Recovering a logic from its fragments by meta-fibring

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Abstract

In this paper we address the question of recovering a logic system by combining two or more fragments of it. We show that, in general, by fibring two or more fragments of a given logic the resulting logic is weaker than the original one, because some meta-properties of the connectives are lost after the combination process. In order to overcome this problem, the categories **Mcon** and **Seq** of multiple-conclusion consequence relations and sequent calculi, respectively, are introduced. The main feature of these categories is the preservation, by morphisms, of meta-properties of the consequence relations, which allows, in several cases, to recover a logic by fibring of its fragments. The fibring in this categories is called *meta-fibring*. Several examples of well-known logics which can be recovered by meta-fibring its fragments (in opposition to fibring in the usual categories) are given. Finally, a general semantics for objects in **Seq** (and, in particular, for objects in **Mcon**) is proposed, obtaining a category of logic systems called **Log**. A general theorem of preservation of completeness by fibring in **Log** is also obtained.

1 Introduction: Collapse vs. Anti-Collapse

The development of relevant techniques for combining logic systems stirred up interest of many logicians in recent years. Among the different methods for combining logics, fibring has been revealed as a very valuable tool for combining logic systems, and it was successfully applied in different contexts (see, for instance, [14, 16, 18, 22, 10, 6]). One of the most outstanding features of fibring is the obtainment, under certain conditions, of preservation of meta-properties of the given logic systems through fibring; in particular, the preservation of completeness by fibring (when possible) is considered as one of the main achievements of the method.

A natural question when combining logics is the following: is it possible to recover a logic system by combining two or more fragments of it? For instance, is classical propositional logic over $\neg, \Rightarrow$ the result of fibring the logic of classical negation with the logic of classical implication? In this paper we show that, in general, the answer is ‘no’, unless a stronger notion of morphism (which preserves meta-properties) is adopted.

Surprisingly enough, the question of recovering a logic by fibring its fragments is related to a well-known problem associated to fibring: the so-called *collapsing problem*, which was identified in [12]. The collapsing problem can be stated as follows: if classical propositional logic and intuitionistic propositional logic are combined by unconstrained fibring (that is, no logic symbol is shared) then the result collapses to classical logic (or, equivalently, to two disjoint copies of classical logic). Specifically, both implications coincide, and then intuitionistic implication became classic. This collapse happens exclusively at the semantical level (in the category of Hilbert calculi such a collapse does not happen, cf. [5]). Another examples of collapse are given in [19], as well as a solution to this problem by means of a controlled notion of fibring called *modulated fibring*. Another (and apparently simpler) solution to the collapsing problem is found in [5], using a relaxed fibring technique called *cryptofibring*.

A related form of the collapsing problem was also observed in [2] (see also [3]) where it is shown that, if we join up the usual sequent rules for (classical) conjunction with the rules for (classical) disjunction, the resulting sequent calculus will prove the distributivity between conjunction and disjunction (see Example 5.8 below). The same result holds if we join up the (two-valued) valuation clauses for (classical) conjunction with the valuation clauses for (classical) disjunction. This situation is arguably undesirable, because the arising of new interaction rules between the given connectives contradicts the basic *desideratum* of fibring: “Given logic systems L_1 and L_2, then the combination $L_1 * L_2$ is the smallest logic system for the combined language which is a conservative extension of both L_1 and L_2”.[1]

The collapsing problem could be summarized as follows:

> *Logics obtained by fibring prove too many things in the new combined language, and they must be weakened.*

But this is just one perspective.

An opposite view of this problem is the following: suppose that we want to obtain a logic system from the combination of its basic components (that is, two fragments of it). Returning to the last example, suppose that we want to recover the logic of classical conjunction and classical disjunction from its basic logical component (that its, the rules for conjunction, on the one side, and the rules for disjunction, on the other). In this case, the fibring of the corresponding consequence systems cannot recover the intended distributivity between both connectives, as it was proved in Theorem 3 of [3] (see also Example 5.8 below).

Consider now another example in the same vein: take the rules for classical negation, on the one side, and the rules for classical disjunction. Suppose that, based on the intuition given by the classical matrices for these connectives, as well as by the usual sequent rules, we intend to recover classical logic from these ingredients. Using the usual notion of fibring, the resulting logic will be a weaker system, in which $\varphi \vee \neg\varphi$ is not valid (see Example 5.5 below). The

[1] Adapted from [16].

phenomenon observed in this example as well as in the last example is what we call the *anti-collapsing problem* of fibring: the impossibility of obtaining, in the logics obtained by fibring, intended interaction rules which are justified, for instance, by well-known models or sequent rules.

One more example: by fibring the logic system just containing the rules for classical implication with the logic system just containing the rules for classical negation, the formula $\varphi \Rightarrow (\neg\varphi \Rightarrow \psi)$ does not hold in the resulting logic system (see Example 5.7 below). Moreover, the deduction meta-theorem is no longer valid in the resulting logic (because, for instance, ψ is derivable from $\{\varphi, \neg\varphi\}$). That is, certain (positive) meta-properties of the given logics are missing by fibring.[2] This is another example of anti-collapse.

In contrast to the collapsing problem, the anti-collapsing problem could be summarized as follows:

> *Logics obtained by fibring sometimes prove too few things in the new combined language, and they must be strengthened.*

This is the perspective we adopt in this paper.

Note that if, in order to avoid the collapse, the prescription about combination of logics mentioned above (that is, to define the minimum conservative extension of the given logics) is taken seriously into account, then there exists the risk to obtain a too weak logic system, in which the only valid inferences are the original ones, and any other inference involving formulas in the new combined language is not valid. This means that, in the limit case, the consequence relation $\vdash$ of the combined logic $L = L_1 * L_2$ could be simply the union of $\vdash_1$ and $\vdash_2$, where $\vdash_1$ and $\vdash_2$ are the consequence relations of L_1 and L_2, respectively. Such a logic could hardly be considered as being the *combination* of L_1 and L_2 in the combined language of L_1 and L_2, if we try to recover a logic from the combination of its fragments. Under the perspective of recovering logics by combinations, new theorems (in the new combined language), resulting from interactions, are to be expected in the resulting logic. In other words, some natural interactions between the rules or axioms defining the connectives of the given logics are to be expected in the combined logic. Any process of *combination* presupposes some kind of *interaction* between the factors, and not the mere adjunction of them (chemical reactions constitute a good analogy here).

The anti-collapsing problem arises mainly because the only attribute of the logics that is preserved by fibring is the consequence relation: namely, from the validity of $\Gamma \vdash_{L_1} \varphi$ in the given logic L_1 then $\Gamma \vdash_L \varphi$ must hold in the logic L obtained by fibring. It could be said that $\Gamma \vdash_{L_1} \varphi$ is a meta-property of L_1: a basic one. But, for instance, a meta-property such as the deduction meta-theorem

$$\Gamma, \varphi \vdash_{L_1} \psi \quad \texttt{iff} \quad \Gamma \vdash_{L_1} \varphi \Rightarrow \psi$$

[2] By *positive* meta-properties of a logic we mean assertions involving derivability (i.e., "some formulas are derived from certain premises") as, for instance, the deduction meta-theorem; on the other hand, a *negative* meta-property is an assertion about non-derivability (i.e., "some formulas are *not* derived from certain premises"). Of course negative meta-properties should not be preserved when embedding a logic system into a larger one.

is a more complex meta-property of L_1, that would be also preserved by fibring. But it is not true in most cases.

This phenomenon is explained because the usual notion of morphism between logics is based on the following idea: if $h : L \to L'$ is a morphism and $\Gamma \vdash_L \varphi$ then $h(\Gamma) \vdash_{L'} h(\varphi)$

The proposal of this paper is to define categories of deduction systems in which a morphism $h : L \to L'$ must preserve meta-properties of the logics of the form

If $\Gamma_1 \vdash_L \varphi_1$ and ... and $\Gamma_n \vdash_L \varphi_n$ then $\Gamma \vdash_L \varphi$.

That is: from the meta-property of L

If $\Gamma_1 \vdash_L \varphi_1$ and ... and $\Gamma_n \vdash_L \varphi_n$ then $\Gamma \vdash_L \varphi$

the following meta-property of L' must be inferred:

If $h(\Gamma_1) \vdash_{L'} h(\varphi_1)$ and ... and $h(\Gamma_n) \vdash_{L'} h(\varphi_n)$ then $h(\Gamma) \vdash_{L'} h(\varphi)$.

(by the sake of simplicity, some technical details are omitted here; see Theorem 3.4 below for a precise formulation of this claim). In such categories, when a logic system is embedded by fibring in a larger one then any meta-property is preserved (by the canonical injection). This is why we will call *meta-fibring* the fibring performed in this kind of categories. It should be obvious that, with respect to the collapsing problem of classical and intuitionistic logics above mentioned, meta-fibring makes the things even worse: the collapse will happen even at the proof-theoretical level (see Example 5.9 below). But this is more than expected since, in general, logics obtained by meta-fibring are stronger than those obtained by fibring.

The organization of this paper is as follows: in Sections 2 and 3 the category **Mcon** of Multiple-conclusion deductive systems with morphisms preserving meta-properties is defined. In Section 4 it is proved that there exist unconstrained fibrings in **Mcon**, that is, coproducts, representing fibrings in which no logic symbols are shared. The advantages of the present approach for our purposes are shown in Section 5, in which some examples of anti-collapse of usual fibring are circumvented. In other words, several well-known logics can be recovered by meta-fibring of its fragments, in opposition to fibring in the usual categories. In Section 6 it is shown that in **Mcon** there exist also constrained fibrings by sharing symbols. In order to cope with substructural logics, in Section 7 the category **Seq** of sequent calculi is introduced, as a generalization of **Mcon**. The novelty here is that sequents in **Seq** are formed by (pairs of) sequences of formulas instead of (pairs of) multisets. It is proved that **Seq** also has both forms of fibring. Section 8 is devoted to define a general semantics for sequent calculi (which, in particular, can also be applied to **Mcon**). In Section 9 it is obtained a general theorem stating that the completeness of logic systems satisfying certain conditions (namely, fullness) is preserved by both forms of fibring. Finally, in Section 10 it is given a brief account of what was achieved and what lays ahead.

2 Multiple-conclusion consequence relations

In this section the notion of assertion calculus is introduced. These calculi essentially describe multiple-conclusion consequence relations. Finally, an appropriate notion of meta-property is proposed.

From now on, we will keep fixed a denumerable set $\mathcal{X} = \{X_i \; : \; i \in \mathbb{N}\}$ of symbols called *set variables* (or simply *variables*), and a denumerable set $\Xi = \{\xi_i \; : \; i \in \mathbb{N}\}$ of symbols called *schema variables* such that $\mathcal{X} \cap \Xi = \emptyset$.

Definition 2.1 A *propositional signature* (or simply a *signature*) is a denumerable set $C = \{C_i \; : \; i \in \mathbb{N}\}$ of sets where $(\mathcal{X} \cup \Xi) \cap C_i = C_i \cap C_j = \emptyset$ for every $i, j \in \mathbb{N}$ such that $i \neq j$. The *support* of C is the set $|C| := \bigcup C$. Elements of C_n are called *n-ary connectives* of C. Elements in C_0 are called *constants* of C. The algebra of type C freely generated by Ξ is denoted by $L(C)$. Elements of $L(C)$ are called *formulas.* □

Since we will only deal with structural logics, the schema variables will play the role usually assigned to the propositional variables, that is, as generators of the language. It should be noted that, when considering logic systems in which propositional variables represent "concrete" information (as, for instance, in knowledge representation applications), new symbols for the propositional variables should be included in the signature as constants, and then formulas without schema variables would correspond to the "concrete" formulas (see Example 5.9 below). On the other hand, the set variables X_i are included in the present framework in order to represent (arbitrary) sets of formulas within the formal language for sequent calculi to be defined above.

Definition 2.2 Let C be a propositional signature. A *general assertion* over C is an expression $\langle A; \Gamma | \Delta; B\rangle$ where Γ, Δ are finite subsets of $L(C)$ and A, B are finite sets of variables such that $\Gamma \cup \Delta \cup A \cup B \neq \emptyset$. An *assertion* over C is a general assertion such that the sets A and B of variables are empty. An assertion will be denoted simply by $\langle \Gamma | \Delta \rangle$. Sometimes we will write $\Gamma \succ \Delta$ and $A; \Gamma \succ \Delta; B$ instead of $\langle \Gamma | \Delta \rangle$ and $\langle A; \Gamma | \Delta; B\rangle$, respectively. Let $GenA(C)$ and $Asse(C)$ be the sets of general assertions and of assertions over C, respectively. □

As usual, we will frequently write Γ, Γ' and Γ, φ instead of $\Gamma \cup \Gamma'$ and $\Gamma \cup \{\varphi\}$ inside assertions and general assertions. Moreover, we will write X instead of $\{X\}$, and so X, Y will stand for $\{X, Y\}$, for variables X and Y. Thus, for instance, $\langle \Gamma, \Gamma' | \Delta, \varphi\rangle$ will denote the assertion $\langle \emptyset; \Gamma \cup \Gamma' | \Delta \cup \{\varphi\}; \emptyset\rangle$ and $\langle X, Y, Z; \Gamma, \varphi | \Delta, \psi, \Delta'; Z\rangle$ or even $X, Y, Z; \Gamma, \varphi \succ \Delta, \psi, \Delta'; Z$ will denote the general assertion $\langle \{X, Y, Z\}; \Gamma \cup \{\varphi\} | \Delta \cup \Delta' \cup \{\psi\}; \{Z\}\rangle$.

Definition 2.3 Let C be a signature. An *assertion rule* over C is a pair $r = \langle \Upsilon, \langle A; \Gamma | \Delta; B \rangle \rangle$ such that $\Upsilon \cup \{\langle A; \Gamma | \Delta; B \rangle\}$ is a finite subset of $GenA(C)$. If $\Upsilon = \emptyset$ then r is called an *axiom*. A rule r will frequently be written as $r = \langle prem(r), conc(r) \rangle$, where $prem(r)$ and $conc(r)$ are the *premises* and the *conclusion* of the rule r, respectively.

An *assertion calculus* (over C) is a pair $\mathcal{A} = \langle C, \mathcal{R} \rangle$ such that C is a signature and $\mathcal{R}$ is a set of assertion rules over C. □

For simplicity, an assertion rule of the form

$$\langle \{\langle A_1; \Gamma_1 | \Delta_1; B_1 \rangle, \ldots, \langle A_n; \Gamma_n | \Delta_n; B_n \rangle\}, \langle A; \Gamma | \Delta; B \rangle \rangle$$

will be denoted by

$$\frac{A_1; \Gamma_1 \succ \Delta_1; B_1 \quad \ldots \quad A_n; \Gamma_n \succ \Delta_n; B_n}{A; \Gamma \succ \Delta; B},$$

and an axiom $\langle \emptyset, \langle A; \Gamma | \Delta; B \rangle \rangle$ will be denoted by

$$\overline{A; \Gamma \succ \Delta; B}\ .$$

Given a signature C, a *substitution* over C is a map $\sigma : \Xi \to L(C)$. We denote by $\hat{\sigma} : L(C) \to L(C)$ the unique homomorphic extension of σ to $L(C)$. An *instantiation* over C is a map $\varrho : \mathcal{X} \to \wp_F(L(C) \cup \mathcal{X})$, where $\wp_F(L(C) \cup \mathcal{X})$ denotes the set of finite subsets of $L(C) \cup \mathcal{X}$. If $\varrho(X) \in \wp_F(L(C))$ for every $X \in \mathcal{X}$ then ϱ is a *basic instantiation* over C. Given a substitution σ and an instantiation ϱ, a map $(\sigma, \varrho) : GenA(C) \to GenA(C)$ is defined as follows: given $\langle A; \Gamma | \Delta; B \rangle$ consider the sets below.

$$A' = \{Y \in \mathcal{X} \ : \ Y \in \varrho(X) \ \text{ for some } X \in A\};$$

$$\Gamma' = \{\varphi \in L(C) \ : \ \varphi \in \varrho(X) \ \text{ for some } X \in A\};$$

$$\Delta' = \{\varphi \in L(C) \ : \ \varphi \in \varrho(X) \ \text{ for some } X \in B\};$$

$$B' = \{Y \in \mathcal{X} \ : \ Y \in \varrho(X) \ \text{ for some } X \in B\}.$$

Then

$$(\sigma, \varrho)(\langle A; \Gamma | \Delta; B \rangle) = \langle A'; \hat{\sigma}(\Gamma \cup \Gamma') | \hat{\sigma}(\Delta \cup \Delta'); B' \rangle.$$

With these definitions, the notion of *derivation* in an assertion calculus can be introduced.

Definition 2.4 Let $\mathcal{A} = \langle C, \mathcal{R} \rangle$ be an assertion calculus, and let $A; \Gamma \succ \Delta; B$ be a general assertion over C. We say that $A; \Gamma \succ \Delta; B$ is *derivable in* $\mathcal{A}$, denoted by $A; \Gamma \vdash_{\mathcal{A}} \Delta; B$, if there is a finite sequence

$$\langle A_1; \Gamma_1 | \Delta_1; B_1 \rangle \ldots \langle A_n; \Gamma_n | \Delta_n; B_n \rangle$$

in $GenA(C)$ such that $\langle A_n; \Gamma_n | \Delta_n; B_n \rangle = \langle A; \Gamma | \Delta; B \rangle$ and, for every $1 \leq i \leq n$, there is a rule r in $\mathcal{R}$, a substitution σ and an instantiation ϱ over C such that:

- $(\sigma, \varrho)(prem(r)) \subseteq \{\langle A_1; \Gamma_1 | \Delta_1; B_1\rangle, \ldots, \langle A_{i-1}; \Gamma_{i-1} | \Delta_{i-1}; B_{i-1}\rangle\}$; and
- $\langle A_i; \Gamma_i | \Delta_i; B_i\rangle = (\sigma, \varrho)(conc(r))$. □

Given an assertion calculus $\mathcal{A}$, a multiple-conclusion consequence relation $\vdash_{\mathcal{A}}$ over C is obtained by using the notion of derivation of Definition 2.4: $\Gamma \vdash_{\mathcal{A}} \Delta$ iff $\Gamma \succ \Delta$ is derivable in $\mathcal{A}$.

Of course, the notion of derivation in an assertion calculus can be easily extended by using assertions as premises.

Definition 2.5 Given a set $\Omega = \{\langle A_1; \Gamma_1 | \Delta_1; B_1\rangle, \ldots, \langle A_n; \Gamma_n | \Delta_n; B_n\rangle\}$ of general assertions, we say that a general assertion $A; \Gamma \succ \Delta; B$ is *derivable in* $\mathcal{A}$ *from* Ω, denoted by

$$\frac{A_1; \Gamma_1 \succ \Delta_1; B_1 \quad \ldots \quad A_n; \Gamma_n \succ \Delta_n; B_n}{A; \Gamma \succ \Delta; B}$$

if there is a finite sequence of general assertions

$$\langle \overline{A}_1; \overline{\Gamma}_1 | \overline{\Delta}_1; \overline{B}_1\rangle \ldots \langle \overline{A}_m; \overline{\Gamma}_m | \overline{\Delta}_m; \overline{B}_m\rangle$$

such that $\langle \overline{A}_m; \overline{\Gamma}_m | \overline{\Delta}_m; \overline{B}_m\rangle = \langle A; \Gamma | \Delta; B\rangle$ and, for every $1 \leq i \leq m$, either $\langle \overline{A}_i; \overline{\Gamma}_i | \overline{\Delta}_i; \overline{B}_i\rangle \in \Omega$, or there is a rule r in $\mathcal{R}$, a substitution σ and an instantiation ϱ over C such that:

- $(\sigma, \varrho)(prem(r)) \subseteq \{\langle \overline{A}_1; \overline{\Gamma}_1 | \overline{\Delta}_1; \overline{B}_1\rangle, \ldots, \langle \overline{A}_{i-1}; \overline{\Gamma}_{i-1} | \overline{\Delta}_{i-1}; \overline{B}_{i-1}\rangle\}$; and
- $\langle \overline{A}_i; \overline{\Gamma}_i | \overline{\Delta}_i; \overline{B}_i\rangle = (\sigma, \varrho)(conc(r))$.

If $A; \Gamma \succ \Delta; B$ is derivable in $\mathcal{A}$ from Ω we say that $\vdash_{\mathcal{A}}$ has the *meta-property*:

$$\begin{array}{ll} \texttt{for all} & \Gamma'_i, \Delta'_i, \Gamma', \Delta' : \\ & \texttt{if}\ \Gamma'_1, \Gamma_1 \vdash_{\mathcal{A}} \Delta_1, \Delta'_1,\ \ldots,\ \Gamma'_n, \Gamma_n \vdash_{\mathcal{A}} \Delta_n, \Delta'_n \\ & \texttt{then}\ \Gamma', \Gamma \vdash_{\mathcal{A}} \Delta, \Delta'. \end{array}$$

We also say that

$$\frac{A_1; \Gamma_1 \succ \Delta_1; B_1 \quad \ldots \quad A_n; \Gamma_n \succ \Delta_n; B_n}{A; \Gamma \succ \Delta; B}$$

is a derived rule of $\mathcal{A}$.

□

Clearly $A; \Gamma \succ \Delta; B$ is derivable in $\mathcal{A}$ iff it is derivable in $\mathcal{A}$ from the empty set, that is,

$$\frac{}{A; \Gamma \succ \Delta; B} \ .$$

Given substitutions σ, σ' over C and instantiations ϱ, ϱ' over C, the (set-theoretic) composite $(\sigma, \varrho) \circ (\sigma', \varrho') := (\sigma \cdot \sigma', \varrho \cdot \varrho')$ is given as follows: $\sigma \cdot \sigma'$ is the substitution over C such that $\sigma \cdot \sigma'(\xi) = \hat{\sigma}(\sigma'(\xi))$ for $\xi \in \Xi$. On the other hand, $\varrho \cdot \varrho'$ is the instantiation over C such that

$$\varrho \cdot \varrho'(X) = \bigcup_{s \in \varrho'(X)} \overline{\varrho}(s)$$

where, for $s \in L(C) \cup \mathcal{X}$,

$$\overline{\varrho}(s) = \begin{cases} \varrho(s) & \text{if } s \in \mathcal{X} \\ \{s\} & \text{if } s \in L(C) \end{cases} .$$

Then $(\sigma \cdot \sigma', \varrho \cdot \varrho')(\langle A; \Gamma|\Delta; B\rangle) = (\sigma, \varrho)((\sigma', \varrho')(\langle A; \Gamma|\Delta; B\rangle))$ for every general assertion $\langle A; \Gamma|\Delta; B\rangle$. Using this, it can be easily proved the following results, stating the structurality of derivations in assertion calculi:

Proposition 2.6 Let $\mathcal{A}$ be an assertion calculus and let $\Omega \cup \{\langle A; \Gamma|\Delta; B\rangle\}$ be a finite subset of $GenA(C)$ such that $\langle A; \Gamma|\Delta; B\rangle$ is derivable in $\mathcal{A}$ from Ω. Then $(\sigma, \varrho)(\langle A; \Gamma|\Delta; B\rangle)$ is derivable in $\mathcal{A}$ from $(\sigma, \varrho)(\Omega)$, for every substitution σ over C and every instantiation ϱ over C.

Corollary 2.7 Let $\Gamma \succ \Delta$ be an assertion over C, and let $\mathcal{A}$ be an assertion calculus over C. Then, $\Gamma \vdash_{\mathcal{A}} \Delta$ implies that $\hat{\sigma}(\Gamma) \vdash_{\mathcal{A}} \hat{\sigma}(\Delta)$, for every substitution σ over C.

Using Proposition 2.6 it follows that, given a calculus $\mathcal{A}$ with a meta-property of the form

$$\begin{array}{ll} \texttt{for all} & \Gamma'_i, \Delta'_i, \Gamma', \Delta' : \\ & \texttt{if } \Gamma'_1, \Gamma_1 \vdash_{\mathcal{A}} \Delta_1, \Delta'_1, \ \ldots, \ \Gamma'_n, \Gamma_n \vdash_{\mathcal{A}} \Delta_n, \Delta'_n \\ & \texttt{then } \Gamma', \Gamma \vdash_{\mathcal{A}} \Delta, \Delta' \end{array}$$

then, for every finite sets of formulas $\Gamma'_i, \Delta'_i, \Gamma', \Delta'$, it holds in $\mathcal{A}$:

$$\frac{\Gamma'_1, \Gamma_1 \succ \Delta_1, \Delta'_1 \quad \ldots \quad \Gamma'_n, \Gamma_n \succ \Delta_n, \Delta'_n}{\Gamma', \Gamma \succ \Delta, \Delta'} .$$

Example 2.8 Multiple-conclusion consequence relations satisfy in general the following meta-property of *weakening*:

$$\begin{array}{ll} \texttt{for all} & \Gamma', \Delta' : \\ & \texttt{if } \Gamma \vdash_{\mathcal{A}} \Delta \texttt{ then } \Gamma', \Gamma \vdash_{\mathcal{A}} \Delta, \Delta' \end{array}$$

for every assertion $\Gamma \succ \Delta$. As a consequence of weakening,

$$\text{if } \Gamma \vdash_{\mathcal{A}} \Delta \text{ then } \Gamma', \Gamma \vdash_{\mathcal{A}} \Delta, \Delta'$$

for every finite $\Gamma, \Gamma', \Delta, \Delta'$.

In order to guarantee that $\vdash_{\mathcal{A}}$ satisfies weakening it is enough to require the following rules (as primitives or as derived rules) in $\mathcal{A}$:

$$\frac{X \succ Y}{X, Z \succ Y} \qquad \frac{X \succ Y}{X \succ Y, Z}$$

□

Example 2.9 Consider a signature C such that $\Rightarrow \in C_2$. The usual rules for classical implication can be described by the following assertion rules over C:

$$\frac{X_1 \succ \xi_1; X_2 \qquad X_1; \xi_2 \succ X_2}{X_1; (\xi_1 \Rightarrow \xi_2) \succ X_2} \qquad\qquad \frac{X_1; \xi_1 \succ \xi_2; X_2}{X_1 \succ (\xi_1 \Rightarrow \xi_2); X_2} \; .$$

On the other hand, the *Cut* rule can be represented (in any signature) as the following assertion rule:

$$(Cut) \;\; \frac{X_1 \succ \xi_1; X_2 \qquad X_3; \xi_1 \succ X_4}{X_1, X_3 \succ X_2, X_4} \; .$$

The reader should note that the use of variables and schema variables for sets of formulas and for formulas, respectively, is a formalization of the usual description of rules in sequent calculi. Thus, the equivalent of the assertion rules above in a sequent calculus are the following schema-rules:

$$\frac{\Gamma \succ \varphi, \Delta \qquad \Gamma, \psi \succ \Delta}{\Gamma, (\varphi \Rightarrow \psi) \succ \Delta} \qquad \frac{\Gamma, \varphi \succ \psi, \Delta}{\Gamma \succ (\varphi \Rightarrow \psi), \Delta} \qquad \frac{\Gamma \succ \varphi, \Delta \qquad \Gamma', \varphi \succ \Delta'}{\Gamma, \Gamma' \succ \Delta, \Delta'}$$

where Γ, Γ', Δ and Δ' stand for arbitrary finite sets of formulas, and φ, ψ stand for arbitrary formulas. This is a consequence of Proposition 2.6. □

3 The category of multiple-conclusion consequence relations

This section is devoted to define the adequate categories we will work out. In particular, the fundamental notion of morphism between multiple-conclusion relations will be introduced in Definition 3.3.

Definition 3.1 The category **Sig** of (propositional) signatures is defined as follows: its objects are propositional signatures and, given signatures C^1 and C^2, a morphism $h : C^1 \to C^2$ in **Sig** is a function $h : |C^1| \to L(C^2)$ such that $h(\xi) = \xi$ for $\xi \in \Xi$ and $h(c)$ is a formula which depends at most on schema variables $\xi_1, \ldots, \xi_n$ whenever $c \in C^1_n$. In particular, $h(c) \in C^2_0$ if $c \in C^1_0$. The composition $f \circ g : C^1 \to C^3$ of $g : C^1 \to C^2$ and $f : C^2 \to C^3$ in **Sig** is the morphism obtained by the function $\hat{f} \circ g : |C^1| \to L(C^3)$, where the function $\hat{f} : L(C^2) \to L(C^3)$ is obtained from f in the natural way:

- $\hat{f}(\xi) = \xi$ for $\xi \in \Xi$; $\hat{f}(c) = f(c)$ for $c \in C^2_0$;
- $\hat{f}(c(\varphi_1, \ldots, \varphi_n)) = f(c)(\hat{f}(\varphi_1), \ldots, \hat{f}(\varphi_n))$ for $c \in C^2_n$.

The identity morphism $id_C : C \to C$ for a signature C is the function $id_C : |C| \to L(C)$ such that $id_C(c) = c(\xi_1, \ldots, \xi_n)$ if $c \in C_n$. In particular, $id_C(c) = c$ if $c \in C_0$. □

Definition 3.2 If $\langle A; \Gamma|\Delta; B\rangle$ is a general assertion over C^1 and $h : C^1 \to C^2$ is a signature morphism then $\hat{h}(\langle A; \Gamma|\Delta; B\rangle)$ is defined as the general assertion $\langle A; \hat{h}(\Gamma)|\hat{h}(\Delta); B\rangle$ over C^2. And given an assertion rule r over C^1 then $\hat{h}(r)$ is the assertion rule $\langle \hat{h}(prem(r)), \hat{h}(conc(r))\rangle$ over C^2. □

Note that in the definition above it is tacitly assumed that $\hat{h}(X) = X$ for every $X \in \mathcal{X}$. This fact will be used in the sequel (for instance, in the proof of Theorem 3.4 above).

Definition 3.3 The category **Mcon** of (propositional) multiple-conclusion consequence relations is defined as follows: its objects are assertion calculi of the form $\mathcal{A} = \langle C, \mathcal{R}\rangle$ such that C is a propositional signature. Given assertion calculi $\mathcal{A}_i = \langle C^i, \mathcal{R}_i\rangle$ $(i = 1, 2)$, a morphism $h : \mathcal{A}_1 \to \mathcal{A}_2$ in **Mcon** is a morphism $h : C^1 \to C^2$ in **Sig** such that, for every rule $r \in \mathcal{R}_1$, the general assertion $\hat{h}(conc(r))$ is derivable in $\mathcal{A}_2$ from the set $\hat{h}(prem(r))$ of general assertions. The composition of morphisms and the identity maps in **Mcon** are defined as in **Sig**. □

The basic feature of morphisms in **Mcon** is that rules of the source calculus are preserved in the target calculus as much as possible. This justifies the fact that, by the very definition, $\hat{h}(\xi) = h(\xi) = \xi$ for $\xi \in \Xi$ and $\hat{h}(X) = X$ for $X \in \mathcal{X}$: the symbols denoting variable components within the rules (that is, variables of sets of formulas and schema variables) are keep fixed through morphisms. This ensures that the meaning of the rule is transferred from the source calculus into the target calculus as much as possible: every variable component of a rule r can be freely substituted for 'concrete' instances within the language of r, therefore this variable components should appear in $\hat{h}(r)$ in the same form, in order to be freely substituted by 'concrete' instances within the target language.

The following proposition shows that, indeed, the notion of morphism in **Mcon** ensures the preservation of meta-properties:

Theorem 3.4 Let $h : \mathcal{A}_1 \to \mathcal{A}_2$ be a morphism in **Mcon**. Suppose that

$$\frac{A_1; \Gamma_1 \succ \Delta_1; B_1 \quad \ldots \quad A_n; \Gamma_n \succ \Delta_n; B_n}{A; \Gamma \succ \Delta; B}$$

holds in $\mathcal{A}_1$. Then

$$\frac{A_1; \hat{h}(\Gamma_1) \succ \hat{h}(\Delta_1); B_1 \quad \ldots \quad A_n; \hat{h}(\Gamma_n) \succ \hat{h}(\Delta_n); B_n}{A; \hat{h}(\Gamma) \succ \hat{h}(\Delta); B}$$

holds in $\mathcal{A}_2$.

Proof: Given a substitution $\sigma : \Xi \to L(C^1)$ and an instantiation $\varrho : \mathcal{X} \to \wp_F(L(C^1) \cup \mathcal{X})$ over C^1, let $\sigma' : \Xi \to L(C^2)$ and $\varrho' : \mathcal{X} \to \wp_F(L(C^2) \cup \mathcal{X})$ such that $\sigma'(\xi) := \hat{h}(\sigma(\xi))$ for $\xi \in \Xi$ and $\varrho'(X) := \hat{h}(\varrho(X))$ for $X \in \mathcal{X}$. Then $\hat{h}(\hat{\sigma}(\varphi)) = \hat{\sigma'}(\hat{h}(\varphi))$ for every $\varphi \in L(C^1)$. Using this, the proof is done by induction on the length of a derivation in $\mathcal{A}_1$ of $A; \Gamma \succ \Delta; B$ from $\{\langle A_1; \Gamma_1 | \Delta_1; B_1\rangle, \ldots, \langle A_n; \Gamma_n | \Delta_n; B_n\rangle\}$. We left the details to the reader (a detailed proof can be found in [7]). ■

The last theorem guarantees that a morphism between consequence relations preserves intrinsic characteristics of the source system. In particular, if we consider the inclusion morphism (the canonical injection) from a deduction system into an extension of it, not only the consequence relation will be preserved by the inclusion morphism, but also some other relevant characteristic of the given logic. This feature has a deep impact on the strength of the combination process of fibring: as we shall see in Section 5 through several representative examples, the preservation of meta-properties will allow to reconstruct a logic by (meta)fibring two or more fragments of it. In particular, the existence of a morphism between two assertion calculi requires that the source and target logics must be compatible in some sense. In more precise terms: define an assertion rule r over a signature C as being *structural* if there are no occurrences of connectives in r. For instance, (Cut) (recall Example 2.9) and the weakening rules stated in Example 2.8 are structural assertion rules. By the very definition, structural rules do not depend on the signature in which they are defined. Thus, for instance, (Cut) can be considered in any calculus defined over any signature, because it does not depends on connectives. As a consequence of the definitions, $\hat{h}(r) = r$ for every structure rule r and every signature morphism. Thus, if $h : \mathcal{A}_1 \to \mathcal{A}_2$ is a **Mcon**-morphism then $\mathcal{A}_2$ must be capable of deriving every structural rule of $\mathcal{A}_1$. This is a basic requirement in order to define a morphism between two assertion calculi: as much as structural rules are concerned, the target calculus must be stronger than the source.

Example 3.5 Consider the following signatures for classical logic:

- C^1 such that $C^1_0 = \{\bot\}$; $C^1_2 = \{\Rightarrow\}$; $C^1_n = \emptyset$ in any other case.
- C^2 such that $C^2_0 = \{\top\}$; $C^2_1 = \{\neg\}$; $C^2_2 = \{\vee\}$; $C^2_n = \emptyset$ in any other case.

We can define the assertion calculus $\mathcal{A}_1 = \langle C^1, \mathcal{R}_1\rangle$ for propositional classical logic over C^1, where $\mathcal{R}_1$ consists of the following rules (here X, Y, Z, W denote variables and ξ, ξ' denote schema variables):

$$\frac{}{X; \xi \succ \xi; Y} \qquad \frac{X \succ Y}{X, Z \succ Y} \qquad \frac{X \succ Y}{X \succ Y, Z} \qquad \frac{X; \xi \succ Y \quad Z \succ \xi; W}{X, Z \succ Y, W}$$

$$\frac{}{X; \bot \succ Y} \qquad \frac{X \succ \xi; Y \quad X; \xi' \succ Y}{X; (\xi \Rightarrow \xi') \succ Y} \qquad \frac{X; \xi \succ \xi'; Y}{X \succ (\xi \Rightarrow \xi'); Y}$$

Note that the first four rules are structural.

Consider now the following assertion calculus over C^2, called $\mathcal{A}_2$: add to the first four rules of $\mathcal{A}_1$ the rules

$$\frac{X \succ \xi; Y}{X; \neg\xi \succ Y} \qquad \frac{X; \xi \succ Y}{X \succ \neg\xi; Y}$$

$$\frac{X; \xi \succ Y \quad X; \xi' \succ Y}{X; \xi \vee \xi' \succ Y} \qquad \frac{X \succ \xi, \xi'; Y}{X \succ \xi \vee \xi'; Y}$$

Clearly, $\mathcal{A}_2$, is adequate for classical logic over C^2. Consider now the following morphisms in **Sig**:

- $h : C^1 \to C^2$ such that $h(\bot) = \neg\top$ and $h(\Rightarrow) = (\neg\xi_1 \vee \xi_2)$;
- $h' : C^2 \to C^1$ such that $h'(\top) = (\bot \Rightarrow \bot)$; $h(\xi) = \xi$ (for $\xi \in \mathcal{V}$); $h'(\neg) = (\xi_1 \Rightarrow \bot)$, and $h'(\vee) = ((\xi_1 \Rightarrow \bot) \Rightarrow \xi_2)$.

It easy to see that both morphisms in **Sig** are indeed morphisms $h : \mathcal{A}_1 \to \mathcal{A}_2$ and $h' : \mathcal{A}_2 \to \mathcal{A}_1$ in **Mcon**. In fact: the following derivations in $\mathcal{A}_2$

$$\frac{X \succ \top; Y}{X; \neg\top \succ Y} \qquad \frac{\dfrac{X \succ \xi; Y \ (Hyp.)}{X; \neg\xi \succ Y} \quad X; \xi' \succ Y \ (Hyp.)}{X; \neg\xi \vee \xi' \succ Y} \qquad \frac{\dfrac{X; \xi \succ \xi'; Y \ (Hyp.)}{X \succ \neg\xi, \xi'; Y}}{X \succ \neg\xi \vee \xi'; Y}$$

show that $h : \mathcal{A}_1 \to \mathcal{A}_2$ is a morphism in **Mcon**. On the other hand, the derivations in $\mathcal{A}_1$

$$\frac{X; \bot \succ \bot; Y}{X \succ \bot \Rightarrow \bot; Y} \qquad \frac{X \succ \xi; Y \ (Hyp.) \quad X; \bot \succ Y}{X; \xi \Rightarrow \bot \succ Y} \qquad \frac{\dfrac{X; \xi \succ Y \ (Hyp.)}{X; \xi \succ \bot; Y}}{X \succ \xi \Rightarrow \bot; Y}$$

$$\frac{\dfrac{\dfrac{X; \xi \succ Y \ (Hyp.)}{X; \xi \succ \bot; Y}}{X \succ \xi \Rightarrow \bot; Y} \quad X; \xi' \succ Y \ (Hyp.)}{X; (\xi \Rightarrow \bot) \Rightarrow \xi' \succ Y} \qquad \frac{\dfrac{X \succ \xi, \xi'; Y \ (Hyp.) \quad X; \bot \succ \xi'; Y}{X; \xi \Rightarrow \bot \succ \xi'; Y}}{X \succ (\xi \Rightarrow \bot) \Rightarrow \xi'; Y}$$

show that $h' : \mathcal{A}_2 \to \mathcal{A}_1$ is also a morphism in **Mcon**. □

4 Fibring multiple-conclusion relations

In this section the concept of (categorial) fibring multiple-conclusion relations is introduced. As usual, this construction can be characterized as a coproduct.

From [13] the following result is known:

Proposition 4.1 The category **Sig** has finite coproducts.

Given signatures C^1 and C^2, the coproduct of C^1 and C^2 will be denoted by $C^1 \oplus C^2$, with canonical injections $i_1 : C^1 \to C^1 \oplus C^2$ and $i_2 : C^1 \to C^1 \oplus C^2$. It is worth noting that $C^1 \oplus C^2$ is simply obtained as the disjoint union of C^1 and C^2 at all levels, that is: $(C^1 \oplus C^2)_k$ is the disjoint union of C^1_k and C^2_k, for every $k \in \mathbb{N}$.

Definition 4.2 Let $\mathcal{A}_j = \langle C^j, \mathcal{R}_j \rangle$ be two assertion calculi ($j = 1, 2$). The (categorial) *fibring* of $\mathcal{A}_1$ and $\mathcal{A}_2$ is the assertion calculus $\mathcal{A}_1 \oplus \mathcal{A}_2 = \langle C, \mathcal{R} \rangle$ defined as follows:

- $C = C^1 \oplus C^2$;
- $\mathcal{R} = \{\hat{i}_1(r_1) \ : \ r_1 \in \mathcal{R}_1\} \cup \{\hat{i}_2(r_2) \ : \ r_2 \in \mathcal{R}_2\}$

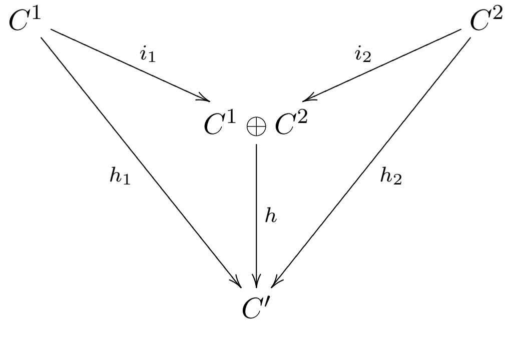
Here i_1 and i_2 are the canonical injections of the coproduct $C^1 \oplus C^2$ of C^1 and C^2, and $\hat{i}_j(r_j)$ is defined as in Definition 3.2 for $j = 1, 2$. □

As expected, it is obtained the following result:

Proposition 4.3 Let $\mathcal{A}_j = \langle C^j, \mathcal{R}_j \rangle$ be two assertion calculi ($j = 1, 2$). Then $\mathcal{A}_1 \oplus \mathcal{A}_2$ is the coproduct in **Mcon** of $\mathcal{A}_1$ and $\mathcal{A}_2$ with canonical injections induced by the injections i_1 and i_2 from **Sig**.

Proof: Let $\mathcal{A}_1 \oplus \mathcal{A}_2 = \langle C, \mathcal{R} \rangle$ as in Definition 4.2. We begin by proving that $i_j : C^j \to C^1 \oplus C^2$ induces a morphism $i_j : \mathcal{A}_j \to \mathcal{A}_1 \oplus \mathcal{A}_2$ in **Mcon** for $j = 1, 2$. Let r_j be an assertion rule of $\mathcal{R}_j$; then $\hat{i}_j(r_j) \in \mathcal{R}$ and so it is a derived rule of $\mathcal{A}_1 \oplus \mathcal{A}_2$. Therefore $i_j : \mathcal{A}_j :\to \mathcal{A}_1 \oplus \mathcal{A}_2$ is a morphism in **Mcon** for $j = 1, 2$.

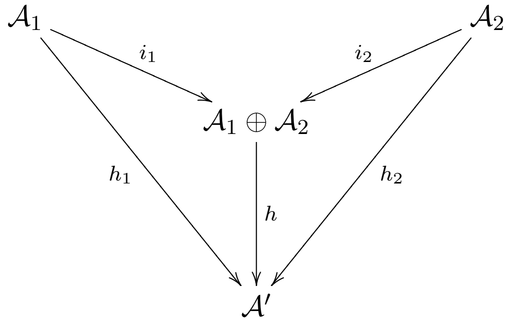
Let $\mathcal{A}' = \langle C', \mathcal{R}' \rangle$ be an assertion calculus and let $h_j : \mathcal{A}_j \to \mathcal{A}'$ be morphisms in **Mcon** ($j = 1, 2$). Then there is an unique morphism $h : C^1 \oplus C^2 \to C'$ such that



commutes in **Sig**. It is enough to prove that h induces a morphism $h : \mathcal{A}_1 \oplus \mathcal{A}_2 \to \mathcal{A}'$ in **Mcon**. So, let $r \in \mathcal{R}$. Then $r = \hat{i}_j(r_j)$ for some $j \in \{1, 2\}$ and some $r_j \in \mathcal{R}_j$. Therefore

$$\hat{h}(r) = \hat{h}(\hat{i}_j(r_j)) = \hat{h}_j(r_j)$$

is a derived rule in $\mathcal{A}'$, since $h_j : \mathcal{A}_j \to \mathcal{A}'$ is a morphism in **Mcon** and r_j is a rule of $\mathcal{A}_j$ (for $j = 1, 2$). This shows that h is a morphism $h : \mathcal{A}_1 \oplus \mathcal{A}_2 \to \mathcal{A}'$ such that



commutes in **Mcon**. The uniqueness of h in **Mcon** follows from the uniqueness of h in **Sig**. ■

5 Some examples

Assume the point of view of trying to recover a logic from its fragments. In this section some examples will show the convenience of our approach with respect to the traditional presentation of deduction systems and their morphisms. It is important to notice that all the examples considered in this section describe multiple-conclusion consequence relations $\vdash$ in which the right-hand side of $\vdash$ consists in a set with at most one formula. This is deliberate: we are comparing the results of fibring deduction systems using our proposal, with those obtained by fibring in the usual categories of deduction systems. In order to compare the results we are looking to the usual (single-conclusion) consequence relations. This is why we use this particular case of assertion calculi in the examples below. The reader should note that it is possible to obtain the same results by considering assertion calculi which define (genuine) multiple-conclusion consequence relations.

The first three examples will serve as a basis for the others given in this section.

Example 5.1 (Intuitionistic and classical negation.)
Let $C^{\neg}$ be a signature just containing a negation symbol $\neg \in C^{\neg}_1$. Consider the assertion calculus $\mathcal{A}^i_{\neg} = \langle C^{\neg}, \mathcal{R}^i_{\neg} \rangle$ given by the following rules (here X, Y denote variables and ξ, ξ' denote schema variables):

$$\frac{}{X; \xi \succ \xi} \qquad \frac{X \succ \xi}{X, Y \succ \xi} \qquad \frac{X \succ}{X, Y \succ} \qquad \frac{X \succ}{X \succ \xi}$$

$$\frac{X \succ \xi \qquad Y;\xi \succ \xi'}{X,Y \succ \xi'} \qquad \frac{X \succ \xi \qquad Y;\xi \succ}{X,Y \succ}$$

$$\frac{X;\xi \succ}{X \succ \neg\xi} \qquad \frac{X \succ \xi \qquad Y \succ \neg\xi}{X,Y \succ}$$

Clearly, $\mathcal{A}^i_\neg$ is an adequate assertion calculus for the intuitionistic negation. Note that

$$(l\neg) \quad \frac{X \succ \xi}{X;\neg\xi \succ}$$

is a derived rule of $\mathcal{A}^i_\neg$, as the following derivation shows:

$$\frac{X \succ \xi\ (Hyp.) \qquad \neg\xi \succ \neg\xi}{X;\neg\xi \succ}$$

Note that the right-hand side of every demonstrable assertion in both $\mathcal{A}^i_\neg$ has at most one formula.

As expected, it is enough to extend $\mathcal{A}^i_\neg$ by adding appropriate rules in order to obtain an assertion calculus for classical negation.

Thus, consider the assertion calculus $\mathcal{A}_\neg = \langle C^\neg, \mathcal{R}_\neg \rangle$ such that $\mathcal{R}_\neg$ is obtained from $\mathcal{R}^i_\neg$ by adding the following rule:

$$\frac{X;\neg\xi \succ}{X \succ \xi}\ .$$

It is easy to see that the assertion calculus $\mathcal{A}_\neg$ is adequate for the classical negation. Now we will analyze some basic features of $\mathcal{A}_\neg$, which will be used later on.

Firstly, note that, just as in $\mathcal{A}^i_\neg$, the right-hand side of every demonstrable assertion in $\mathcal{A}_\neg$ has at most one formula. Assume the following notation: $\neg^n\varphi$ denotes the formula of $L(C^\neg)$ obtained by applying n negations to the formula φ (and so $\neg^0\varphi$ is φ itself). It is clear that there are no theorems in $\vdash_{\mathcal{A}_\neg}$, that is: there is no formula φ such that $\vdash_{\mathcal{A}_\neg} \varphi$. Moreover, $\Gamma \vdash_{\mathcal{A}_\neg} \Delta$ iff one of the following situations happens:

- Δ has at most one formula and there is some formula ψ and some k, n such that $n - k$ is odd and both $\neg^k\psi$, $\neg^n\psi$ belongs to Γ; or
- $\Delta = \{\varphi\}$ such that $\neg^{2k}\varphi \in \Gamma$ for some $k \geq 0$; or
- $\Delta = \{\varphi\}$ such that $\varphi = \neg^{2k}\psi$ for some $\psi \in \Gamma$ and some $k \geq 0$.

In $\mathcal{A}_\neg$ the following rule can be derived:

$$(EM) \quad \frac{X;\xi \succ \xi' \quad Y;\neg\xi \succ \xi'}{X,Y \succ \xi'}\ .$$

In order to prove this, and using the rule ($l\neg$) above (which, of course, is also a derived rule of $\mathcal{A}_\neg$), it is enough to consider the following derivation in $\mathcal{A}_\neg$:

$$\dfrac{\dfrac{\dfrac{X;\xi \succ \xi'\ (Hyp.)}{X;\xi,\neg\xi' \succ}}{X;\neg\xi' \succ \neg\xi} \qquad \dfrac{\dfrac{Y;\neg\xi \succ \xi'\ (Hyp.)}{Y;\neg\xi,\neg\xi' \succ}}{Y;\neg\xi' \succ \xi}}{\dfrac{X,Y;\neg\xi' \succ}{X,Y \succ \xi'}}$$

The derived rule (EM) of $\mathcal{A}_\neg$ plays an important role when combing the logic of classical negation with other fragments of classical logics, as we shall see below. □

Example 5.2 (Classical disjunction and conjunction.)
Let $C^\vee$ be a signature just containing a symbol $\vee \in C^\vee_2$ for disjunction. The properties of classical disjunction $\vee$ can be captured by the following assertion calculus over $C^\vee$, called $\mathcal{A}_\vee$ (here X, Y, Z denote variables and ξ, ξ', ξ'' denote schema variables):

Take the first six rules of the assertion calculus $\mathcal{A}^i_\neg$ of Example 5.1 and add the following:

$$\frac{X \succ \xi}{X \succ \xi \vee \xi'} \qquad \frac{X \succ \xi'}{X \succ \xi \vee \xi'}$$

$$\frac{X \succ \xi \vee \xi' \quad Y;\xi \succ \xi'' \quad Z;\xi' \succ \xi''}{X,Y,Z \succ \xi''} \qquad \frac{X \succ \xi \vee \xi' \quad Y;\xi \succ \quad Z;\xi' \succ}{X,Y,Z \succ}$$

Note that, as in the calculi of Example 5.1, the right-hand side of every demonstrable assertion in $\mathcal{A}_\vee$ has at most one formula. It is easy to see that the following rules (to be used in Example 5.8) are derivable in $\mathcal{A}_\vee$:

$$(l\vee 1)\ \frac{X;\xi \succ \xi'' \quad Y;\xi' \succ \xi''}{X,Y;\xi \vee \xi' \succ \xi''} \qquad (l\vee 2)\ \frac{X;\xi \succ \quad Y;\xi' \succ}{X,Y;\xi \vee \xi' \succ}$$

Let $Var(\varphi)$ be the set of schema variables occurring in the formula φ of $L(C^\vee)$. Clearly, $\Gamma \vdash_{\mathcal{A}_\vee} \varphi$ iff there is some formula γ in Γ such that $Var(\gamma) \subseteq Var(\varphi)$. Then, there are no theorems in $\vdash_{\mathcal{A}_\vee}$, that is: there is no formula φ such that $\vdash_{\mathcal{A}_\vee} \varphi$. Additionally, for no Γ it is the case that $\Gamma \vdash_{\mathcal{A}_\vee}$.

If $C^\wedge$ is a signature just containing a symbol $\wedge \in C^\wedge_2$ for conjunction, then it is easy to define an assertion calculus $\mathcal{A}_\wedge$ over $C^\wedge$ representing the logic of classical conjunction. In fact, it is enough to take again the first six rules of $\mathcal{A}^i_\neg$ (see Example 5.1) and add the following:

$$\frac{X;\xi,\xi' \succ \xi''}{X;\xi \wedge \xi' \succ \xi''} \qquad \frac{X;\xi,\xi' \succ}{X;\xi \wedge \xi' \succ} \qquad \frac{X \succ \xi \quad Y \succ \xi'}{X,Y \succ \xi \wedge \xi'}$$

It is easy to prove that, for instance, the following rule (to be used in Example 5.8) is derivable in $\mathcal{A}_\wedge$:

$$(A)\quad \frac{X;\xi\wedge\xi',\xi'\succ\xi''}{X;\xi\wedge\xi'\succ\xi''}$$

In fact, consider the following derivation in $\mathcal{A}_\wedge$:

$$\frac{\dfrac{X;\xi,\xi'\succ\xi'}{X;\xi\wedge\xi'\succ\xi'}\qquad X;\xi\wedge\xi',\xi'\succ\xi''\ (Hyp.)}{X;\xi\wedge\xi'\succ\xi''}$$

□

Example 5.3 (Intuitionistic and classical implication.)
Let $C^{\Rightarrow}$ be a signature just containing a symbol $\Rightarrow\in C_2^{\Rightarrow}$ for implication. The following assertion calculus over $C^{\Rightarrow}$, called $\mathcal{A}^i_{\Rightarrow}$, is adequate for describing the properties of intuitionistic implication $\Rightarrow$ (here X,Y denote variables and ξ,ξ' denote schema variables):

Consider the first six rules of the assertion calculus $\mathcal{A}^i_{\neg}$ of Example 5.1 and add the following rules:

$$\frac{X;\xi\succ\xi'}{X\succ\xi\Rightarrow\xi'}\qquad\qquad \frac{X\succ\xi\Rightarrow\xi'\quad\ Y\succ\xi}{X,Y\succ\xi'}$$

As in the examples above, the right-hand side of every demonstrable assertion in $\mathcal{A}^i_{\Rightarrow}$ has at most one formula. On the other hand, adding to $\mathcal{A}^i_{\Rightarrow}$ the assertion rule (Peirce's rule)

$$(PR)\quad \frac{X\succ(\xi\Rightarrow\xi')\Rightarrow\xi}{X\succ\xi}$$

produces an assertion calculus called $\mathcal{A}_{\Rightarrow}$, which is adequate for the implicational fragment of classical logic. □

Example 5.4 (Recovering classical logic by meta-fibring of its fragments: case I.)
Consider the fibring $\mathcal{A}_{\neg\vee}:=\mathcal{A}_{\neg}\oplus\mathcal{A}_{\vee}$ of the calculi defined in Examples 5.1 and 5.2. Then $\mathcal{A}_{\neg\vee}$ recovers propositional classical logic over the signature $C^{\neg\vee}:=C^{\neg}\oplus C^{\vee}$ such that $|C^{\neg\vee}|=\{\neg,\vee\}$. That is, $\mathcal{A}_{\neg\vee}$ can be seen as a sequent-calculus presentation of propositional classical logic over $\vee,\neg$. Moreover, the derived rule (EM) of $\mathcal{A}_{\neg}$ (see Example 5.1) hold in $\mathcal{A}_{\neg\vee}$, because of Theorem 3.4 and the fact that the signature canonical injection is, in this case, the inclusion map. Therefore $\vdash_{\mathcal{A}_{\neg\vee}}(\xi\vee\neg\xi)$ holds, as the following derivation shows:

$$\frac{\dfrac{\xi\succ\xi}{\xi\succ(\xi\vee\neg\xi)}\qquad \dfrac{\neg\xi\succ\neg\xi}{\neg\xi\succ(\xi\vee\neg\xi)}}{\succ(\xi\vee\neg\xi)}$$

□

Example 5.5 (Classical logic cannot be recovered by fibring of its fragments: the consequence relations case.)
By considering the category of standard consequence relations (see [4, 13, 11]) then $(\xi \vee \neg\xi)$ is not a valid formula of the fibring $L = \langle C^{\neg\vee}, \vdash\rangle$ of the consequence systems corresponding to classical negation $L_\neg = \langle C^\neg, \vdash_1\rangle$ and classical disjunction $L_\vee = \langle C^\vee, \vdash_2\rangle$. In order to see this, consider the following matrices:

$\vee$	T	$F1$	F
T	T	T	T
$F1$	T	F	F
F	T	F	F

	$\neg$
T	F
$F1$	$F1$
F	T

where T is the only distinguished value. The matrices above are sound for $L_{\neg\vee}$: this is obvious by considering the characterization of the (single-conclusion) consequence relation of classical negation and of classical disjunction given in Examples 5.1 and 5.2. Thus, if $\models$ denotes the (semantical) consequence relation associated to the matrices above then:

- $\Gamma \vdash_1 \varphi$ implies that $\Gamma \models \varphi$, for every $\Gamma \cup \{\varphi\} \subseteq L(C^\neg)$; and
- $\Gamma \vdash_2 \varphi$ implies that $\Gamma \models \varphi$, for every $\Gamma \cup \{\varphi\} \subseteq L(C^\vee)$.

Since, in the category of standard consequence relations, the consequence relation $\vdash$ of the fibring L is the least upper bound of $\{\vdash_1, \vdash_2\}$ in the (complete) lattice of standard consequence relations over signature $C^\neg \oplus C^\vee$ then: $\Gamma \vdash \varphi$ implies that $\Gamma \models \varphi$, for every $\Gamma \cup \{\varphi\} \subseteq L(C^\neg \oplus C^\vee)$. On the other hand, $\not\models (\xi \vee \neg\xi)$: it is enough to take a valuation v such that $v(\xi) = F1$. Therefore, $\nvdash (\xi \vee \neg\xi)$.

One reason for this situation is that the meta-property

$$\frac{\Gamma, \varphi \vdash_1 \psi \quad \Delta, \neg\varphi \vdash_1 \psi}{\Gamma, \Delta \vdash_1 \psi}$$

of the consequence relation $\vdash_1$ associated to classical negation (which is represented by the derived rule (EM), recall Example 5.1) is not preserved by fibring in the category of consequence relations. It should be noticed that the same result holds in the category of Hilbert calculi, that is: classical logic over $\{\neg, \vee\}$ cannot be recovered by fibring of its fragments in the category of Hilbert calculi. As long as we are interested in recovering a logic from its fragments, this result is a strong evidence in favor of considering a category of deduction systems with morphisms preserving meta-properties, instead of a category with morphisms just preserving inferences. □

Example 5.6 (Recovering classical logic by meta-fibring of its fragments: case II.)
Consider now the (meta)fibring $\mathcal{A}_{\neg\Rightarrow} := \mathcal{A}_\neg \oplus \mathcal{A}^i_\Rightarrow$ of the calculi defined in

Examples 5.1 and 5.3. Then $\mathcal{A}_{\neg\Rightarrow}$ recovers propositional classical logic over the signature just containing the connectives $\{\neg, \Rightarrow\}$. In fact, the formula $((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$ (Peirce's law), which is satisfied by classical implication but not for intuitionistic implication, can be derived in $\mathcal{A}_{\neg\Rightarrow}$ as follows: firstly, consider the following meta-properties of $\mathcal{A}_{\neg\Rightarrow}$ (the second one is already valid in $\mathcal{A}^i_{\Rightarrow}$ and then it is transferred to $\mathcal{A}_{\neg\Rightarrow}$ by Theorem 3.4):

$$\frac{}{X; \neg(\xi \Rightarrow \xi') \succ \xi} \quad \text{and} \quad \frac{}{X; (\xi \Rightarrow \xi'), \xi \succ \xi'} \ .$$

The following derivations prove these meta-properties:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{X; \xi, \neg\xi' \succ \xi}{X; \xi, \neg\xi', \neg\xi \succ}}{X; \xi, \neg\xi \succ \xi'}}{X; \neg\xi \succ (\xi \Rightarrow \xi')}}{X; \neg\xi, \neg(\xi \Rightarrow \xi') \succ}}{X; \neg(\xi \Rightarrow \xi') \succ \xi} \qquad \frac{X; (\xi \Rightarrow \xi') \succ (\xi \Rightarrow \xi') \quad \xi \succ \xi}{X; (\xi \Rightarrow \xi'), \xi \succ \xi'}$$

Then, the following derivation in $\mathcal{A}_{\neg\Rightarrow}$ shows that $\Rightarrow$ satisfies Peirce's law:

$$\cfrac{\cfrac{\neg(\xi \Rightarrow \xi') \succ \xi \quad ((\xi \Rightarrow \xi') \Rightarrow \xi), (\xi \Rightarrow \xi') \succ \xi}{((\xi \Rightarrow \xi') \Rightarrow \xi) \succ \xi}}{\succ ((\xi \Rightarrow \xi') \Rightarrow \xi) \Rightarrow \xi}$$

Using Peirce's law, it is proved that the assertion rule (PR) (which, in Example 5.3, was added to $\mathcal{A}^i_{\Rightarrow}$ in order to obtain $\mathcal{A}_{\Rightarrow}$) is derived in $\mathcal{A}_{\neg\Rightarrow}$:

$$\frac{X \succ (\xi \Rightarrow \xi') \Rightarrow \xi \ \ (Hyp.) \qquad \succ ((\xi \Rightarrow \xi') \Rightarrow \xi) \Rightarrow \xi}{X \succ \xi}$$

This explains why, in order to get classical logic from $\neg$ and $\Rightarrow$, it is enough to consider classical negation combined with intuitionistic implication, instead of classical implication. □

It should be noticed that $\mathcal{A}^i_{\neg\Rightarrow} := \mathcal{A}^i_{\neg} \oplus \mathcal{A}^i_{\Rightarrow}$ (see Examples 5.1 and 5.3) produces the $\{\Rightarrow, \neg\}$-fragment of intuitionistic logic.

Example 5.7 (Classical logic cannot be recovered by fibring of its fragments: the Hilbert calculi case.)
By considering the category **Hil** of Hilbert propositional calculi (see for instance [18, 22, 4, 13, 11]) which is frequently used to define fibring, then the result of Example 5.6 cannot be obtained. That is, if we compute the fibring of the Hilbert calculi corresponding to classical implication and to classical negation, respectively, we cannot recover classical logic. Specifically, consider the Hilbert calculus $\mathcal{H}_{\Rightarrow}$ over the signature $C^{\Rightarrow}$ of Example 5.3 defined by the following axioms and inference rules (here, ξ, ξ' and ξ'' are schema variables):

- $\vdash \xi \Rightarrow (\xi' \Rightarrow \xi)$
- $\vdash (\xi \Rightarrow (\xi' \Rightarrow \xi'')) \Rightarrow ((\xi \Rightarrow \xi') \Rightarrow (\xi \Rightarrow \xi''))$
- $\vdash ((\xi \Rightarrow \xi') \Rightarrow \xi) \Rightarrow \xi$ (Peirce's law)
- $\dfrac{\xi \Rightarrow \xi' \quad \xi}{\xi'}$

Now consider the Hilbert calculus $\mathcal{H}_\neg$ over the signature $C^\neg$ of Example 5.1 defined by the following inference rules (again, ξ, and ξ' are schema variables):

- $\dfrac{\neg\neg\xi}{\xi}$
- $\dfrac{\xi}{\neg\neg\xi}$
- $\dfrac{\xi \quad \neg\xi}{\xi'}$

Let $\mathcal{H}_{\neg\Rightarrow}$ be the fibring in **Hil** of $\mathcal{H}_\neg$ and $\mathcal{H}_\Rightarrow$ (which consists simply on putting together all the axioms and inference rules of both systems).

Consider now the following matrices introduced by Urbas (cf. [20], Theorem 8):

$\to$	1	$^6/_7$	$^5/_7$	$^4/_7$	$^3/_7$	$^2/_7$	$^1/_7$	0
1	1	$^6/_7$	$^5/_7$	$^4/_7$	$^3/_7$	$^2/_7$	$^1/_7$	0
$^6/_7$	1	1	$^5/_7$	$^4/_7$	$^5/_7$	$^4/_7$	$^1/_7$	$^1/_7$
$^5/_7$	1	$^6/_7$	1	$^4/_7$	$^6/_7$	$^2/_7$	$^4/_7$	$^2/_7$
$^4/_7$	1	$^6/_7$	$^5/_7$	1	$^3/_7$	$^6/_7$	$^5/_7$	$^3/_7$
$^3/_7$	1	1	1	$^4/_7$	1	$^4/_7$	$^4/_7$	$^4/_7$
$^2/_7$	1	1	$^5/_7$	1	$^5/_7$	1	$^5/_7$	$^5/_7$
$^1/_7$	1	$^6/_7$	1	1	$^6/_7$	$^6/_7$	1	$^6/_7$
0	1	1	1	1	1	1	1	1

	$\neg$
1	0
$^6/_7$	$^5/_7$
$^5/_7$	$^2/_7$
$^4/_7$	$^3/_7$
$^3/_7$	$^4/_7$
$^2/_7$	$^5/_7$
$^1/_7$	1
0	1

where 1 is the only distinguished value. Then, the matrices above constitute a matrix semantics sound for $\mathcal{H}_{\neg\Rightarrow}$, as the reader can easily check: it is enough to check the validity of every axiom in $\mathcal{H}_{\neg\Rightarrow}$ as well as the preservation of validity with respect to every inference rule. Moreover, it is not hard to see that $\xi \Rightarrow (\neg\xi \Rightarrow \xi')$ is not a tautology for the matrices above (taking, for instance, a valuation v such that $v(\xi) = \frac{6}{7}$ and $v(\xi') = \frac{4}{7}$). Therefore this formula is not derivable in $\mathcal{H}_{\neg\Rightarrow}$ and then $\mathcal{H}_{\neg\Rightarrow}$ is a system strictly contained in classical propositional logic over the signature $\{\neg, \Rightarrow\}$. Note that the semantical consequence relation $\models$ associated to the matrices above satisfies the following meta-property: $\varphi, \neg\varphi \models \psi$ for every φ, ψ (since this logic extends $\mathcal{H}_{\neg\Rightarrow}$, which enjoy such a meta-property). This shows that this semantical consequence relation does not satisfy the deduction meta-theorem. By the same argument, $\mathcal{H}_{\neg\Rightarrow}$ does not satisfy the deduction meta-theorem.

It should be noticed that the matrices above are sound for *every* Hilbert calculi adequate for classical implication and classical negation. Thus, the fact that classical logic over $\{\Rightarrow, \neg\}$ cannot be recovered from the fibring in **Hil** of its fragments does not depend on any specific axiomatization. Moreover, this phenomenon also happens if we consider fibring in the usual category of consequence systems.

Again, it can be seen that the deduction systems of the usual categories (**Hil**, in this example), in which the morphisms just preserve inferences, loss important meta-properties when embedded in larger deduction systems. As a consequence of this, the deduction systems obtained by fibring in these categories are weaker than could be expected, if we have in mind the recovering of a logic from its fragments. This example suggest that the category **Mcon** of deduction systems seems more appropriate for attaining this goal. $\square$

Example 5.8 (Combining conjunction and disjunction.)
Let $\mathcal{A}_{\wedge\vee} := \mathcal{A}_{\wedge} \oplus \mathcal{A}_{\vee}$ (recall Example 5.2). By using the derived meta-properties of $\mathcal{A}_{\wedge}$ and $\mathcal{A}_{\vee}$ mentioned in Example 5.2, which are transferred to $\mathcal{A}_{\wedge\vee}$ because of Theorem 3.4, it can be proved that

$$\xi \wedge (\xi' \vee \xi'') \vdash_{\mathcal{A}_{\wedge\vee}} (\xi \wedge \xi') \vee \xi''$$

In fact, consider the following derivation, adapted from [3] (note the use of rules $(l\vee 1)$ and (A), cf. Example 5.2):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\xi \succ \xi \qquad \xi' \succ \xi'}{\xi, \xi' \succ \xi \wedge \xi'}}{\xi, \xi' \vee \xi'', \xi' \succ \xi \wedge \xi'}}{\xi \wedge (\xi' \vee \xi''), \xi' \succ \xi \wedge \xi'}}{\xi \wedge (\xi' \vee \xi''), \xi' \succ (\xi \wedge \xi') \vee \xi''} \qquad \dfrac{\xi'' \succ \xi''}{\xi'' \succ (\xi \wedge \xi') \vee \xi''}}{\dfrac{\xi \wedge (\xi' \vee \xi''), \xi' \vee \xi'' \succ (\xi \wedge \xi') \vee \xi''}{\xi \wedge (\xi' \vee \xi'') \succ (\xi \wedge \xi') \vee \xi''}}$$

But it is known that the above property is equivalent to the distributivity between $\vee$ and $\wedge$. That is, the logic of classical conjunction and classical disjunction can be recovered by fibring in **Mcon** the respective fragments. On the other hand, as was proved in [3], the fibring of the logic of classical conjunction with the logic of classical disjunction (in the usual categories of logic systems such as **Hil**) produces a logic which cannot prove the distributivity law between $\vee$ and $\wedge$. If one wants to recover the classical logic of conjunction and disjunction from its fragments, then the use of stronger morphisms between logics (as in **Mcon**) which preserves meta-properties is, again, more appropriate. $\square$

Example 5.9 (The collapsing problem.)
Recall the *collapsing problem* of classical and intuitionistic logic mentioned in Section 1. As expected, the use of meta-fibring instead of fibring produces the collapse even at the proof-theoretical level. To see that, consider the assertion

calculi $\mathcal{A}^i_\Rightarrow$ and $\mathcal{A}_\Rightarrow$ of Example 5.3. As it was done in [5] with the fibring of the Hilbert calculi of classical and intuitionistic logics, the respective signatures must me enriched by adding, as constants, denumerable sets of propositional variables $\{p^i_n : n \in \mathbb{N}\}$ to $\mathcal{A}^i_\Rightarrow$ and $\{p^c_n : n \in \mathbb{N}\}$ to $\mathcal{A}_\Rightarrow$. Since both implications satisfy the deduction meta-theorem in $\mathcal{A}^i_\Rightarrow \oplus \mathcal{A}_\Rightarrow$, it follows by Gabbay's argument (cf. [15]) that both implications collapse in $\mathcal{A}^i_\Rightarrow \oplus \mathcal{A}_\Rightarrow$. In fact, if $\Rightarrow_i$ and $\Rightarrow_c$ denote the intuitionistic and classical implications of $\mathcal{A}^i_\Rightarrow \oplus \mathcal{A}_\Rightarrow$, respectively, then $\varphi \Rightarrow_i \psi \vdash \varphi \Rightarrow_i \psi$ implies that $\varphi \Rightarrow_i \psi,\ \varphi \vdash \psi$ (by the deduction meta-theorem for $\Rightarrow_i$) implies that $\varphi \Rightarrow_i \psi \vdash \varphi \Rightarrow_c \psi$ (by the deduction meta-theorem for $\Rightarrow_c$). The derivation $\varphi \Rightarrow_c \psi \vdash \varphi \Rightarrow_i \psi$ is proved analogously. □

Example 5.10 (Modal logic K.)
The same phenomenon pointed out in Example 5.7 occurs when we try to recover modal logic K from two fragments by means of fibring in **Hil**: the $\{\Rightarrow, \Box\}$-fragment $\mathcal{H}^K_{\Rightarrow\Box}$, and the $\{\neg\}$-fragment $\mathcal{H}_\neg$ of classical logic defined in Example 5.7. Note that K is described in the signature with support $\{\neg, \Rightarrow, \Box\}$.

Clearly, $\mathcal{H}^K_{\Rightarrow\Box}$ can be defined in **Hil** as follows:

- $\vdash \xi \Rightarrow (\xi' \Rightarrow \xi)$
- $\vdash (\xi \Rightarrow (\xi' \Rightarrow \xi'')) \Rightarrow ((\xi \Rightarrow \xi') \Rightarrow (\xi \Rightarrow \xi''))$
- $\vdash ((\xi \Rightarrow \xi') \Rightarrow \xi) \Rightarrow \xi$
- $\vdash \Box(\xi \Rightarrow \xi') \Rightarrow (\Box\xi \Rightarrow \Box\xi')$
- $\dfrac{\xi \Rightarrow \xi' \quad \xi}{\xi'}$
- $\dfrac{\vdash \xi}{\vdash \Box\xi}$ (NEC)

Here, (NEC) is a global rule, that is, it only applies to theorems. If $\mathcal{H}^K_{\neg\Rightarrow\Box}$, the fibring in **Hil** of $\mathcal{H}^K_{\Rightarrow\Box}$ and $\mathcal{H}_\neg$ is obtained, then K it is not recovered, because $\xi \Rightarrow (\neg\xi \Rightarrow \xi')$ is not a theorem of $\mathcal{H}^K_{\neg\Rightarrow\Box}$. To prove this, the matrices of Example 5.7 can be taken again and it can be defined $\Box(x) := x$ for every $x \in \{1, \frac{6}{7}, \frac{5}{7}, \frac{4}{7}, \frac{3}{7}, \frac{2}{7}, \frac{1}{7}, 0\}$.

On the other hand, it can be considered the assertion calculus $\mathcal{A}^K_{\Rightarrow\Box}$ obtained from $\mathcal{A}_\Rightarrow$ (see the end of Example 5.3) by adding the following assertion rules:

$$\frac{X \succ \Box(\xi \Rightarrow \xi') \qquad Y \succ \Box\xi}{X, Y \succ \Box\xi'} \qquad\qquad \frac{\succ \xi}{\succ \Box\xi}$$

Clearly, this assertion calculus represents the $\{\Rightarrow, \Box\}$-fragment of K. Then the fibring $\mathcal{A}^K_{\neg\Rightarrow\Box} := \mathcal{A}^K_{\Rightarrow\Box} \oplus \mathcal{A}_\neg$ (see Example 5.1) in **Mcon** produces an assertion calculus for modal logic K. □

Example 5.11 (Modal logic T.)
Consider modal logic T over the signature consisting of connectives $\neg, \Rightarrow, \Box, \Diamond$. The positive fragment of T can be described by adding to the assertion calculus $\mathcal{A}^K_{\Rightarrow\Box}$ (see Example 5.10) the following assertion rule:

$$\frac{X \succ \xi}{X \succ \Diamond\xi}$$

Let $\mathcal{A}^T_{\Rightarrow\Box}$ be the resulting assertion calculus. Then modal logic T (in the given signature) can be recovered through the fibring $\mathcal{A}^T_{\neg\Rightarrow\Box} := \mathcal{A}^T_{\Rightarrow\Box} \oplus \mathcal{A}_\neg$, plus the following interaction rules:

$$\frac{X \succ \Diamond\xi}{X \succ \neg\Box\neg\xi} \qquad\qquad \frac{X \succ \neg\Box\neg\xi}{X \succ \Diamond\xi}$$

$\Box$

In the last example, the fibring in **Mcon** was not enough to recover a given logic (modal logic T) from two fragments of it. This is a different situation than the ones described in other examples above, in which a given logic was recovered by the meta-fibring of its fragments. In the present case, what is missing is a definitional axiom $\Diamond\varphi \Leftrightarrow \neg\Box\neg\varphi$. This kind of axioms hardly could be obtained by fibring, even by meta-fibring: note that there is no information about $\Diamond$ in the modal positive fragment of T allowing to infer the definability of $\Diamond$ in terms of $\Box$ and $\neg$ when classical negation is added.

6 Constrained fibring in Mcon

In [18], together with the notion of (categorial) fibring as a coproduct, a more sophisticated form of categorial fibring called *constrained fibring* was introduced, which generalizes the latter. The idea is that some connectives in the given logics can be shared during the process of fibring. This is a very frequent situation, in which it can be combined, for instance, certain modal classical logic with some paraconsistent logic. In this case, it should be reasonable to share the implication $\Rightarrow$, the disjunction $\vee$ and the conjunction $\wedge$, since these connectives usually have classical properties in paraconsistent logics.

In this section, we show that it is possible to define constrained fibring within the category **Mcon** and then the notion of fibring introduced in Section 4 will appear as a particular case. The construction follows exactly the same steps as that of [18].

Firstly, the *forgetful functor* N : **Mcon** $\to$ **Sig** is introduced, which is defined in the obvious manner: $N(\langle C, \mathcal{R}\rangle) = C$ and $N(h) = h$ if $h : \mathcal{A} \to \mathcal{A}'$. We recall now the following notion from category theory:

Definition 6.1 *Given a functor $F : \mathcal{C} \to \mathcal{D}$, a* cocartesian lifting *of a morphism $f : F(c) \to d$ in $\mathcal{D}$ is a morphism $f^* : c \to c'$ in $\mathcal{C}$ such that $F(f^*) = f$ and* *satisfies the following universal property: for every morphism $g : c \to c''$ in $\mathcal{C}$, and every morphism $h : d \to F(c'')$ in $\mathcal{D}$ verifying $h \circ f = F(g)$, there is a unique morphism $h^* : c' \to c''$ in $\mathcal{C}$ with $F(h^*) = h$ and $h^* \circ f^* = g$. The functor F is said to be a* cofibration *if every morphism $f : F(c) \to d$ in $\mathcal{D}$ admits a cocartesian lifting.*

Proposition 6.2 The forgetful functor N is a cofibration, that is: every morphism $h : N(\mathcal{A}) \to C'$ in **Sig** admits a cocartesian lifting.

Proof: Let $\mathcal{A} = \langle C, \mathcal{R} \rangle$ be a assertion calculus and let $h : C \to C'$ be a signature morphism. The assertion calculus $\mathcal{A}' = \langle C', \mathcal{R}' \rangle$ such that

$$\mathcal{R}' := \hat{h}(\mathcal{R}) = \{\hat{h}(r) \ : \ r \in \mathcal{R}\}$$

(recall Definition 3.2) is the codomain of the cocartesian lifting $h : \mathcal{A} \to \mathcal{A}'$ of h through N. The details are left to the reader. ■

Given a signature morphism $h : C \to C'$ and a assertion calculus $\mathcal{A} = \langle C, \mathcal{R} \rangle$ defined over the signature C, we denote by $h_N(\mathcal{A})$ the codomain of the cocartesian lifting of h through N. That is, $h_N(\mathcal{A}) = \langle C', \mathcal{R}' \rangle$, where $\mathcal{R}'$ is as in the proof of Proposition 6.2.

A signature morphism $h : C \to C'$ is called *literal* if, for every $n \in \mathbb{N}$ and every $c \in C_n$, there exists $c' \in C'_n$ such that $h(c) = c'(\xi_1, \ldots, \xi_n)$. Literal signature morphisms correspond to the usual signature morphisms in categorial fibring, that is, to morphisms in the slice category $\mathbf{Set}/\mathbb{N}$ (see, for instance, [18]). It is easy to see that a literal signature morphism $h : C \to C'$ is a monomorphism iff the restriction of h to C_n is an injective function, for every $n \in \mathbb{N}$.

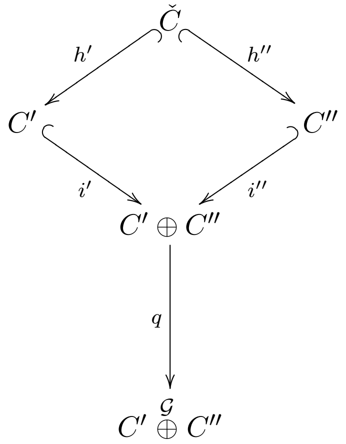
Definition 6.3 Let C' and C'' two signatures. A *sharing constraint* over C' and C'' is a source diagram $\mathcal{G}$ in **Sig** of the form

$$C' \xleftarrow{h'} \check{C} \xrightarrow{h''} C''$$

for some signature $\check{C}$ and signature monomorphisms h' and h'', such that both h' and h'' are literal. The pushout of the diagram $\mathcal{G}$ (if it exists) it will denoted by $C' \overset{\mathcal{G}}{\oplus} C''$. □

From category theory it is known that a pushout can be obtained as a coproduct followed by a coequalizer, provided that these constructions exist in the given category.

Then, given a sharing constraint $\mathcal{G}$ in **Sig**, consider the diagram



where i' and i'' are the canonical injections of the coproduct $C' \oplus C''$, and $C' \overset{\mathcal{G}}{\oplus} C''$ is the codomain of the coequalizer q of $i' \circ h'$ and $i'' \circ h''$ (provided it exists). Therefore

$$\langle C' \overset{\mathcal{G}}{\oplus} C'', \{q \circ i', q \circ i''\}\rangle$$

is the pushout of $\mathcal{G}$ in **Sig**. The following result, whose proof is left to the reader, states that in fact there exists the desired coequalizers in **Sig**:

Proposition 6.4 (1) Let $C' \xleftarrow{h'} \check{C} \xrightarrow{h''} C''$ be a sharing constraint in **Sig** and let $\langle C' \oplus C'', \{i', i''\}\rangle$ be the coproduct of C', C'' in **Sig**. Then $i' \circ h'$ and $i'' \circ h''$ are both literal.
(2) Let $h, h' : C \to C'$ be literal morphisms in **Sig**. Then there exists the coequalizer $\langle C'', \{C' \overset{h''}{\to} C''\}\rangle$ of $C \underset{h'}{\overset{h}{\rightrightarrows}} C'$ in **Sig**.

Using this, the constrained fibring in **Mcon** is defined as follows:

Definition 6.5 With notation as above, let $\mathcal{A}' = \langle C', \mathcal{R}'\rangle$ and $\mathcal{A}'' = \langle C'', \mathcal{R}''\rangle$ be two assertion calculi and let $\mathcal{G}$ be a sharing constraint over C' and C''. Then, their *$\mathcal{G}$-constrained fibring by sharing symbols* is the assertion calculus

$$\mathcal{A}' \overset{\mathcal{G}}{\oplus} \mathcal{A}'' := q_N(\mathcal{A}' \oplus \mathcal{A}''),$$

where q is the coequalizer in **Sig** of $i' \circ h'$ and $i'' \circ h''$. □

Note that Definition 6.5 makes sense because of Proposition 6.4 and because $N(\mathcal{A}' \oplus \mathcal{A}'') = C' \oplus C''$. As usual, the unconstrained fibring can be obtained as a special case of the constrained fibring by taking an appropriate sharing constraint: it is enough to take $\check{C}$ as the initial signature C^0 such that $C^0_n = \emptyset$ for every $n \in \mathbb{N}$; $h' : C^0 \to C'$ and $h'' : C^0 \to C''$ are the obvious (unique) morphisms.

Example 6.6 Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be assertion calculi for two modal logics extending propositional classical logic, defined over signatures C^1 and C^2 with support $\{\neg_1, \Rightarrow_1, \Box_1\}$ and $\{\neg_2, \Rightarrow_2, \Box_2\}$, respectively. Assume that we want to compute the fibring of $\mathcal{A}_1$ with $\mathcal{A}_2$ in **Mcon**. Since both negations and both implications are assumed to be classical, it make sense to share these connectives. Thus, let $\mathcal{G}$ be the sharing constraint $C^1 \xleftarrow{h_1} \check{C} \xrightarrow{h_2} C^2$ in **Sig** such that $\check{C}_1 = \{\neg\}$, $\check{C}_2 = \{\Rightarrow\}$ and $\check{C}_n = \emptyset$ in any other case; $h_j(\neg) = \neg_j \xi_1$ and $h_j(\Rightarrow) = (\xi_1 \Rightarrow_j \xi_2)$ for $j = 1, 2$. This ensures that the $\mathcal{G}$-constrained fibring by sharing symbols $\mathcal{A}_1 \overset{\mathcal{G}}{\oplus} \mathcal{A}_2$ is a bimodal logic defined over a signature with support $\{\neg, \Rightarrow, \Box_1, \Box_2\}$ in which $\neg$ and $\Rightarrow$ are classical. □

7 Propositional sequent calculi

Note that, since assertions are formed by sets, an assertion calculus $\mathcal{A}$ satisfies automatically the meta-property of *exchanging*:

$$\frac{A; \Gamma, \varphi, \psi \succ \Delta; B}{A; \Gamma, \psi, \varphi \succ \Delta; B} \qquad \frac{A, X, Y; \Gamma \succ \Delta; B}{A, Y, X; \Gamma \succ \Delta; B}$$

as well as the same properties to the right-hand side of the assertions.

However, this feature sometimes is not desirable in a sequent calculus. Moreover, in substructural logics (such as linear logic) sequents are formed by multisets or even finite sequences of formulas more than finite sets of formulas. Thus, in order to describe sequent calculi in general, it is necessary to change the notion of general assertion.

Definition 7.1 A *general sequent* over C is an expression $\langle \Sigma | \Psi \rangle$ where Σ and Ψ are two (not simultaneously empty) finite sequences in $L(C) \cup \mathcal{X}$. A *sequent* over C is a expression $\langle \Sigma | \Psi \rangle$ where Σ and Ψ are two (not simultaneously empty) finite sequences in $L(C)$. As done above, sometimes we will write $\Sigma \succ \Psi$ instead of $\langle \Sigma | \Psi \rangle$. Let $GenS(C)$ and $Seq(C)$ be the sets of general sequents and of sequents over C, respectively. □

From now on, we will use commas to indicate the concatenation of finite sequences. Thus, we can write, for instance the following general sequent:

$$\Sigma, \varphi, \psi, X \succ \psi, \Psi, \Sigma', Y, Z$$

(where $\varphi, \psi \in L(C)$; $X, Y, Z \in \mathcal{X}$ and Σ, Ψ, Σ' are finite sequences in $L(C) \cup \mathcal{X}$) with the obvious meaning.

Definition 7.2 Let C be a signature. A *sequent rule* over C is a pair $r = \langle \Upsilon, \langle \Sigma | \Psi \rangle \rangle$ such that $\Upsilon \cup \{\langle \Sigma | \Psi \rangle\}$ is a finite subset of $GenS(C)$. If $\Upsilon = \emptyset$ then r is called an *axiom*. A rule r can be written as $r = \langle prem(r), conc(r) \rangle$, where $prem(r)$ and $conc(r)$ are the *premises* and the *conclusion* of the rule r, respectively. A *sequent calculus* (over C) is a pair $\mathcal{S} = \langle C, \mathcal{R} \rangle$ such that C is a signature and $\mathcal{R}$ is a set of sequent rules over C. □

Now it is necessary to redefine the notion of instantiation. A *sequent instantiation* over C is a map $\varrho : \mathcal{X} \to (L(C) \cup \mathcal{X})^*$, where $(L(C) \cup \mathcal{X})^*$ denotes the set of finite sequences in $L(C) \cup \mathcal{X}$. If $\varrho(X) \in L(C)^*$ for every $X \in \mathcal{X}$ (that is, if no variables occur in $\varrho(X)$) then ϱ is called a *basic sequent instantiation* over C. Given a substitution σ and an instantiation ϱ, a map $(\sigma, \varrho) : GenS(C) \to GenS(C)$ is defined as follows: for $s \in L(C) \cup \mathcal{X}$ let

$$\overline{\sigma}(s) = \begin{cases} \hat{\sigma}(s) & \text{if } s \in L(C) \\ \\ s & \text{if } s \in \mathcal{X} \end{cases} .$$

Now, for $s \in L(C) \cup \mathcal{X}$, suppose that $\varrho(s) = s_1 \dots s_k$. Then $(\sigma, \varrho)(s) = \overline{\sigma}(s_1) \dots \overline{\sigma}(s_k)$. Finally,

$$(\sigma, \varrho)(\langle s_1, \dots, s_n | s'_1, \dots, s'_m \rangle) = \langle (\sigma, \varrho)(s_1), \dots, (\sigma, \varrho)(s_n) | (\sigma, \varrho)(s'_1), \dots, (\sigma, \varrho)(s'_m) \rangle.$$

From these definitions, it is easy to introduce the notion of *derivation* in a sequent calculus. This notion is similar to the concept of derivation in an assertion calculus introduced in Definition 2.4, but now using pairs (σ, ϱ) applied to the rules, where σ is a substitution and ϱ is a sequent instantiation. The notion of derivation from sets of premises is defined as expected:

Definition 7.3 Let $\mathcal{S} = \langle C, \mathcal{R} \rangle$ be a sequent calculus, and let $\Upsilon \cup \{\Sigma \succ \Psi\}$ be a finite set of general sequents over C. We say that $\Sigma \succ \Psi$ is *derivable in* $\mathcal{S}$ *from* Υ if there is a finite sequence $\langle \Sigma_1 | \Psi_1 \rangle \dots \langle \Sigma_n | \Psi_n \rangle$ in $GenS(C)$ such that $\langle \Sigma_n | \Psi_n \rangle = \langle \Sigma | \Psi \rangle$ and, for every $1 \leq i \leq n$, either $\langle \Sigma_i | \Psi_i \rangle \in \Upsilon$ or there is a rule r in $\mathcal{R}$, a substitution σ and an instantiation ϱ over C such that:

- $(\sigma, \varrho)(prem(r)) \subseteq \{\langle \Sigma_1 | \Psi_1 \rangle, \dots, \langle \Sigma_{i-1} | \Psi_{i-1} \rangle\}$; and
- $\langle \Sigma_i | \Psi_i \rangle = (\sigma, \varrho)(conc(r))$.

We also say that $\mathcal{S}$ has the *derived rule*:

$$\texttt{if } \Sigma_1 \Longrightarrow_{\mathcal{S}} \Psi_1, \ \dots, \ \Sigma_n \Longrightarrow_{\mathcal{S}} \Psi_n \ \texttt{ then } \Sigma \Longrightarrow_{\mathcal{S}} \Psi$$

if $\Sigma \succ \Psi$ is derivable in $\mathcal{S}$ from the set $\{\Sigma_1 \succ \Psi_1, \dots, \Sigma_n \succ \Psi_n\}$ of general sequents. □

If a general sequent $\Sigma \succ \Psi$ is derivable in a sequent calculus $\mathcal{S}$ we will write $\Sigma \Longrightarrow_{\mathcal{S}} \Psi$. And if $\Sigma \succ \Psi$ is derived in $\mathcal{S}$ from a set $\{\langle \Sigma_1 | \Psi_1 \rangle, \dots, \langle \Sigma_n | \Psi_n \rangle\}$ of general sequents we will write

$$\frac{\Sigma_1 \succ \Psi_1 \quad \dots \quad \Sigma_n \succ \Psi_n}{\Sigma \succ \Psi} .$$

Remark 7.4 As in the case for general assertions, given substitutions σ, σ' over C and sequent instantiations ϱ, ϱ' over C, it is possible to characterize the (set-theoretic) composite $(\sigma, \varrho) \circ (\sigma', \varrho') := (\sigma \cdot \sigma', \varrho \cdot \varrho')$ as follows: $\sigma \cdot \sigma'$

is the substitution over C such that $\sigma \cdot \sigma'(\xi) = \hat{\sigma}(\sigma'(\xi))$ for $\xi \in \Xi$. On the other hand, $\varrho \cdot \varrho'$ is the instantiation over C such that, if $\varrho'(X) = s_1 \dots s_k$ then $\varrho \cdot \varrho'(X) = \overline{\varrho}(s_1), \dots, \overline{\varrho}(s_k)$ where, for $s \in L(C) \cup \mathcal{X}$,

$$\overline{\varrho}(s) = \begin{cases} \varrho(s) & \text{if } s \in \mathcal{X} \\ \\ s & \text{if } s \in L(C) \end{cases}.$$

Then $(\sigma{\cdot}\sigma', \varrho{\cdot}\varrho')(\langle \Sigma | \Psi \rangle) = (\sigma, \varrho)((\sigma', \varrho')(\langle \Sigma | \Psi \rangle))$ for every $\langle \Sigma | \Psi \rangle \in \mathit{GenS}(C)$. □

The following result of structurality of sequent calculi (analogous to Proposition 2.6 concerning assertion calculi) follows easily using the characterization of composition introduced in Remark 7.4:

Proposition 7.5 Let $\{\langle \Sigma_1 | \Psi_1 \rangle, \dots, \langle \Sigma_n | \Psi_n \rangle, \langle \Sigma | \Psi \rangle\}$ be a finite set of general sequent over C, and let $\mathcal{S}$ be a sequent calculus over C. It holds: if $\langle \Sigma | \Psi \rangle$ is derived in $\mathcal{S}$ from $\{\langle \Sigma_1 | \Psi_1 \rangle, \dots, \langle \Sigma_n | \Psi_n \rangle\}$ then $(\sigma, \varrho)\langle \Sigma | \Psi \rangle$ is derived in $\mathcal{S}$ from $\{(\sigma, \varrho)(\langle \Sigma_1 | \Psi_1 \rangle), \dots, (\sigma, \varrho)(\langle \Sigma_n | \Psi_n \rangle)\}$, for every substitution σ and every sequent instantiation ϱ over C.

Let $\langle s_1, \dots, s_n | s'_1, \dots, s'_m \rangle$ be a general sequent over a signature C, and let $h : C \to C'$ be a signature morphism. Then we define

$$\hat{h}(\langle s_1, \dots, s_n | s'_1, \dots, s'_m \rangle) := \langle \hat{h}(s_1), \dots, \hat{h}(s_n) | \hat{h}(s'_1), \dots, \hat{h}(s'_m) \rangle$$

such that $\hat{h}(X) := X$ if $X \in \mathcal{X}$. Thus, given a sequent rule r over C then $\hat{h}(r)$ is the sequent rule $\langle \hat{h}(\mathit{prem}(r)), \hat{h}(\mathit{conc}(r)) \rangle$ over C'. Using this, it is possible to adapt Definition 3.3 and define a morphism $h : \mathcal{S} \to \mathcal{S}'$ between sequent calculi as being a signature morphism $h : C \to C'$ such that

$$\frac{\hat{h}(\mathit{prem}(r))}{\hat{h}(\mathit{conc}(r))}$$

is a derived rule of $\mathcal{S}'$ for every rule r of $\mathcal{S}$. This defines a category of sequent calculi called **Seq**.

Obviously, an analogous of Theorem 3.4 can be obtained in **Seq**:

Theorem 7.6 Let $h : \mathcal{S}_1 \to \mathcal{S}_2$ be a morphism in **Seq**. Then h preserves every derived rule of $\mathcal{S}_1$, that is: if

$$\frac{\Sigma_1 \succ \Psi_1 \quad \dots \quad \Sigma_n \succ \Psi_n}{\Sigma \succ \Psi}$$

is a derived rule of $\mathcal{S}_1$ then

$$\frac{\hat{h}(\Sigma_1 \succ \Psi_1) \quad \dots \quad \hat{h}(\Sigma_n \succ \Psi_n)}{\hat{h}(\Sigma \succ \Psi)}$$

is a derived rule of $\mathcal{S}_2$.

Of course, the results about fibring assertion calculi obtained in the sections above can be adapted adequately: the constrained fibring in **Seq** is defined using the forgetful functor $F : \mathbf{Seq} \to \mathbf{Sig}$ such that $F(\langle C, \mathcal{R}\rangle) = C$ and $F(h) = h$ if $h : \mathcal{A} \to \mathcal{A}'$. In fact, as in Proposition 6.2, it is easy to prove that the functor F is a cofibration: given $\mathcal{S} = \langle C, \mathcal{R}\rangle$ and a morphism $h : F(\mathcal{S}) \to C'$ in **Sig** then $h_F(\mathcal{S}) := \langle C', \hat{h}(\mathcal{R})\rangle$ is the codomain of the cocartesian lifting $h : \mathcal{S} \to h_F(\mathcal{S})$ of h through F. The unconstrained fibring is defined analogously to the case of assertion calculi. Thus, we arrive to the following result:

Proposition 7.7 There exist both forms of fibring in **Seq**: if $\mathcal{S}'$ and $\mathcal{S}''$ are two sequent calculi then their unconstrained fibring $\mathcal{S}' \oplus \mathcal{S}''$ is the coproduct in **Seq** of $\mathcal{S}'$ and $\mathcal{S}''$, which is defined analogously to the construction of Definition 4.2. And given a sharing diagram $\mathcal{G}$ in **Sig** then the $\mathcal{G}$-constrained fibring by sharing symbols of $\mathcal{S}'$ and $\mathcal{S}''$ is the sequent calculus

$$\mathcal{S}' \overset{\mathcal{G}}{\oplus} \mathcal{S}'' := q_F(\mathcal{S}' \oplus \mathcal{S}'')$$

defined analogously to the construction of Definition 6.5.

8 Semantics

In this section we address the definition of a semantics for general sequents. As a particular case, a semantics for general assertions will follow.

Definition 8.1 Let C be a signature. A *C-structure* (or *a structure over* C) is a tuple

$$M = \langle D, [\![\cdot]\!]_M, \mathsf{R}, \otimes, \odot, \top, \bot\rangle$$

such that:

- D is set such that $\top, \bot \in D$;
- $[\![\cdot]\!]_M : L(C) \to D$ is a function;
- $\mathsf{R} \subseteq D \times D$ is a relation;
- $\otimes : D \times D \to D$ is a function such that $\langle D, \otimes, \top\rangle$ is a monoid;
- $\odot : D \times D \to D$ is a function such that $\langle D, \odot, \bot\rangle$ is a monoid.

The class of C-structures will be denoted by $Str(C)$. □

Definition 8.2 Let M be a C-structure. Let σ and ϱ be a substitution and a basic sequent instantiation over C, respectively.
(a) The maps $[\![\cdot]\!]_M^{(\sigma,\varrho)_L} : L(C) \cup \mathcal{X} \to D$ and $[\![\cdot]\!]_M^{(\sigma,\varrho)_R} : L(C) \cup \mathcal{X} \to D$ are defined as follows:

- if $\varphi \in L(C)$ then $[\![\varphi]\!]_M^{(\sigma,\varrho)_L} = [\![\varphi]\!]_M^{(\sigma,\varrho)_R} = [\![\hat{\sigma}(\varphi)]\!]_M$;

- if $X \in \mathcal{X}$ and $\varrho(X) = \varphi_1 \ldots \varphi_n$ then

  $[\![X]\!]_M^{(\sigma,\varrho)_L} = [\![\hat{\sigma}(\varphi_1)]\!]_M \otimes \cdots \otimes [\![\hat{\sigma}(\varphi_n)]\!]_M$ and

  $[\![X]\!]_M^{(\sigma,\varrho)_R} = [\![\hat{\sigma}(\varphi_1)]\!]_M \odot \cdots \odot [\![\hat{\sigma}(\varphi_n)]\!]_M$.

(b) We say that M *satisfies a general sequent* $\langle s_1, \ldots, s_n | s'_1, \ldots, s'_m \rangle$ *at* (σ, ϱ), denoted by $M \models_{\sigma,\varrho} \langle s_1, \ldots, s_n | s'_1, \ldots, s'_m \rangle$, if

$$([\![s_1]\!]_M^{(\sigma,\varrho)_L} \otimes \cdots \otimes [\![s_n]\!]_M^{(\sigma,\varrho)_L}) \ \mathsf{R}\ ([\![s'_1]\!]_M^{(\sigma,\varrho)_R} \odot \cdots \odot [\![s'_m]\!]_M^{(\sigma,\varrho)_R}).$$

And we say that M *satisfies a general sequent* $\langle \Sigma | \Psi \rangle$, denoted by $M \models \langle \Sigma | \Psi \rangle$, if $M \models_{\sigma,\varrho} \langle \Sigma | \Psi \rangle$ for every (σ, ϱ).
(c) Let r be a sequent rule. We say that M *satisfies* r *at* (σ, ϱ), denoted by $M \models_{\sigma,\varrho} r$, if the following holds:

if $M \models_{\sigma,\varrho} \langle \Sigma | \Psi \rangle$ for every $\langle \Sigma | \Psi \rangle \in prem(r)$ then $M \models_{\sigma,\varrho} conc(r)$.

(d) Let $\mathcal{S} = \langle C, \mathcal{R} \rangle$ be a sequent calculus over C. We say that M *is a model of* $\mathcal{S}$, denoted by $M \models \mathcal{S}$, if $M \models_{\sigma,\varrho} r$ for every $r \in \mathcal{R}$ and every (σ, ϱ). The class of models of $\mathcal{S}$ will be denoted by $Mod(\mathcal{S})$. Note that $Mod(\mathcal{S}) \subseteq Str(C)$. □

The definition of C-structure is very ample, and allows to take into consideration diverse interpretations of the rules. Note that, in particular, no restrictions are imposed to R. In principle, it would seem that R should be a pre-order. This is related to the fact that the sequent calculi should satisfy reflexivity and cut rule, but it is not always the case.

By virtue of its generality, the notion of C-structure allows to represent well-known adequate semantics for several logic systems, as the following examples show.

Examples 8.3 (1) Consider a signature C such that $C_0 = \{\top, \bot\}$; $C_1 = \{\neg\}$; $C_2 = \{\Rightarrow, \vee, \wedge\}$ and $C_n = \emptyset$ in any other case. The class of *standard C-structures for propositional classical logic* is the class $\mathcal{M}_{CL}$ of C-structures of the form

$$M = \langle B, [\![\cdot]\!]_M, \leq, \wedge, \vee, \top, \bot \rangle$$

such that $\langle B, \leq, \wedge, \vee, \top, \bot \rangle$ is a Boolean algebra and $[\![\cdot]\!]_M : L(C) \to B$ is a homomorphism of C-algebras.
(2) Consider the signature C of item (1). The class of *standard C-structures for propositional intuitionistic logic* is the class $\mathcal{M}_{IL}$ of C-structures of the form

$$M = \langle H, [\![\cdot]\!]_M, \leq, \wedge, \vee, \top, \bot \rangle$$

such that $\langle H, \leq, \wedge, \vee, \top, \bot \rangle$ is a Heyting algebra and $[\![\cdot]\!]_M : L(C) \to H$ is a homomorphism of C-algebras.
(3) Consider a signature C such that $C_0 = \{\mathbf{1}, \bot, \top, \mathbf{0}\}$; $C_1 = \{(\cdot)^\perp, !, ?\}$; $C_2 = \{\multimap, \oplus, \&, \otimes, \sqcup\}$ and $C_n = \emptyset$ in any other case. The class of *standard*

C-structures for propositional classical linear logic is the class $\mathcal{M}_{CLL}$ of C-structures of the form (see [21, 9]):

$$M_\mu = \langle Q, [\![\cdot]\!]_{M_\mu}, \leq, \otimes, \sqcup, \mathbf{1}, \bot \rangle$$

such that

- $\langle Q, \leq, \wedge, \vee, \otimes, \bot, \top, \mathbf{0} \rangle$ is a Girard quantale with cyclic dualizing element $\bot$, supremum $\vee$, infimum $\wedge$, top element $\top$ and bottom element $\mathbf{0}$;
- $x^\bot := (x \multimap \bot)$ (where $x \multimap \cdot$ is the right adjoint of the endomorphism $x \otimes \cdot : Q \to Q$), $\mathbf{1} := \bot^\bot$ and $x \sqcup y := (x^\bot \otimes y^\bot)^\bot$ for every $x, y \in Q$;
- $\mu : Q \to Q$ is an open modality in Q;
- $[\![\cdot]\!]_{M_\mu} : L(C) \to Q$ is a homomorphism of C-algebras such that
  - $[\![\varphi \oplus \psi]\!]_{M_\mu} = [\![\varphi]\!]_{M_\mu} \vee [\![\psi]\!]_{M_\mu}$ and $[\![\varphi \& \psi]\!]_{M_\mu} = [\![\varphi]\!]_{M_\mu} \wedge [\![\psi]\!]_{M_\mu}$;
  - $[\![!\varphi]\!]_{M_\mu} = \mu([\![\varphi]\!]_{M_\mu})$ and $[\![?\varphi]\!]_{M_\mu} = (\mu([\![\varphi]\!]^\bot_{M_\mu}))^\bot$.

(4) Consider a signature C such that $C_1 = \{\neg, \Box\}$; $C_2 = \{\Rightarrow\}$ and $C_n = \emptyset$ in any other case. The class of *standard C-structures for propositional modal logic S4* is the class $\mathcal{M}_{S4}$ of C-structures of the form

$$M_R = \langle \wp(W), [\![\cdot]\!]_{M_R}, \subseteq, \cap, \cup, W, \emptyset \rangle$$

such that W is a nonempty set, $\wp(W)$ is the power set of W, $\langle W, R \rangle$ is a reflexive, transitive Kripke frame and $[\![\cdot]\!]_{M_R} : L(C) \to \wp(W)$ is a homomorphism of C-algebras such that

- $[\![\neg\varphi]\!]_{M_R} = W \setminus [\![\varphi]\!]_{M_R}$;
- $[\![\varphi \Rightarrow \psi]\!]_{M_R} = (W \setminus [\![\varphi]\!]_{M_R}) \cup [\![\psi]\!]_{M_R}$;
- $[\![\Box\varphi]\!]_{M_R} = \{w \in W \ : \ w' \in [\![\varphi]\!]_{M_R}$ for every $w \in W$ such that $wRw'\}$. □

Using the semantical notions introduced above, it is possible to define the concept of semantical consequence in a sequent calculus.

Definition 8.4 Let $\mathcal{M}$ be a class of C-structures.
(a) Let $\Upsilon \cup \{\Sigma \succ \Psi\}$ be finite set of general sequents over C. We say that $\langle \Sigma | \Psi \rangle$ *is a consequence of* Υ *in* $\mathcal{M}$, denoted by $\Upsilon \models^{\mathcal{M}} \langle \Sigma | \Psi \rangle$, if $M \models_{\sigma,\varrho} \langle \Upsilon, \langle \Sigma | \Psi \rangle \rangle$ for every $M \in \mathcal{M}$ and every (σ, ϱ) over C.
(b) We say that a general sequent $\langle \Sigma, \Psi \rangle$ over C is *valid in* $\mathcal{M}$, denoted by $\models^{\mathcal{M}} \langle \Sigma, \Psi \rangle$, if $M \models \langle \Sigma, \Psi \rangle$ for every $M \in \mathcal{M}$. That is, if $\langle \Sigma, \Psi \rangle$ is a consequence of $\emptyset$ in $\mathcal{M}$. □

Definition 8.5 Let $\mathcal{S}$ be a sequent calculus over C and $\mathcal{M} \subseteq Str(C)$.
(a) $\mathcal{S}$ is said to be *sound for* $\mathcal{M}$ if, for every finite subset $\Upsilon \cup \{\langle \Sigma | \Psi \rangle\}$ of $Seq(C)$: if $\Sigma \succ \Psi$ is derivable in $\mathcal{S}$ from Υ then $\Upsilon \models^{\mathcal{M}} \langle \Sigma | \Psi \rangle$.
(b) $\mathcal{S}$ is said to be *complete for* $\mathcal{M}$ if, for every finite subset $\Upsilon \cup \{\langle \Sigma | \Psi \rangle\}$ of $Seq(C)$: if $\Upsilon \models^{\mathcal{M}} \langle \Sigma | \Psi \rangle$ then $\Sigma \succ \Psi$ is derivable in $\mathcal{S}$ from Υ.
(c) $\mathcal{S}$ is said to be *adequate for* $\mathcal{M}$ if it is sound and complete for $\mathcal{M}$. □

The next result justifies the definition of soundness stated above:

Proposition 8.6 $\mathcal{S}$ is sound for $\mathcal{M}$ iff $\mathcal{M} \subseteq Mod(\mathcal{S})$.

Proof: In order to prove the 'only if' part, let r be a rule of $\mathcal{S}$, let σ be a substitution, let $M \in \mathcal{M}$ and let ϱ be a basic sequent instantiation over C such that $M \models_{\sigma,\varrho} prem(r)$. Since $(\sigma, \varrho)(conc(r)) \in Seq(C)$ is derivable in $\mathcal{S}$ from $(\sigma, \varrho)(prem(r))$ then $(\sigma, \varrho)(prem(r)) \models^{\mathcal{M}} (\sigma, \varrho)(conc(r))$, by soundness of $\mathcal{S}$. In particular, taking $\sigma'(\xi) = \xi$ for every $\xi \in \Xi$ and $\varrho'(X) = \xi_1$ for every $X \in \mathcal{X}$ it follows that $M \models_{\sigma',\varrho'} (\sigma, \varrho)(conc(r))$, because $M \in \mathcal{M}$ is such that $M \models_{\sigma',\varrho'} (\sigma, \varrho)(prem(r))$. Therefore $M \models_{\sigma,\varrho} conc(r)$. This shows that $\mathcal{M} \subseteq Mod(\mathcal{S})$.

Suppose now that $\mathcal{M} \subseteq Mod(\mathcal{S})$ and let $\Upsilon \cup \{\langle \Sigma | \Psi \rangle\}$ be a finite subset of $Seq(C)$ such that $\Sigma \succ \Psi$ is derivable in $\mathcal{S}$ from Υ. Let (σ, ϱ) and $M \in \mathcal{M}$ such that $M \models_{\sigma,\varrho} \langle \Sigma' | \Psi' \rangle$ for every $\langle \Sigma' | \Psi' \rangle \in \Upsilon$. By induction on the length l of a derivation in $\mathcal{S}$ of $\Sigma \succ \Psi$ from Υ it will be proved that $M \models_{\sigma,\varrho} \langle \Sigma | \Psi \rangle$. If $l = 1$ then there are two cases: either $\langle \Sigma | \Psi \rangle = (\sigma', \varrho')(\langle \Sigma' | \Psi' \rangle)$ (for some axiom $\langle \emptyset, \langle \Sigma' | \Psi' \rangle \rangle$ of $\mathcal{S}$ and some (σ', ϱ')) or $\langle \Sigma | \Psi \rangle \in \Upsilon$. Clearly $M \models_{\sigma,\varrho} \langle \Sigma | \Psi \rangle$ in both cases. Suppose now that the result is true for every sequent derived in $\mathcal{S}$ from Υ in l steps, and assume that $\langle \Sigma | \Psi \rangle$ is derived in $\mathcal{S}$ from Υ in $l + 1$ steps. If $\langle \Sigma | \Psi \rangle$ is obtained from an axiom of $\mathcal{S}$ or if it is an element of Υ the result is obviously true. Suppose that $\langle \Sigma | \Psi \rangle$ is obtained from a rule r of $\mathcal{S}$ such that $\langle \Sigma | \Psi \rangle = (\sigma', \varrho')(conc(r))$ for some (σ', ϱ'). Then $M \models_{\sigma,\varrho} (\sigma', \varrho')(\langle \Sigma'' | \Psi'' \rangle)$ for every $\langle \Sigma'' | \Psi'' \rangle \in prem(r)$, by induction hypothesis. That is, $M \models_{\sigma \cdot \sigma', \varrho \cdot \varrho'} \langle \Sigma'' | \Psi'' \rangle$ for every $\langle \Sigma'' | \Psi'' \rangle \in prem(r)$ (recall the characterization of composition given in Remark 7.4). Since $M \models_{\sigma \cdot \sigma', \varrho \cdot \varrho'} r$ (by hypothesis) it follows that $M \models_{\sigma \cdot \sigma', \varrho \cdot \varrho'} conc(r)$. That is, $M \models_{\sigma,\varrho} \langle \Sigma | \Psi \rangle$. ■

Examples 8.7 Recall Examples 8.3. Then, if we consider the usual sequent calculi for each logic, then they are sound and complete for the corresponding class of C-structures. In the case of propositional classical linear logic, it must be considered a two-sided sequent calculus (see [21, 9]). □

9 Completeness preservation by fibring

In this section a natural notion of logic system is introduced, and a general theorem of completeness preservation by fibring is obtained. The steps to be followed are in line with standard proofs of completeness preservation by categorial fibring (see, for instance, [22, 10]).

Definition 9.1 Let C be a signature. A *logic system* (over C) is a tuple $\mathcal{L} = \langle C, \mathcal{M}, \mathcal{R}\rangle$ such that $\langle C, \mathcal{R}\rangle$ is a sequent calculus over C and $\mathcal{M} \subseteq \mathit{Str}(C)$. A logic system $\mathcal{L}$ is said to be:

- *sound* if $\mathcal{M} \subseteq \mathit{Mod}(\langle C, \mathcal{R}\rangle)$ (that is, if $\langle C, \mathcal{R}\rangle$ is sound for $\mathcal{M}$);
- *full* if $\mathcal{M} = \mathit{Mod}(\langle C, \mathcal{R}\rangle)$;
- *complete* if $\langle C, \mathcal{R}\rangle$ is complete for $\mathcal{M}$;
- *adequate* if $\langle C, \mathcal{R}\rangle$ is adequate for $\mathcal{M}$. □

The following result, which states a sufficient condition for obtaining completeness, will be useful:

Theorem 9.2 If a logic system $\mathcal{L}$ is full then it is complete.

Proof: Assume that $\mathcal{L} = \langle C, \mathcal{M}, \mathcal{R}\rangle$ is full, that is, $\mathcal{M} = \mathit{Mod}(\mathcal{S})$, where $\mathcal{S} = \langle C, \mathcal{R}\rangle$. Let $\Upsilon \cup \{\Sigma \succ \Psi\}$ be a finite subset of $\mathit{Seq}(C)$ such that $\Sigma \succ \Psi$ is not derivable in $\mathcal{S}$ from Υ. Consider the C-structure

$$M_c = \langle L(C)^*, [\![\cdot]\!]_{M_c}, \mathsf{R}, \star, \star, \varepsilon, \varepsilon\rangle$$

such that:

- $L(C)^*$ is the set of finite sequences in $L(C)$;
- ε is the empty sequence in $L(C)$;
- $[\![\cdot]\!]_{M_c} : L(C) \to L(C)^*$ is given by $[\![\varphi]\!]_{M_c} = \varphi$ (considered as a finite sequence formed exactly by the formula φ);
- $\mathsf{R} \subseteq L(C)^* \times L(C)^*$ is the following relation: $\Sigma\,\mathsf{R}\,\Psi$ iff $\langle \Sigma|\Psi\rangle$ is derivable in $\mathcal{S}$ from Υ;
- $\star : L(C)^* \times L(C)^* \to L(C)^*$ is the concatenation function.

Clearly, M_c is a C-structure because $\langle L(C)^*, \star, \varepsilon\rangle$ is a monoid. Moreover, $M_c \in \mathit{Mod}(\mathcal{S})$ and then $M_c \in \mathcal{M}$. Now, take the identity substitution σ such that $\sigma(\xi) = \xi$ (for every $\xi \in \Xi$) and take the basic sequent instantiation ϱ such that $\varrho(X) = \xi_1$ (for every $X \in \mathcal{X}$). Then $M_c \models_{\sigma,\varrho} \Upsilon$ but it is not the case that $M_c \models_{\sigma,\varrho} \langle \Sigma|\Psi\rangle$, because $\langle \Sigma|\Psi\rangle$ is not derivable in $\mathcal{S}$ from Υ. Therefore $\langle \Sigma|\Psi\rangle$ is not a consequence of Υ in $\mathcal{M}$. ■

In order to define the notion of morphism between logic systems, it is necessary to introduce the following concept.

Definition 9.3 Let $M' = \langle D, [\![\cdot]\!]_{M'}, \mathsf{R}, \otimes, \odot, \top, \bot\rangle$ be a structure in $\mathit{Str}(C')$ and let $h : C \to C'$ be a morphism in **Sig**. The *reduct* of M' along h, denoted by $M'|_h$, is the C-structure $M'|_h = \langle D, [\![\cdot]\!]_{M'|_h}, \mathsf{R}, \otimes, \odot, \top, \bot\rangle$ such that $[\![\cdot]\!]_{M'|_h} = [\![\cdot]\!]_{M'} \circ \hat{h}$. That is, $[\![\varphi]\!]_{M'|_h} = [\![\hat{h}(\varphi)]\!]_{M'}$ for every $\varphi \in L(C)$. □

Lemma 9.4 Let $M' \in Str(C')$ and $h : C \to C'$ in **Sig**.
(1) Given a substitution σ and a basic sequent instantiation ϱ over C let σ' and ϱ' be the substitution and the basic sequent instantiation over C' defined, respectively, as follows: $\sigma'(\xi) = \hat{h}(\sigma(\xi))$, for every $\xi \in \Xi$, and $\varrho'(X) = \hat{h}(\varphi_1)\ldots\hat{h}(\varphi_k)$ if $\varrho(X) = \varphi_1 \ldots \varphi_k$, for every $X \in \mathcal{X}$. Then $[\![s]\!]^{(\sigma,\varrho)_L}_{M'|_h} = [\![\hat{h}(s)]\!]^{(\sigma',\varrho')_L}_{M'}$ and $[\![s]\!]^{(\sigma,\varrho)_R}_{M'|_h} = [\![\hat{h}(s)]\!]^{(\sigma',\varrho')_R}_{M'}$ for every $s \in L(C) \cup \mathcal{X}$.
(2) Let $\langle\Sigma|\Psi\rangle \in GenS(C)$. Given a substitution σ and a basic sequent instantiation ϱ over C let σ' and ϱ' as in item (1). Then $M'|_h \models_{\sigma,\varrho} \langle\Sigma|\Psi\rangle$ iff $M' \models_{\sigma',\varrho'} \hat{h}(\langle\Sigma|\Psi\rangle)$.
(3) Let r be a sequent rule over C. Then $M' \in Mod(\hat{h}(r))$ implies that $M'|_h \in Mod(r)$.

Proof: (1) Since $\hat{\sigma}'(\hat{h}(\varphi)) = \hat{h}(\hat{\sigma}(\varphi))$ for every $\varphi \in L(C)$ the result follows easily from the definitions.
(2) Immediate from the definitions, using item (1).
(3) Suppose that $M' \in Mod(\hat{h}(r))$ for a given sequent rule r over C. Let (σ, ϱ) such that $M'|_h \models_{\sigma,\varrho} prem(r)$. Consider σ' and ϱ' as in item (1). Then $M' \models_{\sigma',\varrho'} \hat{h}(prem(r))$, by item (2) and then $M' \models_{\sigma',\varrho'} \hat{h}(conc(r))$, by hypothesis. Using again item (2) it follows that $M'|_h \models_{\sigma,\varrho} conc(r)$. ■

Corollary 9.5 Let $M' \in Str(C')$ and $h : C \to C'$ in **Sig**. Let $\langle C, \mathcal{R}\rangle$ be a sequent calculus over C. Then: $M' \in Mod(\langle C', \hat{h}(\mathcal{R})\rangle)$ implies that $M'|_h \in Mod(\langle C, \mathcal{R}\rangle)$.

Inspired by [10] it is defined the following:

Definition 9.6 Let $\mathcal{L} = \langle C, \mathcal{M}, \mathcal{R}\rangle$ and $\mathcal{L}' = \langle C', \mathcal{M}', \mathcal{R}'\rangle$ be two logic systems. A *logic system morphism* $h : \mathcal{L} \to \mathcal{L}'$ is a signature morphism $h : C \to C'$ such that:

1. $h : \langle C, \mathcal{R}\rangle \to \langle C', \mathcal{R}'\rangle$ is a morphism in **Seq**;
2. for every $M' \in Str(C')$: $M' \in \mathcal{M}'$ implies $M'|_h \in \mathcal{M}$;
3. for every $M' \in \mathcal{M}'$: $M'|_h \in Mod(\langle C, \mathcal{R}\rangle)$ implies $M' \in Mod(\langle C', \hat{h}(\mathcal{R})\rangle)$.

□

Definition 9.7 The category **Log** of logic systems is defined as follows: its objects are logic systems (cf. Definition 9.1), and its morphisms are logic systems morphisms (cf. Definition 9.6). Composition of morphisms and identity maps are defined as in **Sig**. □

Now (unconstrained) fibring of logic systems will be defined, showing that they coincide with coproducts in **Log**. Again, the following notion is adapted from [10].

Definition 9.8 Let $\mathcal{L}' = \langle C', \mathcal{M}', \mathcal{R}' \rangle$ and $\mathcal{L}'' = \langle C'', \mathcal{M}'', \mathcal{R}'' \rangle$ be two logic systems. The *unconstrained fibring* of $\mathcal{L}'$ and $\mathcal{L}''$ is the logic system $\mathcal{L} = \langle C, \mathcal{M}, \mathcal{R} \rangle$ such that:

- $C = C' \oplus C''$ is the coproduct in **Sig** of C' and C'' with canonical injections $i' : C' \to C$ and $i'' : C'' \to C$;
- $\mathcal{R} = \hat{i}'(\mathcal{R}') \cup \hat{i}''(\mathcal{R}'')$ is the set of sequent rules of the coproduct $\langle C', \mathcal{R}' \rangle \oplus \langle C'', \mathcal{R}'' \rangle$ in **Seq**;
- $\mathcal{M} = \{M \in Str(C) \ : \ M|_{i'} \in \mathcal{M}' \text{ and } M|_{i''} \in \mathcal{M}'' \text{ and}$
  $M|_{i'} \in Mod(\langle C', \mathcal{R}' \rangle) \text{ implies } M \in Mod(\langle C, \hat{i}'(\mathcal{R}') \rangle) \text{ and}$
  $M|_{i''} \in Mod(\langle C'', \mathcal{R}'' \rangle) \text{ implies } M \in Mod(\langle C, \hat{i}''(\mathcal{R}'') \rangle)\}$.

The unconstrained fibring of $\mathcal{L}'$ and $\mathcal{L}''$ will be denoted by $\mathcal{L}' \oplus \mathcal{L}''$. □

Proposition 9.9 With notation as in Definition 9.8, $\mathcal{L} = \mathcal{L}' \oplus \mathcal{L}''$ is the coproduct in **Log** of $\mathcal{L}'$ and $\mathcal{L}''$ with canonical injections obtained from $i' : C' \to C$ and $i'' : C'' \to C$.

Proof: By Definition 9.8, the canonical injections are morphisms $i' : \mathcal{L}' \to \mathcal{L}$ and $i'' : \mathcal{L}'' \to \mathcal{L}$ in **Log**. Let $\check{\mathcal{L}} = \langle \check{C}, \check{\mathcal{M}}, \check{\mathcal{R}} \rangle$ be a logic system and let $j' : \mathcal{L}' \to \check{\mathcal{L}}$ and $j'' : \mathcal{L}'' \to \check{\mathcal{L}}$ be morphisms in **Log**. Then, there is a (unique) morphism $h : C \to \check{C}$ such that $h \circ i' = j'$ and $h \circ i'' = j''$. It is necessary to prove that h is a morphism in **Log**. By definition of fibring in **Seq** the morphism h is a morphism in **Seq**, then condition (1) of Definition 9.6 is satisfied. Let $\check{M} \in \check{\mathcal{M}}$ and let $M = \check{M}|_h$. Then $M|_{i'} = \check{M}|_{j'} \in \mathcal{M}'$, because j' is a morphism in **Log**. Analogously it is proved that $M|_{i''} = \check{M}|_{j''} \in \mathcal{M}''$. Suppose now that $M|_{i'} \in Mod(\langle C', \mathcal{R}' \rangle)$. Since $h \circ i' = j'$ is a morphism in **Log** then $\check{M} \in Mod(\langle \check{C}, \hat{h}(\hat{i}'(\mathcal{R}')) \rangle)$ and so $M \in Mod(\langle C, \hat{i}'(\mathcal{R}') \rangle)$, by Corollary 9.5. Analogously, if $M|_{i''} \in Mod(\langle C'', \mathcal{R}'' \rangle)$ then $M \in Mod(\langle C, \hat{i}''(\mathcal{R}'') \rangle)$. Thus $M \in \mathcal{M}$ and so h satisfies condition (2) of Definition 9.6. Now, suppose that $\check{M} \in \check{\mathcal{M}}$ such that $\check{M}|_h \in Mod(\langle C, \mathcal{R} \rangle)$. Then $\check{M}|_h \in Mod(\langle C, \hat{i}'(\mathcal{R}') \rangle)$ and so $\check{M}|_{j'} \in Mod(\langle C', \mathcal{R}' \rangle)$, by Corollary 9.5 and because $h \circ i' = j'$. Since j' is a morphism in **Log** it follows that $\check{M} \in Mod(\langle \check{C}, \hat{j}'(\mathcal{R}') \rangle)$. That is, $\check{M} \in Mod(\langle \check{C}, \hat{h}(\hat{i}'(\mathcal{R}')) \rangle)$. Analogously, since $\check{M}|_h \in Mod(\langle C, \hat{i}''(\mathcal{R}'') \rangle)$, it can be proved that $\check{M} \in Mod(\langle \check{C}, \hat{h}(\hat{i}''(\mathcal{R}'')) \rangle)$. This means that $\check{M} \in Mod(\langle \check{C}, \hat{h}(\mathcal{R}) \rangle)$ and then h satisfies condition (3) of Definition 9.6. That is, $h : \mathcal{L} \to \check{\mathcal{L}}$ is a morphism in **Log** such that $h \circ i' = j'$ and $h \circ i'' = j''$ in **Log**. The uniqueness of h in **Log** follows from the uniqueness of h in **Sig**. ■

With respect to constrained fibring, it is obtained the following:

Proposition 9.10 Let $G : \mathbf{Log} \to \mathbf{Sig}$ be the forgetful functor (defined in the obvious way). Then G is a cofibration.

Proof: Given a logic system $\mathcal{L} = \langle C, \mathcal{M}, \mathcal{R} \rangle$ and a morphism $h : G(\mathcal{L}) \to C'$ in **Sig**, consider the logic system $h_G(\mathcal{L}) := \langle C', \mathcal{M}', \mathcal{R}' \rangle$ such that $\mathcal{R}' = \hat{h}(\mathcal{R})$ and

$$\mathcal{M}' = \{M' \in Str(C') \;:\; M'|_h \in \mathcal{M}, \text{ and } \\ M'|_h \in Mod(\langle C, \mathcal{R}\rangle) \text{ implies } M' \in Mod(\langle C', \hat{h}(\mathcal{R})\rangle)\}.$$

Clearly $h : \mathcal{L} \to h_G(\mathcal{L})$ is a morphism in **Log**. Let $\check{\mathcal{L}} = \langle \check{C}, \check{\mathcal{M}}, \check{\mathcal{R}}\rangle$ be a logic system, $g : \mathcal{L} \to \check{\mathcal{L}}$ be a morphism in **Log** and $f : C' \to \check{C}$ be a morphism in **Sig** such that $f \circ h = g$ in **Sig**. It is enough to prove that $f : h_G(\mathcal{L}) \to \check{\mathcal{L}}$ is a morphism in **Log**. The first requirement of Definition 9.6 is clearly satisfied by f. The rest of the proof is very similar to the proof of Proposition 9.9. ■

From this, the following result is obtained:

Corollary 9.11 There exists constrained fibring in **Log**: if $\mathcal{L}'$ and $\mathcal{L}''$ are two logic systems and $\mathcal{G}$ is a sharing diagram in **Sig** then the $\mathcal{G}$-constrained fibring of $\mathcal{L}'$ and $\mathcal{L}''$ by sharing symbols is the logic system

$$\mathcal{L}' \overset{\mathcal{G}}{\oplus} \mathcal{L}'' := q_G(\mathcal{L}' \oplus \mathcal{L}'')$$

defined analogously to the construction of Definition 6.5.

Theorem 9.12 *(Soundness and Completeness preservation I)*
Let $\mathcal{L}' = \langle C', \mathcal{M}', \mathcal{R}'\rangle$ and $\mathcal{L}'' = \langle C'', \mathcal{M}'', \mathcal{R}''\rangle$ be two logic systems, and let $\mathcal{L} = \langle C, \mathcal{M}, \mathcal{R}\rangle$ be the unconstrained fibring of $\mathcal{L}'$ and $\mathcal{L}''$. Then:
(1) If both $\mathcal{L}'$ and $\mathcal{L}''$ are sound then $\mathcal{L}$ is sound.
(2) If both $\mathcal{L}'$ and $\mathcal{L}''$ are full (and then complete) then $\mathcal{L}$ is full, and then complete.

Proof: (1) Suppose that $\mathcal{M}' \subseteq Mod(\langle C', \mathcal{R}'\rangle)$ and $\mathcal{M}'' \subseteq Mod(\langle C'', \mathcal{R}''\rangle)$, and let $M \in \mathcal{M}$. Then, by Definition 9.8, $M|_{i'} \in \mathcal{M}'$, that is, $M|_{i'} \in Mod(\langle C', \mathcal{R}'\rangle)$ and so $M \in Mod(\langle C, \hat{\imath}'(\mathcal{R}')\rangle)$. Analogously it is proved that $M \in Mod(\langle C, \hat{\imath}''(\mathcal{R}'')\rangle$. Then $M \in Mod(\langle C, \mathcal{R}\rangle)$ and so $\mathcal{L}$ is sound.
(2) Suppose that $\mathcal{M}' = Mod(\langle C', \mathcal{R}'\rangle)$ and $\mathcal{M}'' = Mod(\langle C'', \mathcal{R}''\rangle)$. Then, by Definition 9.8,

$$\mathcal{M} = \{M \in Str(C) \;:\; M|_{i'} \in \mathcal{M}' \text{ and } M|_{i''} \in \mathcal{M}'' \text{ and} \\ M|_{i'} \in Mod(\langle C', \mathcal{R}'\rangle) \text{ implies } M \in Mod(\langle C, \hat{\imath}'(\mathcal{R}')\rangle) \text{ and} \\ M|_{i''} \in Mod(\langle C'', \mathcal{R}''\rangle) \text{ implies } M \in Mod(\langle C, \hat{\imath}''(\mathcal{R}'')\rangle)\}.$$

Thus, by hypothesis,

$$\mathcal{M} = \{M \in Str(C) \;:\; M|_{i'} \in Mod(\langle C', \mathcal{R}'\rangle) \text{ and } M|_{i''} \in Mod(\langle C'', \mathcal{R}''\rangle) \text{ and} \\ M|_{i'} \in Mod(\langle C', \mathcal{R}'\rangle) \text{ implies } M \in Mod(\langle C, \hat{\imath}'(\mathcal{R}')\rangle) \text{ and} \\ M|_{i''} \in Mod(\langle C'', \mathcal{R}''\rangle) \text{ implies } M \in Mod(\langle C, \hat{\imath}''(\mathcal{R}'')\rangle)\}.$$

Therefore

$$\mathcal{M} = \{M \in Str(C) \;:\; M|_{i'} \in Mod(\langle C', \mathcal{R}'\rangle) \text{ and } M|_{i''} \in Mod(\langle C'', \mathcal{R}''\rangle) \text{ and} \\ M \in Mod(\langle C, \hat{\imath}'(\mathcal{R}')\rangle) \text{ and } M \in Mod(\langle C, \hat{\imath}''(\mathcal{R}'')\rangle)\},$$

that is, $\mathcal{M} = Mod(\langle C, \mathcal{R}\rangle)$, by Corollary 9.5. Then $\mathcal{L}$ is full and so it is complete. ■

Lemma 9.13 Let $\mathcal{L} = \langle C, \mathcal{M}, \mathcal{R}\rangle$ be a logic system, and let $h : C \to C'$ be a morphism in **Sig**. Then:
(1) If $\mathcal{L}$ is sound then $h_G(\mathcal{L})$ is sound.
(2) If $\mathcal{L}$ is full (and then complete) then $h_G(\mathcal{L})$ is full, and then complete.

Proof: Analogous to the proof of Theorem 9.12, but now using the definition of $\mathcal{M}$ given in the proof of Proposition 9.10. ■

Theorem 9.14 *(Soundness and Completeness preservation II)*
Let $\mathcal{L}' = \langle C', \mathcal{M}', \mathcal{R}'\rangle$ and $\mathcal{L}'' = \langle C'', \mathcal{M}'', \mathcal{R}''\rangle$ be two logic systems, let $\mathcal{G}$ be a sharing diagram in **Sig** and let $\mathcal{L}$ be the $\mathcal{G}$-constrained fibring of $\mathcal{L}'$ and $\mathcal{L}''$ by sharing symbols. Then:
(1) If both $\mathcal{L}'$ and $\mathcal{L}''$ are sound then $\mathcal{L}$ is sound.
(2) If both $\mathcal{L}'$ and $\mathcal{L}''$ are full (and then complete) then $\mathcal{L}$ is full, and then complete.

Proof: Immediate, from Lemma 9.13. ■

From the results above, it can be seen that both forms of fibring always preserve soundness. On the other hand, if the logic systems are complete because they are full, then both forms of fibring preserve fullness and then completeness.

10 Concluding Remarks

In this paper the problem of recovering a logic by fibring of its fragments was addressed. To this end, the usual notion of logic morphism (i.e., a translation which preserves deductions) was substituted for a stronger one, ensuring that meta-properties of the form

`If` $\Gamma_1 \vdash \varphi_1$ `and` $\ldots$ `and` $\Gamma_n \vdash \varphi_n$ `then` $\Gamma \vdash \varphi$

are preserved, provided that the variable symbols (for formulas and sets of formulas) occurring in the meta-property are kept fixed (cf. Theorem 3.4). The importance of this kind of meta-properties in the analysis of logics from the point of view of Universal Logic was already studied in [1]. The problem of preservation of general meta-properties of a logic system by morphisms was also addressed in [8], under a different perspective.

The basic framework underlying our analysis was a formal meta-language of rules appropriated for describing sequent calculi, together with two categories associated to it: **Mcon**, in which the formulas are grouped through sets (encompassing a large class of sequent calculi), and **Seq**, in which the formulas are grouped through finite sequences and, in particular, multisets (if suitable structural rules are added). The latter category is appropriate for substructural sequent calculi, and it is a generalization of the former, which basically define multiple-conclusion consequence relations.

It was proved that there exist both forms of fibring in these categories: unconstrained (that is, no symbols are shared) and constrained (by sharing symbols). Both kind of fibrings are called *meta-fibrings*, because meta-properties are preserved. It is worth noting that the concept of (categorial) fibring used in **Mcon** and **Seq** is the same as in [18]: the unique difference is the notion of morphism between calculi we adopt, which allows to preserve meta-properties.

By introducing a general semantics for sequent calculi, the category **Log** of logic systems was defined and, by using natural notions of soundness and completeness, it was proved that both forms of fibring preserve completeness (provided that the logic systems are full). On the other hand, it was proved that soundness is always preserved by both forms of fibring.

It should be noted that, in general, semantical aspects of a given logic system are mainly useful for finding counter-examples (i.e., impossibility of certain derivations): is under this perspective that the rather general semantical structures for sequent calculi should be considered.

Together with the technical development concerning the conceptual framework mentioned above, some interesting examples of recovering logics from its fragments were offered, in conformity with our objectives. The choice of meta-fibring instead of fibring was shown to be essential for attaining this goal.

A possible explanation for this phenomenon is that sequent calculi or natural deduction calculi are defined in a modular way, by describing the features of each connective independently of the others. Thus, it is possible to define separately the logic of each connective (possibly with the adding of structural rules that do not depend on any connective), and then recover the whole logic simply by putting together all the rules. This is basically what was done in the examples given above. On the other hand, Hilbert calculi, by its nature, are defined from mixed axioms. Take as an illustrative example the well-known Hilbert calculi for classical propositional logic over $\{\neg, \Rightarrow\}$ due to Hilbert and Bernays, and popularized by [17]. This systems is formed by the first two axioms given in Example 5.7 above together with *Modus Ponens* and the mixed axiom $(\neg\xi \Rightarrow \neg\xi') \Rightarrow ((\neg\xi \Rightarrow \xi') \Rightarrow \xi)$. The presence of a mixed axiom (that is, a formula involving both connectives) is unavoidable in order to obtain classical propositional logic over this signature, as shown in Example 5.7.

However, it should be noted that some logics cannot be recovered from single-connective logics: modal logics, for instance, must be defined by rules describing the behavior of the modalities in terms of their interactions with the other connectives. That is, rules involving more than one connective (mixed rules) are necessary. Moreover, in some cases the meta-fibring of the fragments is not enough to recover a modal logic, as was shown in Example 5.11.

This paper should be considered as a first attempt to solve the problem of recovering a logic from its fragments, from the point of view of fibring. As it can be observed, several important question remain open. In the first place, it is no clear how a mixed formula (an interaction between connectives) could be generated from the different fragments in a controlled way. On the other hand, Example 5.11 shows that it is not always possible to obtain a mixed formula from the logics of their connectives. A detailed study of this issue will contribute to determine the limitations and the scope of meta-fibring as a tool for recovering a logic from its fragments.

In a different line of research, it would be interesting to study the case for first-order sequent calculi, in which rules have provisos. In order to do this, the notion of proviso (and its management) given in [10] could be adapted.

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