σ_C,

$$\mathfrak{A}, v \vDash \psi \qquad \Longleftrightarrow \qquad \overline{\Delta^H} \vdash^{\approx}_{C} \psi\,.$$

Define, thus, the following relation in the set C of constants : $c \simeq d$ iff $\overline{\Delta^H} \vdash^{\approx}_{C} (c \approx d)$. Then $\simeq$ is an equivalence relation. Let $\widetilde{c} = \{d \in C \ : \ c \simeq d\}$ for $c \in C$, and let $A = \{\widetilde{c} \ : \ c \in C\}$. The structure $\mathfrak{A}$ over Σ_C with domain A is defined as follows:

(i) if c is an individual constant in Σ_C then $I_{\mathfrak{A}}(c) = \widetilde{d}$, where $d \in C$ is such that $\overline{\Delta^H} \vdash^{\approx}_{C} (c \approx d)$;

(ii) if f is a function symbol, then $I_{\mathfrak{A}}(f) : A^n \to A$ is such that $I_{\mathfrak{A}}(f)(\widetilde{c}_1, \ldots, \widetilde{c}_n) = \widetilde{c}$ where $c \in C$ is such that $\overline{\Delta^H} \vdash^{\approx}_{C} (f(c_1, \ldots, c_n) \approx c)$;

(iii) if P is a predicate symbol, then

$$\langle \widetilde{c}_1, \ldots, \widetilde{c}_n\rangle \in I_{\mathfrak{A}}(P) \qquad \Longleftrightarrow \qquad \overline{\Delta^H} \vdash^{\approx}_{C} P(c_1, \ldots, c_n)\,.$$

The proof that $I_{\mathfrak{A}}$ is well-defined is similar to that for classical logic (see, for instance, [10]).

Now, let $v : S_{L^{\approx}(\mathfrak{A})} \to \{0, 1\}$ be the mapping defined as follows:

$$v(\psi) = 1 \qquad \Longleftrightarrow \qquad \overline{\Delta^H} \vdash^{\approx}_{C} \psi^*$$

where ψ^* is the sentence over Σ_C obtained from ψ by substituting every occurrence of a constant $\bar{\widetilde{c}}$ by the constant $\widetilde{c}$. Thus, for every sentence ψ over Σ_C,

$$\mathfrak{A}, v \vDash \psi \qquad \Longleftrightarrow \qquad \overline{\Delta^H} \vdash^{\approx}_{C} \psi\,.$$

Then, it is proved that v is a $\mathbf{QmbC}_{\approx}$-valuation over $\mathfrak{A}$. Finally, the reduct $\langle \overline{\mathfrak{A}}, \bar{v}\rangle$ of $\langle \mathfrak{A}, v\rangle$ to Σ is an interpretation for $\mathbf{QmbC}_{\approx}$ over Σ such that $\overline{\mathfrak{A}}, \bar{v} \vDash \Delta$ but $\overline{\mathfrak{A}}, \bar{v} \nvDash \varphi$, showing that $\Delta \nvDash^{\approx} \varphi$. □

Finally, it is easy to adapt the definitions and results of the previous section to the logic $\mathbf{QmbC}_{\approx}$. Thus, the logic $\mathbf{QmbC}_{\approx}$ is compact (by using a notion of compactness similar to that of Definition 6.6), and the two versions of the Lowenhëim-Skolem Theorem hold for $\mathbf{QmbC}_{\approx}$. From this, it is easy to prove the following:

Proposition 7.6. *If a theory* Γ *of* $\mathbf{QmbC}_{\approx}$ *has arbitrarily large finite models, then it has an infinite model.*

Proof. Given a theory Γ of $\mathbf{QmbC}_{\approx}$ over a signature Σ with arbitrarily large finite models, consider a denumerable set $C = \{c_n \ : \ n \geq 0\}$ of new individual constants. Let Σ_C be the signature obtained from Σ by adding the set C of individual constants and let Δ be the following theory over Σ_C:

$$\Delta = \Gamma \cup \{{\sim}(c_n \approx c_m) \ : \ n < m\}.$$

Under the given hypothesis over Γ, it is easy to prove that every finite subset of Δ is satisfiable. By the Compactness Theorem for $\mathbf{QmbC}_{\approx}$, the theory Δ has a model $\langle \mathfrak{A}, v\rangle$ and so the domain A of $\mathfrak{A}$ must be infinite. Let $\langle \mathfrak{A}', v'\rangle$ such that $\mathfrak{A}'$ is the reduct of $\mathfrak{A}$ to Σ and v' is the corresponding restriction of v to $S_{L(\mathfrak{A}')}$. Since $\Gamma \subseteq S_L$ then $\langle \mathfrak{A}', v'\rangle$ is a model of Γ which is infinite. □

8 First-order characterization of other quantified LFIs

In the previous sections **QmbC**, the first-order extension of **mbC**, which constitutes the simplest propositional **LFI** analyzed in [7], has been carefully studied. There exist several propositional extensions of **mbC** proposed and studied in [8] and [7], to which the concepts and techniques employed in the previous sections could be readily applied in order to obtain the corresponding first-order versions. Some extensions of **mbC** will be briefly mentioned below.

(i) The logic **mCi** is the extension of **mbC** obtained by adding the following axiom schemas:

(ci) $\neg\circ\varphi \to (\varphi \wedge \neg\varphi)$

(cc$_n$) $\circ\neg^n\circ\varphi$ (for $n \geq 0$)

(ii) The logic **Ci** is obtained from **mCi** by adding the axiom

(cf) $\neg\neg\varphi \to \varphi$

or, equivalently, by adding to **mbC** the axioms (ci) and (cf).

(iii) The system **Cil** is obtained from **Ci** by adding the axiom

(cl) $\neg(\varphi \wedge \neg\varphi) \to \circ\varphi$

A basic feature of da Costa's C-systems (see [12]) is that 'well-behavior' (that is, consistency) is propagated from simpler to more complex formulas. This motivates the following:

(iv) The logic **Cia** is obtained by the addition of the following axiom schemas to **Ci**:

(ca1) $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \wedge \beta)$

(ca2) $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \vee \beta)$

(ca3) $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \to \beta)$

(v) The logic **Cila** is obtained by the addition of the axiom schema (cl) to **Cia** or, equivalently, of the axioms (ca1)–(ca3) to **Cil**.[4]

Observe that all the extensions of **mbC** presented above consist in the addition of some axiom schemas. On the other hand, the corresponding clauses for the associated bivaluations can be straightforwardly associated to such axioms. From this, it is possible to extend the soundness and completeness theorem of **mbC** to all these propositional systems (see [7]). Being so, the first-order version of each of the **LFI**s introduced above is straightforwardly defined by adding to **QmbC** the corresponding axiom schemas from the list above. Thus, for instance, **QCi** is obtained from **QmbC** by adding axiom schemas (ci) and (cf). Of course, the valuations of the interpretation structures must satisfy the additional clauses for bivaluations required in each case. Thus, a **QCi**-valuation over a structure $\mathfrak{A}$ is a **QmbC** valuation (recall Definition 3.5) satisfying, additionally, the following clauses:

(vCon)' $v(\neg\circ\alpha) = 1 \quad \Longrightarrow \quad v(\alpha) = 1$ and $v(\neg\alpha) = 1$

(vNeg)' $v(\neg\neg\alpha) = 1 \quad \Longrightarrow \quad v(\alpha) = 1.$

From this perspective, the proof of soundness and completeness theorems of **QmbC** stated above can be easily extended to the new quantified **LFI**s. Clearly, all of them can also be equipped with an equality predicate $\approx$, as it was done for **QmbC** in Section 7. The details of these constructions are left to the diligent reader .

9 Related work

There exist several proposals in the literature concerning the development of first-order **LFI**'s. In his famous monograph [12] (see also [13]), da Costa introduced the first-order version C_n^* of each calculus C_n (recall that, as observed in [7], the calculi C_n are special cases of **LFI**'s). The first-order axioms and rules are, as in our case, the classical

[4] It is worth noting that the only difference between **Cila** and C_1 is that the consistency connective $\circ$ was not taken as primitive in C_1, but as an abbreviation given by the formula $\neg(\alpha \wedge \neg\alpha)$. It can be proven that C_1 is equivalent to **Cila** up to translations (see [7]).

ones, with just one difference: da Costa required the propagation of consistency for the quantifiers (in this way, generalizing the propagation of consistency for conjunction and disjunction, by taking into account that the universal and the existencial quantifiers can be seen as arbitrary conjunctions and disjunctions, respectively). The extension of these systems to include the (standard) equality predicate was also considered by da Costa, obtaining the hierarchy of calculi called $C_n^=$. Subsequently, E. Alves obtained several basic results of model theory for such calculi (see [1]).

The semantics for the calculi C_n^* and $C_n^=$ is, as the one proposed in the present paper, 2-valued. Correspondingly, the usual Tarskian first-order structures are equipped with paraconsistent bivaluations. In her Phd thesis [15] and in a series of papers ([16, 17, 18]), I. D'Ottaviano developed the basic model theory of the first-order version of the well-known 3-valued paraconsistent logic J_3, introduced in [19]. It is worth noting that J_3 was reintroduced in [9] as an **LFI** (in a different signature contaning a consistency connective) called **LFI**1 (see also [7]). The semantics proposed by D'Ottaviano is given again by usual Tarskian first-order structures, now equiped with a 3-valued paraconsistent valuation. A generalization of the quasi-truth theory introduced in [22] has been more recently proposed (cf. [11]). The proposed system is a 3-valued first-order paraconsistent **LFI** whose semantics is given by Tarskian first-order structures in which the predicate symbols of arity n are interpreted as triples of pairwise disjoint subsets of D^n (where D is the domain of the Tarskian structure) whose union is D^n. It is proved in [11] that the proposed logic coincides with D'Ottaviano's first-order version of J_3, and so it is a first-order version of **LFI**1, which also coincides with the first-order version of **LFI**1 called **LFI**1* studied in [9]. As a consequence, the logic **LFI**1* coincides with the first-order version of J_3 proposed by D'Ottaviano. By its turn, these systems are conservative extensions of G. Priest's first-order version of the logic of paradox *LP* proposed in [24] (see [11]).

In an independent research line, A. Avron and I. Lev introduced in [2, 3] a generalization of the concept of matrix semantics called non-deterministic matrices, or Nmatrices. In a series of papers, Avron and his collaborators introduced Nmatrices for several logic systems, including all the **LFI**'s studied in [8, 7], as well as for new **LFI**'s proposed by them. In [4] a semantics based on Nmatrices for several first-order **LFI**'s is proposed. The first-order axioms and rules added to the propositional systems, guaranteeing a careful treatment of the substitution lemma, coincide with the ones used in the present paper (despite the fact that they do not include axiom (Ax14)). Semantically, they consider Tarskian first-order structures enriched with valuations over suitable Nmatrices. Since the semantics of the considered Nmatrices coincides with the corresponding bivaluation semantics, both proposals are equivalent. The convenience of enriching the Tarskian structures with bivaluations or with valuations over Nmatrices is a matter of discussion. Moreover, the truth-values as well as the operations of the Nmatrices are obtained from an analysis of the bivaluations, as it was shown in [6], and so both approaches are conceptually very close. It can be argued that the use of bivaluations allows us to consider a model theory closer to the classical one: the structures considered here define a conservative extension of the logic associated to the usual ones, by adding two (non-truth-functional) new connectives, namely the paraconsistent negation $\neg$ and the consistency operator $\circ$. Being so, our treatment of first-order **LFI**'s extends the original proposal of da Costa and Alves to several **LFI**'s, on the one hand, and it is more 'palatable' to the classically-oriented logicians, on the other.

10 Some methodological considerations

This paper introduces first-order extension of several **LFI**'s in such a way that the semantical structures are as close as possible to the classical ones. Some essential results on model theory, showing the validity of the compactness and of the Lowenhëim-Skolem theorems, were also obtained. It was also shown how the underlying first-order language of such systems can be extended with a standard equality predicate, which affords richer theories. For instance, in [5] several paraconsistent versions of ZF set-theory based on different **LFI**'s were investigated. The corresponding first-order logic with equality for each system is based on the results obtained of the present paper.

There is still a good deal of work to be done by employing the methods developed here, or extensions thereof, to investigate, for instance, the validity of other "grand theorems" of model theory, as put in [20]. For example, the Lyndon interpolation theorem, the omitting types theorem and the initial model theorem, not to mention elimination of quantifiers. It may happen that theorems of this type would be of less interest for paraconsistent model theory, and that other, currently unheard of, properties will emerge.

We believe, however, that there is a modest, but solid lesson to be learned from our approach, of interest to the philosophy and methodology of logic: regarded in terms of their methods and mathematical properties, the borders between the so-called classical and non-classical logics are too vague to grant an absolute distinction between classicality and non-classicality when referring to logic.

Indeed, as this paper has attested, with a bit of generalization some well-established constructions in logic can be suited to meet the requirements of more expressive logics which, as in our case, are genuine enlargements of the logic space, and not any exotic concoction. And if the distinction from the mathematical perspective is as faint as it appears, much more is needed on the philosophical side to maintain the demarcation between classical and non-classical logics, at least in some cases. From this perspective, the logics we have treated here, and for whose model theory we have started an investigation, are perfectly classical.

Acknowledgments: We thank the anonymous referees for several suggestions that helped to improve the final version of the paper. This research was financed by FAPESP (Thematic Project LogCons 2010/51038-0, Brazil) and by individual research grants from The National Council for Scientific and Technological Development (CNPq), Brazil.

References

[1] Elias Alves. Paraconsistent logic and model theory. *Studia Logica*, 43(1/2):17–32, 1984.

[2] Arnon Avron and Iddo Lev. Canonical propositional Gentzen-type systems. In R. Gore, A. Leitsch, and T. Nipkow, editors, *Proceedings of the 1st International Joint Conference on Automated Reasoning (IJCAR 2001)*, volume 2083 of *LNAI*, pages 529–544. Springer Verlag, 2001.

[3] Arnon Avron and Iddo Lev. Non-deterministic multi-valued structures. *Journal of Logic and Computation*, 15:241–261, 2005.

[4] Arnon Avron and Anna Zamansky. Many-valued non-deterministic semantics for first-order Logics of Formal (In)consistency. In S. Aguzzoli, A. Ciabattoni, B. Gerla, C. Manara, and V. Marra, editors, *Algebraic and Proof-theoretic Aspects of Non-classical Logics. Papers in Honor of Daniele Mundici on the Occasion of His 60th Birthday*, volume 4460 of *Lecture Notes in Computer Science*, pages 1–24. Springer, 2007.

[5] Walter Carnielli and Marcelo E. Coniglio. Paraconsistent set theory by predicating on consistency. *Journal of Logic and Computation*, 2013. First published online: July 9, 2013. DOI: 10.1093/logcom/ext020.

[6] Walter Carnielli and Marcelo E. Coniglio. Swap structures for **LFI**'s. *CLE e-Prints*, 14(1), 2014. Available at
`http://www.cle.unicamp.br/e-prints/vol_14,n_1,2014.html`.

[7] Walter Carnielli, Marcelo E. Coniglio, and João Marcos. Logics of Formal Inconsistency. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic (2nd. edition)*, volume 14, pages 1–93. Springer, 2007.

[8] Walter A. Carnielli and João Marcos. A taxonomy of C-systems. In W. A. Carnielli, M. E. Coniglio, and I. M. L. D' Ottaviano, editors, *Paraconsistency: The Logical Way to the Inconsistent. Proceedings of the 2nd World Congress on Paraconsistency (WCP 2000)*, volume 228 of *Lecture Notes in Pure and Applied Mathematics*, pages 1–94, New York, 2002. Marcel Dekker.

[9] Walter A. Carnielli, João Marcos, and Sandra de Amo. Formal inconsistency and evolutionary databases. *Logic and Logical Philosophy*, 8:115–152, 2000.

[10] Chen C. Chang and H. Jerome Keisler. *Model Theory*. Elsevier, Third edition, 1990.

[11] Marcelo E. Coniglio and Luiz H. Silvestrini. An alternative approach for quasi-truth. *Logic Journal of the IGPL*, 2013. First published online: August 6, 2013. DOI: 10.1093/ljigpal/jzt026.

[12] Newton C. A da Costa. Sistemas formais inconsistentes (Inconsistent formal systems, in Portuguese). Habilitation thesis, Universidade Federal do Paraná, Curitiba, Brazil, 1963. Republished by Editora UFPR, Curitiba, Brazil,1993.

[13] Newton C. A da Costa. On the theory of inconsistent formal systems (Lecture delivered at the First Latin-American Colloquium on Mathematical Logic, held at Santiago, Chile, July 1970). *Notre Dame Journal of Formal Logic*, 15(4):497–510, 1974.

[14] Newton C. A da Costa and Elias Alves. A semantical analysis of the calculi C_n. *Notre Dame Journal of Formal Logic*, 18(4):621–630, 1977.

[15] Itala M. L. D'Ottaviano. *Sobre uma Teoria de Modelos Trivalente (On a three-valued model theory, in Portuguese)*. PhD thesis, IMECC, State University of Campinas, Brazil, 1982.

[16] Itala M. L. D'Ottaviano. The completeness and compactness of a three-valued first order logic. *Revista Colombiana de Matemáticas*, XIX(1-2):31–42, 1985.

[17] Itala M. L. D'Ottaviano. The model extension theorems for J3-theories. In C. A. Di Prisco, editor, *Methods in Mathematical Logic. Proceedings of the 6th Latin American Symposium on Mathematical Logic held in Caracas, Venezuela, Aug. 1-6, 1983*, volume 1130 of *Lecture Notes in Mathematics*, pages 157–173, Berlin, 1985. Springer-Verlag.

[18] Itala M. L. D'Ottaviano. Definability and quantifier elimination for J3-theories. *Studia Logica*, 46(1):37–54, 1987.

[19] Itala M. L. D'Ottaviano and Newton C. A. da Costa. Sur un problème de Jaśkowski. *Comptes Rendus de l'Académie de Sciences de Paris (A-B)*, 270:1349–1353, 1970.

[20] Wilfrid Hodges and Thomas Scanlon. First-order model theory. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Fall 2013 edition, 2013.

[21] Elliot Mendelson. *Introduction to Mathematical Logic*. Chapman & Hall, New York, Third edition, 1987.

[22] Irene Mikenberg, Newton C. A. da Costa, and Rolando Chuaqui. Pragmatic truth and approximation to truth. *The Journal of Symbolic Logic*, 51(1):201–221, 1986.

[23] Rodrigo Podiacki. Lógicas da inconsistência formal quantificadas (Quantified logics of formal inconsistency, in Portuguese). Masters thesis, IFCH-State University of Campinas, 2008. In Portuguese.

[24] G. Priest. *In Contradiction: a study of the transconsistent*. Oxford: Oxford University Press, 2^{nd} edition, 2006.

[25] Tarcísio G. Rodrigues. Sobre os fundamentos da programação lógica paraconsistente (On the foundations of paraconsistent logic programming, in Portuguese). Masters thesis, IFCH-State University of Campinas, 2010. In Portuguese.

[26] Joseph R. Shoenfield. *Mathematical Logic*. Addison-Wesley, Reading, MA, 1967.

[27] Ryszard Wójcicki. *Lectures on propositional calculi*. Ossolineum, Wroclaw, Poland, 1984. Available at
`http://www.ifispan.waw.pl/studialogica/wojcicki/papers.html`.