Plain Fibring and Direct Union of Logics with Matrix Semantics

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Abstract. In this paper a variation of the fibred semantics of D. Gabbay called *plain fibring* is proposed, with the aim of combining logics given by matrix semantics. It is proved that the plain fibring of matrix logics is also a matrix logic. Moreover, it is proved that any logic obtained by plain fibring is a conservative extension of the original logics. It is also proposed a simpler version of plain fibring of matrix logics called *direct union*. This technique is applied to the study of the class of fuzzy logics defined by t-norms.

Introduction

Among the different techniques for combining logics, fibring has deserved the attention of a great number of logicians. This method was introduced by D. Gabbay in the 90's, with the aim of combining logics having Kripke semantics, such as intuitionistic and modal logics (see [15] and [16]). Essentially, the underlying idea of fibring is the following: given two logics $\mathcal{L}_1$ and $\mathcal{L}_2$ (with Kripke semantics Kr_1 and Kr_2 respectively), a new logic $\mathcal{L}_1 \circledast \mathcal{L}_2$ is defined over the language obtained from the union of the connectives of both $\mathcal{L}_1$ and $\mathcal{L}_2$. So, the formulas of $\mathcal{L}_i$ $(i = 1, 2)$ are also formulas of $\mathcal{L}_1 \circledast \mathcal{L}_2$. But in the new logic there exist new formulas (that we call "hybrid formulas"), obtained by mixing the connectives of the two original logics. Thus, it is possible to evaluate a connective c of the language of $\mathcal{L}_1$ applied to formulas of the language of $\mathcal{L}_2$. This can be done by means of a function which associates, to each world of any model of Kr_1, a world of a model of Kr_2. Using this method, the hybrid formulas of the new language can always be evaluated. Moreover, $\mathcal{L}_1 \circledast \mathcal{L}_2$ is a weak extension of $\mathcal{L}_1$ and $\mathcal{L}_2$, in the sense that all the tautologies of the original logics are tautologies of the new logic.

Right after the introduction of fibring in the literature, the original notion was investigated and modified by several authors. In particular, an interesting adaptation of fibring using the framework of Category Theory was introduced in [19] and widely studied afterwards (see for instance [2], [23], [4], [9] and [3]). However, D. Gabbay used the expression "fibring" for another techniques (see the above mentioned [16]), applied to fuzzy logics and temporal logics among others. In the present article the original notion of fibring is extended to another class of logics: the logics characterized by matrix semantics. Thus, in this paper the term "fibring" will mean "Gabbay's fibring". Moreover, one of the techniques to be introduced here will be called *plain fibring* because its inspiration on the original notion of fibring.

Briefly, the method proposed here is defined as follows: given two matrix logics $\mathcal{L}_1$ and $\mathcal{L}_2$, the mixed language is obtained as usual from the given languages of both logics. Then, since each $\mathcal{L}_i$ has a matrix semantics M_i $(i = 1, 2)$, we define functions $f : M_1 \to M_2$ and $g : M_2 \to M_1$ which allow us "to jump" from a valuation of M_1 to a valuation of M_2 and vice-versa. Considering these functions we can obtain the truth-value of any hybrid formula of the mixed language. The class of valuations obtained in this manner define a consequence relation $\vdash_{M_1 \circledast M_2}$, corresponding to the logic that will be denoted by $\mathcal{L}_1 \circledast \mathcal{L}_2$ (the plain fibring of $\mathcal{L}_1$ and $\mathcal{L}_2$).

The following two sections are devoted to introduce the basic notions about propositional logics, matrix semantics and fibring of modal logics.

1 Preliminaries

Definition 1.
(a) *The* set of propositional variables $\mathcal{V}$ *is any countable set (fixed from now on). Its elements will be denoted by* $p, q, r, p_1, p_2, \ldots$ *. A* signature *is a family* $C = \{C^k\}_{k \in \mathbb{N}}$*, where each* C^k *is a set of* connectives of arity k. *We assume that* $C^k \cap C^n = \emptyset = C^k \cap \mathcal{V}$ *for every* $k \neq n$*. The* domain of the signature C *is the set* $|C| = \bigcup_{k \in \mathbb{N}} C^k$*. Given two signatures* C_1 *and* C_2*, we say that* C_1 is included in C_2 *(indicated by* $C_1 \subseteq C_2$*) if, for every* $k \in \mathbb{N}$, $C_1^k \subseteq C_2^k$*. The signatures* $C_1 \cup C_2$ *and* $C_1 \uplus C_2$ *(the union and the disjoint union of* C_1 *and* C_2*, respectively) are defined as expected.*
(b) *A* propositional language with signature C *(denoted by* $L(C)$*) is the algebra of words, freely generated by* C *over* $\mathcal{V}$*, considering each set* C^k *as the set of* k*-ary operations of that algebra.*

It should be noted that it is possible to have $C \neq C'$ such that $L(C) = L(C')$. By simplicity, sometimes we will identify a signature C with its domain $|C|$.

Following the Tarskian perspective, the other component that characterizes a logic is the consequence relation.

Definition 2. *Fixed a signature* C*, a* consequence relation over the signature C *(or* in C*) is a relation* $\vdash \;\subseteq \wp(L(C)) \times L(C)$ *verifying:* [3]

– *If* $\varphi \in \Gamma$ *then* $\Gamma \vdash \varphi$ *(***Extensiveness***).*

[3] As usual, $(\Gamma, \alpha) \in \vdash$ will be denoted by $\Gamma \vdash \alpha$.

– *If $\Gamma \vdash \varphi$ and $\Sigma \vdash \psi$, for every $\psi \in \Gamma$, then $\Sigma \vdash \varphi$* (**Transitivity**).

It is worth noting that any consequence relation defined as above satisfies the following useful property:

– If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Sigma$ then $\Sigma \vdash \varphi$ (**Monotonicity**).

In fact, If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Sigma$ then $\Sigma \vdash \psi$ for every $\psi \in \Gamma$, by **Extensiveness**. Thus $\Sigma \vdash \varphi$, by **Transitivity**.

A *substitution in* C is any function $\sigma : \mathcal{V} \to L(C)$. Since $L(C)$ is freely generated from $\mathcal{V}$, σ can be extended to a unique endomorphism $\widehat{\sigma} : L(C) \to L(C)$. Using this, we arrive to the following definition.

Definition 3. *A* (propositional) logic *is a pair $\mathcal{L} = \langle C_{\mathcal{L}}, \vdash_{\mathcal{L}} \rangle$, where $C_{\mathcal{L}}$ is a signature and $\vdash_{\mathcal{L}}$ is a consequence relation in $C_{\mathcal{L}}$ in the sense of Definition 2. A logic $\mathcal{L}$ is said to be a* structural logic *if it satisfies:*

– *For every substitution σ in $C_{\mathcal{L}}$ and every $\Gamma \cup \{\varphi\} \subseteq L(C_{\mathcal{L}})$:*
If $\Gamma \vdash_{\mathcal{L}} \varphi$ then $\widehat{\sigma}(\Gamma) \vdash_{\mathcal{L}} \widehat{\sigma}(\varphi)$ (**Structurality**).

On the other hand, $\mathcal{L}$ is called a finitary logic *if it satisfies:*

– *For every $\Gamma \cup \{\varphi\} \subseteq L(C_{\mathcal{L}})$:*
If $\Gamma \vdash_{\mathcal{L}} \varphi$ then $\Gamma' \vdash_{\mathcal{L}} \varphi$ for some finite set $\Gamma' \subseteq \Gamma$ (**Finitariness**).

Finally, $\mathcal{L}$ is a standard logic *if it is structural and finitary.*

The following notions will be used from now on:

Definition 4. *Let $\mathcal{L} = \langle C_{\mathcal{L}}, \vdash_{\mathcal{L}} \rangle$ be a logic, and let $C \subseteq C_{\mathcal{L}}$. The* C-fragment of $\mathcal{L}$ *is the logic $\mathcal{L}|_C := \langle C, \vdash_{\mathcal{L}|_C} \rangle$ where $\vdash_{\mathcal{L}|_C} = \vdash_{\mathcal{L}} \cap\ (\wp(L(C)) \times L(C))$. Thus, for every $\Gamma \cup \{\varphi\} \subseteq L(C)$, $\Gamma \vdash_{\mathcal{L}|_C} \varphi$ if and only if $\Gamma \vdash_{\mathcal{L}} \varphi$. Given two logics $\mathcal{L}$ and $\mathcal{L}'$, we will say that $\mathcal{L}'$* is a strong extension of $\mathcal{L}$ *if $C_{\mathcal{L}} \subseteq C_{\mathcal{L}'}$ and $\vdash_{\mathcal{L}} \subseteq \vdash_{\mathcal{L}'}$. The logic $\mathcal{L}'$ is a* weak extension *of $\mathcal{L}$ if $C_{\mathcal{L}} \subseteq C_{\mathcal{L}'}$ and, for every $\varphi \in L(C_{\mathcal{L}})$, $\vdash_{\mathcal{L}} \varphi$ implies that $\vdash_{\mathcal{L}'} \varphi$. We say that $\mathcal{L}'$ is a* conservative extension of $\mathcal{L}$ *if $C_{\mathcal{L}} \subseteq C_{\mathcal{L}'}$ and $\mathcal{L} = \mathcal{L}'|_{C_{\mathcal{L}}}$. Finally, $\mathcal{L}'$ is a* weak conservative extension of $\mathcal{L}$ *if $C_{\mathcal{L}} \subseteq C_{\mathcal{L}'}$ and, for every $\varphi \in L(C)$, $\vdash_{\mathcal{L}} \varphi$ iff $\vdash_{\mathcal{L}'} \varphi$.*

The proof of the following result is straightforward:

Proposition 1. *The following holds:*
(*a*) *Each C-fragment of any (structural, finitary, standard) logic is also a (structural, finitary, standard) logic.*
(*b*) *Every logic $\mathcal{L}$ is a conservative extension of any of its C-fragments.* □

We briefly recall the basic facts about matrix semantics.

Definition 5. *Given a signature C, a* C-matrix *is a pair $M = \langle \mathbf{A}, D \rangle$, where $\mathbf{A} = \langle A, C \rangle$ is an algebra over C,* [4] *and $D \subseteq A$. The set D is usually referred*

[4] That is, we can identify the set of k-ary operations of $\mathbf{A}$ with C^k. Therefore $L(C)$, considered as an algebra, has the same type of similarity than $\mathbf{A}$.

as the set of designated values of M. *The* M-valuations of $L(C)$ *are the* C-*homomorphisms* $v : L(C) \to A$.

For simplicity, sometimes we will write $M = \langle A, D \rangle$ instead of $M = \langle \mathbf{A}, D \rangle$ in concrete examples. Additionally, the interpretation of a connective c in M will be frequently written as c^M, or will be simply written as c.

Definition 6. *Let* C *be a signature and let* $\mathcal{K}$ *be a class of* C*-matrices. The* matrix semantics induced by $\mathcal{K}$ *(denoted by* $\vdash_{\mathcal{K}}$*) is defined by:* $\Gamma \vdash_{\mathcal{K}} \varphi$ *iff, for every* C*-matrix* $M = \langle \mathbf{A_M}, D_M \rangle$ *belonging to* $\mathcal{K}$ *and every* M*-valuation* v *of* $L(C)$, $v(\Gamma) \subseteq D_M$ *implies that* $v(\varphi) \in D_M$.

A logic $\mathcal{L}$ is said to be *matrix logic* if there exists a class $\mathcal{K}$ of $C_{\mathcal{L}}$-matrices such that $\vdash_{\mathcal{L}} = \vdash_{\mathcal{K}}$. When $\mathcal{K} = \{M\}$ is a singleton then $\vdash_M$ will stand for $\vdash_{\{M\}}$. Clearly, every matrix logic is a logic in the sense of Definition 3. Moreover, the following fundamental result due to J. Łoś and R. Suszko (see [18]) shows that a matrix logic is, in fact, structural:

Proposition 2. *Let* C *be a signature, and let* $\mathcal{K}$ *be a class of* C*-matrices. Then* $\vdash_{\mathcal{K}}$ *is a structural consequence relation in* C *and* $\vdash_{\mathcal{K}} = inf\{\vdash_M : M \in \mathcal{K}\}$.[5] $\square$

Note that $\vdash_{\mathcal{K}}$ do not need to be finitary and, therefore, $\mathcal{L} = \langle C, \vdash_{\mathcal{K}} \rangle$ is not necessarily standard. A sufficient condition for a matrix logic be standard was obtained by R. Wójcicki (see [22]):

Proposition 3. *Every consequence relation induced by a finite set of finite matrices is finitary and then it defines a standard logic.* $\square$

2 Fibring of modal logics

In this section we briefly recall the fibred semantics as defined by D. Gabbay, applied to Kripke semantics.

Definition 7. *A* modal signature *is a signature* C *such that* $C^1 = \{\neg, \Box\}$*;* $C^2 = \{\Rightarrow\}$*;* $C^k = \emptyset$ *in any other case. A* Kripke model *(for modal logics) is a triple* $m = \langle W_m, R_m, h_m \rangle$ *such that* W_m *is a nonempty set (the set of* possible-worlds of m*);* $R_m \subseteq W_m \times W_m$ *(the* accessibility relation of m*); and* $h_m : \mathcal{V} \to \wp(W_m)$ *is a mapping (the* m-valuation*). A* Kripke semantics *is a class* Kr *of Kripke models.*

Fixed a modal signature C and a Kripke semantics Kr, a consequence relation $\vdash_{Kr}$ in C can be defined as usual.

[5] The infimum is taken with respect to the inclusion ordering $\subseteq$.

It is worth noting that the consequence relation of a modal logic can be obtained from several Kripke semantics: for instance, the well-known modal logic $\mathbf{S_5}$ can be obtained by means of Kr_1 (the Kripke semantics such that every R_m is an equivalence relation) or by means of Kr_2 (where there is no restrictions to R_m). This is a relevant fact because the fibring of modal logics depends not only on the consequence relations but also on the given Kripke semantics of each logic. Therefore, in the following definition, the modal logics will be denoted by pairs $\mathcal{L} = \langle C_{\mathcal{L}}, Kr\rangle$.

Definition 8 (Gabbay). *Let $\mathcal{L}_i = \langle C_i, Kr_i\rangle$ $(i = 1, 2)$ be two modal logics. The* fibred signature $C_{\circledast}$ of $\mathcal{L}_1$ and $\mathcal{L}_2$ *is defined by:*

$$C^1_{\circledast} = \{\neg, \Box_1, \Box_2\};\ C^2_{\circledast} = \{\Rightarrow\};\ C^k_{\circledast} = \emptyset \quad \textit{in any other case.}$$

The fibred language of C_1 and C_2 *is $L(C_{\circledast})$. A* fibred model of Kr_1 and Kr_2 *is a triple (f, g, h) such that*

$$f : \biguplus_{m\in Kr_1} W_m \rightarrow \biguplus_{m\in Kr_2} W_m; \qquad g : \biguplus_{m\in Kr_2} W_m \rightarrow \biguplus_{m\in Kr_1} W_m; \qquad h : \mathcal{V} \rightarrow \wp(W),$$

where $W := (\biguplus_{m\in Kr_1} W_m) \uplus (\biguplus_{m\in Kr_2} W_m)$. [6] *The* fibred structure of Kr_1 and Kr_2 *(symbolized by $Kr_1 \circledast Kr_2$) is the class of all the fibred models of Kr_1 and Kr_2. The consequence relation $\vdash_{Kr_1\circledast Kr_2} \subseteq \wp(L(C_{\circledast})) \times L(C_{\circledast})$ is defined from the notion of $(f, g, h) \Vdash_w \varphi$ (for $(f, g, h) \in Kr_1 \circledast Kr_2$, $w \in W$ and $\varphi \in L(C_{\circledast})$) as expected, with the following relevant cases: if $w \in W_m$ and $m \in Kr_2$ then $(f, g, h) \Vdash_w \Box_1\psi$ iff $(f, g, h) \Vdash_{w'} \psi$ for every $w' \in W_{m'}$ such that $g(w)R_{m'}w'$ (the dual case is treated analogously, using f instead of g). The logic $\mathcal{L}_{\circledast} = \langle C_{\circledast}, \vdash_{Kr_1\circledast Kr_2}\rangle$ is called* the fibring of $\mathcal{L}_1$ and $\mathcal{L}_2$. [7]

Example 1. Let $\mathcal{L}_1$ be the modal logic $\mathbf{T}$. Then, Kr_1 can be taken as the class of all the models m such that R_m is reflexive. Let $\mathcal{L}_2$ be the von Wright's temporal logic $\mathbf{VW}$ (with $\Box_{\mathbf{VW}}$ meaning "in all future time"). In this case Kr_2 can be taken as the class of all the models m such that $W_m = \mathbb{N}$ and R_m is a total ordering. The fibred signature $C_{\circledast}$ is given by: $C^1_{\circledast} = \{\neg, \Box_{\mathbf{T}}, \Box_{\mathbf{VW}}\}$; $C^2_{\circledast} = \{\Rightarrow\}$ and $C^k_{\circledast} = \emptyset$ in any other case. Let (f, g, h) be a fibred model, $w \in W_m$, with $m \in Kr_1$, and let φ be the formula $\Box_{\mathbf{VW}}(\Box_{\mathbf{T}} q \Rightarrow p)$. Then:
$(f, g, h) \Vdash_w \Box_{\mathbf{VW}}(\Box_{\mathbf{T}} q \Rightarrow p)$ iff $(f, g, h) \Vdash_{w'} \Box_{\mathbf{T}} q \Rightarrow p$ for every $w' \in W_{m'}$ such that $f(w)R_{m'}w'$. On the other hand, given $m' \in Kr_2$ and $w' \in W_{m'}$ then $(f, g, h) \Vdash_{w'} \Box_{\mathbf{T}} q \Rightarrow p$ iff either $w' \in h(p)$ or $w'' \notin h(q)$ for some $w'' \in W_{m''}$ such that $g(w')R_{m''}w''$.

The logic $\mathcal{L}_{\circledast}$ is a *weak extension* of $\mathcal{L}_1$ and $\mathcal{L}_2$ (i.e., it just preserves the tautologies, cf. Definition 4). On the other hand, by the very definition, $\mathcal{L}_{\circledast}$ distinguishes the different modal connectives, but identifies the implication and the negation of both logics as being the same. In the plain fibring of matrix semantics we propose here there are no identifications of connectives.

[6] Here, and in the rest of the paper, $\biguplus$ denote the disjoint union of sets.

[7] Note that $\mathcal{L}_{\circledast}$ is a logic in the sense of Definition 3.

3 Direct union of matrix logics

In the sequel we introduce an adaptation of fibring to logics defined by means of matrix semantics. The basic motivation is to consider the set of worlds as being an analogous to the algebra of truth-values. Similarly to the modal case, the operations to be defined below will depend on the choice of the matrices characterizing the semantics of the given logics. Thus, from now on a matrix logic will be represented as a pair $\langle C, \mathcal{K}\rangle$ instead of a pair $\langle C, \vdash_{\mathcal{K}}\rangle$, for a class $\mathcal{K}$ of C-matrices. In particular, if $\mathcal{K} = \{M\}$ we will write $\langle C, M\rangle$.

The basic idea is that, given two matrix logics $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_i$ is characterized by a single matrix M_i with domain A_i and designated values D_i $(i = 1, 2)$, it is possible to extend the original operators of the algebra M_i to the disjoint union $A_1 \uplus A_2$ by means of mappings $f_i : A_j \to A_i$ $(i \neq j)$.

In this section we briefly analyze the simpler case in which the domain and designated values of the matrices involved are the same. In such cases, the combined logic can be simply obtained by putting together both matrices. This technique is what we call *direct union* of matrix logics. Formally:

Definition 9. *Let $\mathcal{L}_i = \langle C_i, M_i\rangle$ (with $i = 1, 2$) be two matrix logics, where each $M_i = \langle \mathbf{A_i}, D_i\rangle$ is a C_i-matrix. Assume that $A_1 = A = A_2$ and $D_1 = D = D_2$.* [8] *The* direct union *of* $\mathcal{L}_1$ and $\mathcal{L}_2$ *is the logic $\mathcal{L}_1 + \mathcal{L}_2 = \langle C_1 \uplus C_2, \vdash_{M_1+M_2}\rangle$ where $\vdash_{M_1+M_2}$ is the consequence relation defined by the $C_1 \uplus C_2$-matrix $M_1 + M_2 = \langle \mathbf{A}, D\rangle$ such that, if $c \in C_i^k$ and $a_1, \ldots, a_k \in A$, then $c^{M_1+M_2}(a_1, \ldots, a_k) = c^{M_i}(a_1, \ldots, a_k)$ $(i = 1, 2)$.*

Proposition 4. *Let $\mathcal{L} = \langle C, \vdash\rangle$ be a logic such that $\vdash$ is characterized by a C-matrix M. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two fragments of $\mathcal{L}$ defined over signature C_1 and C_2, respectively, such that $C_1 \uplus C_2 = C$. Then $\mathcal{L}_1 + \mathcal{L}_2 = \mathcal{L}$. In particular, the same result holds if $C_1 \cup C_2 = C$.*

Proof. Straightforward, from the definitions. [9] □

The idea behind plain fibring and direct union is not just to decompose a logic into fragments, but also to combine given logics in order to obtain a bigger logic such that the given logics are fragments of it. A related discussion was addressed in [8], where a stronger notion of logical translations was proposed in order to recover a logic from its fragments through the process of (categorial) fibring. On the other hand, in [6] the generic operations of decomposition and composition of logics were called *splitting* and *splicing* logics, respectively. In the next section we will give an example of the direct union of two fuzzy logics defined through t-norms, in order to obtain a bigger logic which contains both

[8] Note that this *does not mean* that the operations defined in M_1 and M_2 coincide.

[9] There is a minor technical detail to be considered in the case that $C_1 \cup C_2 = C$ and $C_1^k \cap C_2^k \neq \emptyset$ for some $k \in \mathbb{N}$. In this case, there will be duplicate connectives in $\mathcal{L}_1 + \mathcal{L}_2$ and so this logic is not identical to $\mathcal{L}$. However, $\mathcal{L}_1 + \mathcal{L}_2$ can be identified with $\mathcal{L}$ up to logical translations.

logics as its fragments. This example is useful in the realm of abstract algebraic logic, as studied in [11].

Example 2. Let $\mathcal{L}_i = \langle C_i, M_i \rangle$ (with $i = 1, 2$) such that $C_1 = \{\neg, \vee\}$ (negation and disjunction, respectively) and $C_2 = \{\wedge, \Rightarrow\}$ (conjunction and implication, respectively). Suppose that M_1 is the matrix for classical negation and disjunction, and that M_2 is the matrix for classical conjunction and implication, where both matrices are defined over $A = \{1, 0\}$ with $D = \{1\}$. Then $\mathcal{L}_1 + \mathcal{L}_2$ turns out to be the matrix presentation $\mathcal{L}$ of classical propositional logic over A and D and signature $\{\neg, \vee, \wedge, \Rightarrow\}$. The logics $\mathcal{L}_1$ and $\mathcal{L}_2$ are two (simpler) factors of $\mathcal{L}$. By its turn, $\mathcal{L}_1$ and $\mathcal{L}_2$ can of course be split into two elementary logics: $\mathcal{L}_1^1$ (the logic of classical negation) and $\mathcal{L}_1^2$ (the logic of classical disjunction), on one hand; and $\mathcal{L}_2^1$ (the logic of classical conjunction) and $\mathcal{L}_2^2$ (the logic of classical implication), on the other hand. That is, $\mathcal{L}_1 = \mathcal{L}_1^1 + \mathcal{L}_1^2$ and $\mathcal{L}_2 = \mathcal{L}_2^1 + \mathcal{L}_2^2$. Therefore, $\mathcal{L}$ splits into $\mathcal{L}_1^1$, $\mathcal{L}_1^2$, $\mathcal{L}_2^1$ and $\mathcal{L}_2^2$, and so $\mathcal{L} = \mathcal{L}_1^1 + \mathcal{L}_1^2 + \mathcal{L}_2^1 + \mathcal{L}_2^2$.

4 Direct union of logics induced by *t*-norms

By the very definition, the direct union of logics deals with matrix logics, say $\mathcal{L}_1$ and $\mathcal{L}_2$, where $\mathcal{L}_i$ is characterized by a single matrix M_i ($i = 1, 2$), such that M_1 and M_2 share the same domain and set of designated truth-values. An interesting example comes from fuzzy logics, which are usually defined by means of a matrix over the interval $[0, 1]$ with 1 as the unique designated value. The logics induced by t-norms constitute a particular case of this (cf. [17]; see also [11] for a study within the context of abstract algebraic logic). In what follows we briefly recall the main definitions and properties of logics induced by t-norms:

Definition 10. *A* t-norm *is a mapping* $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ *satisfying, for any* $x, y, z \in [0, 1]$*:*
(i) $x * y = y * x$ *(commutativity);*
(ii) $x * (y * z) = (x * y) * z$ *(associativity);*
(iii) *If* $x \leq y$ *then* $x * z \leq y * z$ *(monotonicity);*
(iv) $1 * x = x$ *(identity).*

A t-norm is continuous *if it is a continuous mapping with respect to the usual topologies of* $[0, 1]$ *and* $[0, 1] \times [0, 1]$.

The following are trivial consequences of the definition above:
$(iii)'$ If $x \leq y$ then $z * x \leq z * y$;
$(iv)'$ $0 * x = 0$.

Proposition 5. *Let $*$ be a continuous t-norm. Then*

$$x \Rightarrow y := sup\{z \in [0, 1] \ : \ x * z \leq y\}$$

is the unique operation $\Rightarrow : [0, 1] \times [0, 1] \rightarrow [0, 1]$ *satisfying, for every* $x, y, z \in [0, 1]$*:* $x * z \leq y$ *iff* $z \leq x \Rightarrow y$. □

Given a continuous t-norm $*$, the operation $\Rightarrow$ defined above is called *the residuum of* $*$.

Definition 11. *Let* $*$ *be a continuous* t*-norm and let* $\Rightarrow$ *be its residuum. The* logic generated by $*$ *is the logic* $\mathcal{L}(*) = \langle C_{\mathcal{L}(*)}, \vdash_{\mathcal{L}(*)} \rangle$ *defined as follows:*
(i) $C^1_{\mathcal{L}(*)} = \{*\}$, $C^2_{\mathcal{L}(*)} = \{\Rightarrow\}$, $C^k_{\mathcal{L}(*)} = \emptyset$ *in any other case;*
(ii) $\vdash_{\mathcal{L}(*)}$ *is the consequence relation defined by the matrix* $\langle [0,1], \{1\} \rangle$*, where the matrix operations are the mappings* $*$ *and* $\Rightarrow$*.* [10]

Previous to study the direct union of logics induced by t-norms it is necessary to state some technical results, whose proof is left to the reader.

Lemma 1. *For every* t*-norm* $*$ *it holds, for every* $x, y \in [0,1]$*:*
a) $x \Rightarrow x = 1$.
b) $x * y = 1$ *iff* $x = y = 1$.
c) $x \Rightarrow y = 1$ *iff* $x \leq y$. □

We will state below a fundamental result relating different t-norms: their respective residua are interderivable in a certain sense (see Proposition 7).

Definition 12. *Let* $*$ *be a continuous* t*-norm, and let* $\Rightarrow$ *be its correspondent residuum. The* equivalence operator *is given by* $x\Delta y := (x \Rightarrow y) * (y \Rightarrow x)$.

Note that the equivalence operator Δ induces a (derived) connective Δ in the logic $\mathcal{L}(*)$, called the *equivalence connective*: $\varphi\Delta\psi := (\varphi \Rightarrow \psi) * (\psi \Rightarrow \varphi)$. Using this connective, it is possible to obtain a very interesting result (cf. [11] and [7]):

Proposition 6. *Any logic of the form* $\mathcal{L}(*)$ *for a continuous* t*-norm* $*$ *is algebraizable (in the sense of Blok-Pigozzi, cf. [1]).* □

The direct union of logics can therefore be applied to obtain another interesting result in abstract algebraic logic (see again [11] and [7]):

Proposition 7. *Let* $\mathcal{L}(*_i)$ *be the logic induced by a continuous* t*-norm* $*_i$ *(*$i = 1, 2$*). Let* Δ_i *be the equivalence connective of* $\mathcal{L}(*_i)$ *(*$i = 1, 2$*). If* $\mathcal{L}$ *is the direct union* $\mathcal{L}(*_1) + \mathcal{L}(*_2)$ *then* $p\Delta_1 q \vdash_{\mathcal{L}} p\Delta_2 q$ *and* $p\Delta_2 q \vdash_{\mathcal{L}} p\Delta_1 q$ *for every* $p, q \in \mathcal{V}$*, and so* $\mathcal{L}(*_1) + \mathcal{L}(*_2)$ *is algebraizable in the sense of Blok-Pigozzi.* □

A detailed discussion about combination of algebraizable logics can be found in [7] (see also [11], [12] and [14]).

[10] Here, as usual, we use the same symbol for a connective and its matrix interpretation.

5 Unrestricted plain fibring of matrix logics

A more interesting situation is to combine two matrix logics characterized by single matrices defined over different domains. In this case, each matrix logic is extended to the disjoint union of the domains of the matrices by means of a pair of mappings, and then the direct union of the extensions is computed. The set of matrices obtained in this way is the matrix semantics of the so-called *(unrestricted) plain fibring*.

Definition 13. *Let $\mathcal{L}_i = \langle C_i, M_i \rangle$ (with $i = 1, 2$) be two matrix logics, where each $M_i = \langle \mathbf{A_i}, D_i \rangle$ is a C_i-matrix with domain A_i. The* fibred signature *is given by $C_1 \uplus C_2$ and the* fibred language *is $L(C_1 \uplus C_2)$. A* unrestricted fibred valuation *is a triple (f, g, v), where $(f, g) \in A_2^{A_1} \times A_1^{A_2}$ and $v \in (A_1 \uplus A_2)^{\mathcal{V}}$. Given $\varphi \in L(C_1 \uplus C_2)$ and a unrestricted fibred valuation (f, g, v), we define $(f, g, v)(\varphi) \in A_1 \uplus A_2$ by recursion on the complexity of φ:*

- *If $\varphi \in \mathcal{V}$ then $(f, g, v)(\varphi) = v(\varphi)$;*
- *If $\varphi = c(\beta_1, \ldots, \beta_k)$ then $(f, g, v)(\varphi) = c(\overline{(f, g, v)}(\beta_1), \ldots, \overline{(f, g, v)}(\beta_k))$* [11] *where, for every formula β_j ($j = 1, \ldots, k$):*
  - *If $c \in C_i^k$ and $(f, g, v)(\beta_j) \in A_i$ then $\overline{(f, g, v)}(\beta_j) = (f, g, v)(\beta_j)$ (for $i = 1, 2$);*
  - *If $c \in C_1^k$ and $(f, g, v)(\beta_j) \in A_2$ then $\overline{(f, g, v)}(\beta_j) = g((f, g, v)(\beta_j))$;*
  - *If $c \in C_2^k$ and $(f, g, v)(\beta_j) \in A_1$ then $\overline{(f, g, v)}(\beta_j) = f((f, g, v)(\beta_j))$.*

We say that a unrestricted fibred valuation (f, g, v) satisfies *φ if $(f, g, v)(\varphi) \in D_1 \uplus D_2$. The* unrestricted plain fibred consequence relation *$\vdash_{M_1 \circledast M_2} \subseteq \wp(L(C_1 \uplus C_2)) \times L(C_1 \uplus C_2)$ is defined as follows: $\Gamma \vdash_{M_1 \circledast M_2} \varphi$ if, for every unrestricted fibred valuation (f, g, v) satisfying simultaneously all the formulas of Γ, we have that (f, g, v) satisfies φ. The pair $\mathcal{L}_1 \circledast \mathcal{L}_2 = \langle C_1 \uplus C_2, \vdash_{M_1 \circledast M_2} \rangle$ is the* unrestricted plain fibring *of $\mathcal{L}_1$ and $\mathcal{L}_2$.*

Example 3. Consider the paraconsistent matrix logic P^1, introduced in [20]. The signature of P^1 is such that $|C_{P^1}| = \{\neg_{P^1}, \Rightarrow_{P^1}\}$. The P^1-matrix is $M_{P^1} = \langle \mathbf{A}_{P^1}, \{T, T_1\} \rangle$ such that $A_{P^1} = \{T, T_1, F\}$, and the operations corresponding to $\neg_{P^1}$ and $\Rightarrow_{P^1}$ are defined through the tables below.

	T	T_1	F
$\neg_{P^1}$	F	T	T

$\Rightarrow_{P^1}$	T	T_1	F
T	T	T	F
T_1	T	T	F
F	T	T	T

Now, consider the classical propositional logic CPL defined over signature C_{CPL} such that $|C_{CPL}| = \{\wedge_{CPL}, \neg_{CPL}\}$, and with the usual matrix semantics over $\{0, 1\}$. Let $P^1 \circledast CPL$ be the unrestricted plain fibring of P^1 and CPL, let φ be the formula $\Rightarrow_{P^1}(p, \wedge_{CPL}(\neg_{CPL} r, \wedge_{CPL}(q, \neg_{P^1} r)))$ in the fibred language, where $p, q, r \in \mathcal{V}$, and let (f, g, v) be a unrestricted fibred valuation such that:

[11] Again, we use here the same symbol for a connective and its matrix interpretation.

– $v(p) = T$; $v(q) = 0$; $v(r) = T_1$;
– $f(T) = 1$; $f(T_1) = 1$; $f(F) = 0$;
– $g(1) = T$; $g(0) = F$.

Then, by Definition 13,

$$\begin{aligned}
(f, g, v)(\varphi) &= (f, g, v)(\Rightarrow_{P^1}(p, \wedge_{CPL}(\neg_{CPL} r, \wedge_{CPL}(q, \neg_{P^1} r)))) \\
&= \Rightarrow_{P^1}(T, g(\wedge_{CPL}(\neg_{CPL}(f(T_1)), \wedge_{CPL}(0, f(\neg_{P^1}(T_1)))))) \\
&= \Rightarrow_{P^1}(T, g(\wedge_{CPL}(\neg_{CPL}(1), \wedge_{CPL}(0, f(T))))) \\
&= \Rightarrow_{P^1}(T, g(\wedge_{CPL}(0, \wedge_{CPL}(0, 1)))) \\
&= \Rightarrow_{P^1}(T, g(\wedge_{CPL}(0, 0))) \\
&= \Rightarrow_{P^1}(T, g(0)) = \Rightarrow_{P^1}(T, F) = F.
\end{aligned}$$

6 Some results about unrestricted plain fibring

In this section we present some basic results about the logics obtained by means of unrestricted plain fibring.

Firstly, consider the following questions: is $\mathcal{L}_1 \circledast \mathcal{L}_2$ a logic in the sense of Definition 3? Is it structural? Is it finitary? Is it standard? Proposition 8 below gives us some answers.

Definition 14. *Let $\mathcal{L} = \langle C, M \rangle$ be a matrix logic, where M is a C-matrix with domain A and set of designated values $D \subseteq A$. Let A' and D' be two sets such that $D' \subseteq A'$. Suppose, without loss of generality, that $A \cap A' = \emptyset$. Finally, let $f : A' \to A$ be a mapping. The C-matrix M_f is defined as follows: its domain is $A \uplus A'$; the set of designated values is $D \uplus D'$ and, for $c \in C^k$ and $a_1, \ldots, a_k \in A \uplus A'$, $c^{M_f}(a_1, \ldots, a_k) = c^M(\overline{a_1}, \ldots, \overline{a_k})$ where, for every a_j ($j = 1, \ldots, k$):*
- If $a_j \in A$, then $\overline{a_j} = a_j$.
- If $a_j \in A'$, then $\overline{a_j} = f(a_j)$.

Proposition 8. *Let $\mathcal{L}_i = \langle C_i, M_i \rangle$ (with $i = 1, 2$) be two matrix logics, where each M_i is a C_i-matrix. Then:*
(a) $\mathcal{L}_1 \circledast \mathcal{L}_2$ is a matrix logic and so it is a structural logic.
(b) If the matrix M_i is finite ($i = 1, 2$) then $\vdash_{M_1 \circledast M_2}$ is finitary and so $\mathcal{L}_1 \circledast \mathcal{L}_2$ is a standard logic.

Proof. (*a*) The basic idea is to prove the following: $\vdash_{M_1 \circledast M_2}$ can be characterized by a certain set $\mathcal{K}_\circledast$ of $C_1 \uplus C_2$-matrices of the form $M = \langle \mathbf{A}, D_1 \uplus D_2 \rangle$ such that $A = A_1 \uplus A_2$ and, if $a_1, \ldots, a_k \in A_i$ and $c \in C_i^k$ then $c^M(a_1, \ldots, a_k) = c^{M_i}(a_1, \ldots, c_k)$ ($i = 1, 2$). Thus, the matrices of the set $\mathcal{K}_\circledast$ extend the original matrices. Moreover:
(*i*) For each unrestricted fibred valuation (f, g, v) there exists a $C_1 \uplus C_2$-matrix M in $\mathcal{K}_\circledast$ and a valuation v' in M such that $(f, g, v)(\varphi) = v'(\varphi)$ for every formula $\varphi \in L(C_1 \uplus C_2)$.
(*ii*) Conversely, for each $C_1 \uplus C_2$-matrix $M \in \mathcal{K}_\circledast$ and each valuation v in M there is a unrestricted fibred valuation (f, g, v') such that $(f, g, v')(\varphi) = v(\varphi)$ for

every formula $\varphi \in L(C_1 \uplus C_2)$.

It is clear that, from (i) and (ii), the desired result follows easily. We proceed now to define the set $\mathcal{K}_{\circledast}$ of matrices. Given $(f,g) \in A_2^{A_1} \times A_1^{A_2}$ (assuming, without loss of generality, that $A_1 \cap A_2 = \emptyset$) consider the $C_1 \uplus C_2$-matrix $M_{(f,g)} = (M_1)_g + (M_2)_f$ (recall Definitions 9 and 14). Thus, $M_{(f,g)} = \langle \mathbf{A}, D_1 \uplus D_2 \rangle$ where $A = A_1 \uplus A_2$ and the operations are defined as follows:
- If $c \in C_1^k$ and $a_1, \ldots, a_k \in A$, then $c^{M_{(f,g)}}(a_1, \ldots, a_k) = c^{M_1}(\overline{a_1}, \ldots, \overline{a_k})$ where, for every a_j $(j = 1, \ldots, k)$:
  - If $a_j \in A_1$, then $\overline{a_j} = a_j$.
  - If $a_j \in A_2$, then $\overline{a_j} = g(a_j)$.
- If $c \in C_2^k$ and $a_1, \ldots, a_k \in A$, then $c^{M_{(f,g)}}(a_1, \ldots, a_k) = c^{M_2}(\overline{a_1}, \ldots, \overline{a_k})$ where, for every a_j $(j = 1, \ldots, k)$:
  - If $a_j \in A_2$, then $\overline{a_j} = a_j$.
  - If $a_j \in A_1$, then $\overline{a_j} = f(a_j)$.

Let $\mathcal{K}_{\circledast} = \{M_{(f,g)} \ : \ (f,g) \in A_2^{A_1} \times A_1^{A_2}\}$. Now we will prove that $\mathcal{K}_{\circledast}$ satisfies property (i). Given a unrestricted fibred valuation (f,g,v), consider the matrix $M_{(f,g)}$ and the valuation v' defined in $M_{(f,g)}$ such that $v'(p) = v(p)$ for every $p \in \mathcal{V}$. It is straightforward to see that, for every $\varphi \in L(C_1 \uplus C_2)$, $v'(\varphi) = (f,g,v)(\varphi)$.
In order to prove that $\mathcal{K}_{\circledast}$ satisfies (ii), let $M_{(f,g)} \in \mathcal{K}_{\circledast}$ and let v be a valuation in $M_{(f,g)}$. Then we define $v'(p) = v(p)$ for every $p \in \mathcal{V}$ and so the unrestricted fibred valuation (f,g,v') satisfies the requirements.

From the considerations above we have that $\vdash_{M_1 \circledast M_2}$ is defined by a class of matrices. Using Proposition 2 it follows that $\vdash_{M_1 \circledast M_2}$ is structural.
(b) Follows from item (a) and Proposition 3. □

The result above shows that the unrestricted plain fibring is obtained as follows:
(1) the original logics are extended to the disjoint union of the domains of the matrices using a pair of mappings (f,g) (as in Definition 14); and
(2) the direct union of the extended matrices is computed, obtaining a matrix $M_{(f,g)}$ for the fibred signature.
(3) The unrestricted plain fibring is characterized by the set of all the matrices obtained in this way (by varying the pair (f,g)).

Now we will analyze the relationship between the logic obtained by unrestricted plain fibring and the given logics. Firstly, we obtain two useful lemmas.

Lemma 2. *Let (f,g,v) be a unrestricted fibred valuation, and consider the M_1-valuation $v' : L(C_1) \to A_1$ such that*

$$v'(p) = \begin{cases} g(v(p)) & \text{if } v(p) \in A_2 \\ v(p) & \text{otherwise} \end{cases}$$

for every $p \in \mathcal{V}$. Then $v'(\varphi) = (f, g, v)(\varphi)$ for every $\varphi \in L(C_1) \setminus \mathcal{V}$. An analogous result holds for M_2 (using f instead of g).

Proof. Straightforward, by induction on the complexity of φ and Definition 13. □

Lemma 3. *Let $v : L(C_1) \to A_1$ be a M_1-valuation, and let $v' : \mathcal{V} \to A$ such that $v'(p) = v(p)$ for every $p \in \mathcal{V}$. Then $(f, g, v')(\varphi) = v(\varphi)$ for every $(f, g) \in A_2^{A_1} \times A_1^{A_2}$ and every $\varphi \in L(C_1)$. An analogous result holds for M_2.*

Proof. Again, it follows straightforwardly by induction on the complexity of φ and Definition 13. □

Proposition 9. *Let $\mathcal{L}_i$ be a nontrivial logic [12] induced by the matrix M_i ($i = 1, 2$). Then $\mathcal{L}_1 \circledast \mathcal{L}_2$ is a weak conservative extension of both logics $\mathcal{L}_1$ and $\mathcal{L}_2$, that is: $\vdash_{\mathcal{L}_i} \varphi$ iff $\vdash_{M_1 \circledast M_2} \varphi$, for every $\varphi \in L(C_i)$ ($i = 1, 2$).*

Proof. By hypothesis, no propositional variable is a tautology of $\mathcal{L}_i$ ($i = 1, 2$). In fact, if $\vdash_{\mathcal{L}_i} p$ for some propositional variable then, by structurality, $\vdash_{\mathcal{L}_i} \varphi$ for every formula φ and so, by monotonicity, $\Gamma \vdash_{\mathcal{L}_i} \varphi$ for every $\Gamma \cup \{\varphi\}$. Thus, suppose that $\varphi \in L(C_1)$ is such that $\vdash_{\mathcal{L}_1} \varphi$, and let (f, g, v) be a fibred valuation. Then $\varphi \notin \mathcal{V}$ and so there exists a M_1-valuation v' such that $(f, g, v)(\varphi) = v'(\varphi)$, by Lemma 2. But φ is a M_1-tautology and so $v'(\varphi) \in D_1$. That is, $(f, g, v)(\varphi) \in D$ and then $\vdash_{M_1 \circledast M_2} \varphi$. Analogously, it can be proved that $\vdash_{\mathcal{L}_2} \varphi$ implies that $\vdash_{M_1 \circledast M_2} \varphi$, for every $\varphi \in L(C_2)$.

Conversely, let $\varphi \in L(C_1)$ such that $\vdash_{M_1 \circledast M_2} \varphi$, and let v be a M_1-valuation. Let $v' : \mathcal{V} \to A$ the mapping defined as in Lemma 3, and let $(f, g) \in A_2^{A_1} \times A_1^{A_2}$ (observe that, by hypothesis, both A_1 and A_2 are nonempty, so $A_2^{A_1} \times A_1^{A_2} \neq \emptyset$). Since $(f, g, v')(\varphi) = v(\varphi)$ then $v(\varphi) \in D_1$. Therefore, $\vdash_{\mathcal{L}_1} \varphi$. The proof for $\mathcal{L}_2$ is analogous. □

Note that, if exactly one of the logics (say, $\mathcal{L}_1$) is trivial, then the last result is no longer true. In fact: assuming that $A_1 = D_1 \neq \emptyset$ and $\mathcal{L}_2$ is not trivial then the propositional variable p is a $\mathcal{L}_1$-tautology but not a $\mathcal{L}_2$-tautology. Since $A_2^{A_1} \times A_1^{A_2} \neq \emptyset$ it is easy to define a unrestricted fibred valuation (f, g, v) such that $(f, g, v)(p) = v(p) \in A_2 \setminus D_2$ and then $(f, g, v)(p) \notin D_1 \uplus D_2$.

The Proposition 9 cannot be improved. In fact, $\mathcal{L}_1 \circledast \mathcal{L}_2$ *is not* in general a strong extension of the given logics, as the following example shows.

Example 4. Let $\mathcal{L}_1 = \langle \{\vee\}, \vdash_1 \rangle$ be the disjunction-fragment of the classical propositional logic (induced, therefore, by the matrix $M_1 = \langle \{0, 1\}, \{1\} \rangle$ with its usual truth-table). On the other hand, let $\mathcal{L}_2 = \langle C_2, \vdash_2 \rangle$ be any logic such that $\vdash_2$ is defined by a matrix $M_2 = \langle \{T, T_1, F\}, \{T, T_1\} \rangle$ (the signature C_2 and the operations of M_2 are irrelevant here). Obviously, $p_1 \vdash_1 p_1 \vee p_2$. Now, let $v : \mathcal{V} \to \{0, 1, T, T_1, F\}$ be a mapping such that $v(p_1) = T$, $v(p_2) = 0$, and

[12] A logic is *trivial* if $\Gamma \vdash \varphi$ for every $\Gamma \cup \{\varphi\}$.

let $g : \{T, T_1, F\} \to \{0, 1\}$ such that $g(T) = 0$. Consider now the unrestricted fibred valuation (f, g, v) (where f is any mapping $f : \{0, 1\} \to \{T, T_1, F\}$). Then $(f, g, v)(p_1) = T \in D_1 \uplus D_2$. On the other hand, $(f, g, v)(p_1 \vee p_2) = (f, g, v)(p_1) \vee (f, g, v)(p_2) = g(T) \vee 0 = 0 \vee 0 = 0 \notin D_1 \uplus D_2$. Hence, $p_1 \nvdash_{M_1 \circledast M_2} p_1 \vee p_2$.

7 Plain fibring

A situation as the one described in Example 4 is not desirable, in general, in the context of combination of logics: any logic obtained by a combination process should be a strong extension of the given logics. Sometimes (see, for instance, [16]) it is required that the obtained logic should be a conservative extension of the given logics.[13] The reason of the failure in Example 4 is that we are using unrestricted fibred valuations, and then a designated value could be mapped into a nondesignated one, and vice-versa (note that, in Example 4, we define $g(T) = 0$). In order to obtain a fibred semantics inducing a logic $\mathcal{L}$ which is a conservative extension of both $\mathcal{L}_1$ and $\mathcal{L}_2$, the set of unrestricted fibred valuations must be refined. This is the key idea of the *plain fibring*, to be defined below.

Definition 15. *Let $\mathcal{L}_i = \langle C_i, M_i \rangle$ (with $i = 1, 2$) be two matrix logics as in Definition 13. A pair $(f, g) \in A_2^{A_1} \times A_1^{A_2}$ is* admissible *if it satisfies: $f(x) \in D_2$ iff $x \in D_1$, for every $x \in A_1$; and $g(y) \in D_1$ iff $y \in D_2$, for every $y \in A_2$. A* fibred valuation *is a unrestricted fibred valuation (f, g, v) such that (f, g) is admissible. The* plain fibring *of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the pair $\mathcal{L}_1 \odot \mathcal{L}_2 = \langle C_1 \uplus C_2, \vdash_{M_1 \odot M_2} \rangle$ such that $\vdash_{M_1 \odot M_2}$ is the consequence relation obtained by using fibred valuations. That is, for every $\Gamma \cup \{\varphi\} \subseteq L(C_1 \uplus C_2)$ it holds: $\Gamma \vdash_{M_1 \odot M_2} \varphi$ iff, for every fibred valuation (f, g, v), $(f, g, v)(\Gamma) \subseteq D$ implies that $(f, g, v)(\varphi) \in D$.*

The condition required for a pair (f, g) to be admissible is related to the definition of strict homomorphisms between matrices (cf. [10]). In Proposition 10 it will be proved that, in normal cases, the plain fibring is a conservative extension of its factors.

Two logics $\mathcal{L}_1$ and $\mathcal{L}_2$ as in Definition 15 are said to be *compatible* if there exist admissible pairs in $A_2^{A_1} \times A_1^{A_2}$. Note that $\mathcal{L}_1$ and $\mathcal{L}_2$ are compatible iff:
(i) $D_1 \neq \emptyset$ iff $D_2 \neq \emptyset$; and
(ii) $(A_1 \setminus D_1) \neq \emptyset$ iff $(A_2 \setminus D_2) \neq \emptyset$.
Observe that, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are not compatible, then one of the logics is trivial; therefore any pair of nontrivial logics is compatible. Now, we will prove that Proposition 9 can be improved by considering plain fibrings.

Proposition 10. *Let $\mathcal{L}_i = \langle C_i, M_i \rangle$ (with $i = 1, 2$) be two matrix logics as in Definition 13 such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are compatible. Then $\mathcal{L}_1 \odot \mathcal{L}_2$ is a conservative extension of both $\mathcal{L}_1$ and $\mathcal{L}_2$.*

[13] In [8], however, it is argued against the *desideratum* of obtaining conservative extensions of the given logics through a combination process.

Proof. We just prove that $\mathcal{L}_1 \odot \mathcal{L}_2$ is a conservative extension of $\mathcal{L}_1$, because the proof for $\mathcal{L}_2$ is analogous. Suppose firstly that $D_1 = A_1$. Then $D_2 = A_2$ (because $\mathcal{L}_1$ and $\mathcal{L}_2$ are compatible) and then both $\mathcal{L}_1$ and $\mathcal{L}_1 \odot \mathcal{L}_2$ are trivial, thus the result follows. Suppose now that $D_1 = \emptyset$. Then $D_2 = \emptyset$ and, again, both $\mathcal{L}_1$ and $\mathcal{L}_1 \odot \mathcal{L}_2$ are trivial. Suppose now that both $A_1 \setminus D_1$ and D_1 are nonempty. For every fibred valuation (f, g, v) let v' the M_1-valuation defined as in Lemma 2. Then, for every $\varphi \in L(C_1)$,

$$(f, g, v)(\varphi) \in D \ \text{ iff } \ v'(\varphi) \in D_1. \qquad (*)$$

In fact, if $\varphi \in \mathcal{V}$ then suppose that $v(\varphi) \in A_1$. Then $v'(\varphi) = v(\varphi) = (f, g, v)(\varphi)$ and the result holds. On the other hand, if $v(\varphi) \notin A_1$ then $v'(\varphi) = g(v(\varphi)) \in D_1$ iff $v(\varphi) = (f, g, v)(\varphi) \in D_2$ and the result holds. On the other hand, if $\varphi \notin \mathcal{V}$ then the result follows from Lemma 2. Now, let $\Gamma \cup \{\varphi\} \subseteq L(C_1)$ such that $\Gamma \vdash_{\mathcal{L}_1} \varphi$ and let (f, g, v) be a fibred valuation such that $(f, g, v)(\Gamma) \subseteq D$. Let v' be the M_1-valuation obtained from (f, g, v) as in Lemma 2. Using $(*)$ it follows that $v'(\Gamma) \subseteq D_1$ and then $v'(\varphi) \in D_1$. From $(*)$ again we get $(f, g, v)(\varphi) \in D$. This shows that $\Gamma \vdash_{M_1 \odot M_1} \varphi$.

Conversely, suppose that $\Gamma \cup \{\varphi\} \subseteq L(C_1)$ such that $\Gamma \vdash_{M_1 \odot M_1} \varphi$, and let v be a M_1-valuation such that $v(\Gamma) \subseteq D_1$. Let $(f, g) \in A_2^{A_1} \times A_1^{A_2}$ admissible (recall that $\mathcal{L}_1$ and $\mathcal{L}_2$ are compatible) and let (f, g, v') be the fibred valuation defined as in Lemma 3. Then $(f, g, v')(\Gamma) \subseteq D$ and so $(f, g, v')(\varphi) \in D$, that is, $v(\varphi) \in D_1$. This proves that $\Gamma \vdash_{\mathcal{L}_1} \varphi$. □

By the observation above, Proposition 10 still holds if, in particular, both logics are nontrivial.

It is worth noting that Proposition 8 can be adapted to plain fibring. Thus, given matrix logics $\mathcal{L}_i = \langle C_i, M_i \rangle$ $(i = 1, 2)$ then the plain fibring $\mathcal{L}_1 \odot \mathcal{L}_2$ is the matrix logic characterized by the set of $C_1 \uplus C_2$-matrices

$$M_1 \odot M_2 = \{(M_1)_g + (M_2)_f \ : \ (f, g) \in A_2^{A_1} \times A_1^{A_2} \text{ is admissible}\}$$

(recall Definitions 9, 14 and 15). That is, $\mathcal{L}_1 \odot \mathcal{L}_2 = \langle C_1 \uplus C_2, M_1 \odot M_2 \rangle$.

It should be stressed that each matrix logic $\langle C_1, (M_1)_g \rangle$ coincides with $\mathcal{L}_1$, and each matrix logic $\langle C_2, (M_2)_f \rangle$ coincides with $\mathcal{L}_2$, provided that (f, g) is admissible. In fact:

Proposition 11. *For every admissible pair (f, g) it holds:*
(a) $\vdash_{M_1} = \vdash_{(M_1)_g}$;
(b) $\vdash_{M_2} = \vdash_{(M_2)_f}$.

Proof. (a) Let $\Gamma \cup \{\varphi\} \subseteq L(C_1)$. Suppose that $\Gamma \vdash_{(M_1)_g} \varphi$, and let v be a valuation over M_1 such that $v(\Gamma) \subseteq D_1$. Then v is a valuation over $(M_1)_g$ such that $v(\Gamma) \subseteq D_1 \uplus D_2$ and then $v(\varphi) \in D_1 \uplus D_2$. Thus $v(\varphi) \in D_1$ and so $\Gamma \vdash_{M_1} \varphi$. Conversely, suppose that $\Gamma \vdash_{M_1} \varphi$ and let v' be a valuation over $(M_1)_g$ such that

$v'(\Gamma) \subseteq D_1 \uplus D_2$. The valuation v over M_1 such that $v(p) = v'(p)$ if $v'(p) \in A_1$, and $v(p) = g(v'(p))$ if $v'(p) \in A_2$ is such that, for every $\psi \in L(C_1)$:

$$v'(\psi) \in D_1 \uplus D_2 \text{ iff } v(\psi) \in D_1.$$

Therefore $v(\Gamma) \subseteq D_1$ and then $v(\varphi) \in D_1$. From this we get $v'(\varphi) \in D_1 \uplus D_2$. This shows that $\Gamma \vdash_{(M_1)_g} \varphi$. The proof of item (b) is entirely analogous. □

Example 5. Recall the paraconsistent matrix logic P^1 considered in Example 3. Let $\mathcal{L}_1$ be the fragment of P^1 defined over signature $\{\neg_{P^1}\}$ given by the matrix M_1 with domain $A_1 = \{T, T_1, F\}$ defined below, where $D_1 = \{T, T_1\}$ is the set of designated values.

	T	T_1	F
$\neg_{P^1}$	F	T	T

On the other hand, let $\mathcal{L}_2$ be the fragment of classical propositional logic defined over signature $\{\Rightarrow\}$ given by the usual matrix M_2, with domain $A_2 = \{0, 1\}$ and set $D_2 = \{1\}$ of designated values, displayed below.

$\Rightarrow$	1	0
1	1	0
0	1	1

Now, let $A = \{T, T_1, F, 1, 0\}$ and $D = \{T, T_1, 1\}$, and let $(f, g) \in A_2^{A_1} \times A_1^{A_2}$ such that $f(T) = f(T_1) = 1$, $f(F) = 0$, $g(1) = T$ and $g(0) = F$. Then (f, g) is admissible and $(M_1)_g$ and $(M_2)_f$ are given by the tables below, respectively.

	T	T_1	1	F	0
$\neg$	F	T	F	T	T

$\Rightarrow$	T	T_1	1	F	0
T	1	1	1	0	0
T_1	1	1	1	0	0
1	1	1	1	0	0
F	1	1	1	1	1
0	1	1	1	1	1

Let $\mathcal{L}$ be the logic over $\{\neg, \Rightarrow\}$ characterized by the matrix $M_{(f,g)} = (M_1)_g + (M_2)_f$ given by the two tables above, with $\{T, T_1, 1\}$ as the set of designated values. It is easy to see that the reduced matrix for $\mathcal{L}$ produces the 3-valued logic P^1, because T and 1 are congruent, as well as F and 0. On the other hand, it is clear that the pair $(f, g') \in A_2^{A_1} \times A_1^{A_2}$ such that $g'(1) = T_1$ and $g'(0) = F$ is also admissible; moreover, (f, g) and (f, g') are the unique admissible pairs, and so $\mathcal{L}_1 \odot \mathcal{L}_2$ is characterized by the set of matrices

$$M_1 \odot M_2 = \{(M_1)_g + (M_2)_f,\ (M_1)_{g'} + (M_2)_f\}.$$

It is easy to see that, for instance, the formula $\varphi = (p_1 \Rightarrow p_2) \Rightarrow \neg\neg(p_1 \Rightarrow p_2)$ is not valid in $(M_1)_{g'} + (M_2)_f$ and so φ is not valid in $\mathcal{L}_1 \odot \mathcal{L}_2$.

Example 6. In [13] a hierarchy of paraconsistent logics generalizing P^1 was introduced, called $\{P^n\}_{n\in\mathbb{N}}$. Each logic P^n is defined over the signature $C_{P^n} = \{\neg_{P^n}, \Rightarrow_{P^n}\}$, with semantics given by the matrix $M_{P^n} = \langle \mathbf{A}_{P^n}, \{T_0, T_1, \ldots, T_n\}\rangle$ such that $A_{P^n} = \{T_0, T_1, \ldots, T_n, \mathbf{f}\}$. The corresponding operations are displayed in the tables below.

	T_0	T_h	$\mathbf{f}$
$\neg_{P^n}$	$\mathbf{f}$	T_{h-1}	T_0

$\Rightarrow_{P^n}$	T_0	T_h	$\mathbf{f}$
T_0	T_0	T_0	$\mathbf{f}$
T_h	T_0	T_0	$\mathbf{f}$
$\mathbf{f}$	T_0	T_0	T_0

$(1 \leq h \leq n)$

Also in [13], it was introduced a hierarchy of weakly-intuitionistic logics called $\{I^n\}_{n\in\mathbb{N}}$, generalizing the weakly-intuitionistic logic I^1 introduced in [21] as a dual of P^1. Each I^n is defined over the signature $C_{I^n} = \{\neg_{I^n}, \Rightarrow_{I^n}\}$, with semantics given by the matrix $M_{I^n} = \langle \mathbf{A}_{I^n}, \{\mathbf{t}\}\rangle$ such that $A_{I^n} = \{\mathbf{t}, F_0, F_1, \ldots, F_n\}$. The operations of the matrix M_{I^n} are given by the tables below.

	$\mathbf{t}$	F_0	F_l
$\neg_{I^n}$	F_0	$\mathbf{t}$	F_{l-1}

$\Rightarrow_{I^n}$	$\mathbf{t}$	F_0	F_l
$\mathbf{t}$	$\mathbf{t}$	F_0	F_0
F_0	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
F_l	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$

$(1 \leq l \leq n)$

Note that both P^0 and I^0 coincide with the classical propositional logic over $\{\neg, \Rightarrow\}$ with two-valued matrix semantics. Now we will analyze the plain fibring of I^n with P^k. First note that, given I^n and P^k, the admissible pairs are of the form (f_j, g_i) (for $0 \leq j \leq k$ and $0 \leq i \leq n$) such that $g_i(\mathbf{f}) = F_i$; $f_j(\mathbf{t}) = T_j$; $g_i(T_h) = \mathbf{t}$ and $f_j(F_l) = \mathbf{f}$ for $0 \leq h \leq k$ and $0 \leq l \leq n$. Let $M_{(f_j,g_i)} = (M_{I^n})_{g_i} + (M_{P^k})_{f_j}$. Then the matrix $M_{(f_j,g_i)}$ is defined over the signature $C_{nk} = \{\neg_{In}, \Rightarrow_{In}, \neg_{Pk}, \Rightarrow_{Pk}\}$, having domain $\{\mathbf{t}, T_0, T_1, \ldots, T_k, F_0, F_1, \ldots, F_n, \mathbf{f}\}$ and designated values $\{\mathbf{t}, T_0, T_1, \ldots, T_k\}$. The operations are given below (the truth-tables of the negations consider the cases: $i = 0$ and $i > 0$; $j = 0$ and $j > 0$).

$\Rightarrow^i_{In}$	$\mathbf{t}$	T_0	T_h	F_0	F_l	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	F_0	F_0	F_0
T_0	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	F_0	F_0	F_0
T_h	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	F_0	F_0	F_0
F_0	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
F_l	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$

$\Rightarrow^j_{Pk}$	$\mathbf{t}$	T_0	T_h	F_0	F_l	$\mathbf{f}$
$\mathbf{t}$	T_0	T_0	T_0	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
T_0	T_0	T_0	T_0	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
T_h	T_0	T_0	T_0	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
F_0	T_0	T_0	T_0	T_0	T_0	T_0
F_l	T_0	T_0	T_0	T_0	T_0	T_0
$\mathbf{f}$	T_0	T_0	T_0	T_0	T_0	T_0

	$\mathbf{t}$	T_0	T_h	F_0	F_l	$\mathbf{f}$
$\neg^0_{In}$	F_0	F_0	F_0	$\mathbf{t}$	F_{l-1}	$\mathbf{t}$
$\neg^i_{In}$	F_0	F_0	F_0	$\mathbf{t}$	F_{l-1}	F_{i-1}
$\neg^0_{Pk}$	$\mathbf{f}$	$\mathbf{f}$	T_{h-1}	T_0	T_0	T_0
$\neg^j_{Pk}$	T_{j-1}	$\mathbf{f}$	T_{h-1}	T_0	T_0	T_0

$(1 \leq h \leq k;\ 1 \leq l \leq n)$

Each $M_{(f_j,g_i)}$ defines a matrix logic which is simultaneously paraconsistent (w.r.t. $\neg_{Pk}$) and weakly-intuitionistic (w.r.t. $\neg_{In}$). The plain fibring $I^n \odot P^k$ of I^n and P^k is the matrix logic characterized by the set of matrices

$$M_{I^n} \odot M_{P^k} = \{M_{(f_j,g_i)} \ : \ 0 \leq j \leq k \text{ and } 0 \leq i \leq n\}.$$

The relationships between the logic defined by each matrix $M_{(f_j,g_i)}$, the logic $I^n \odot P^k$ and the logic $I^n P^k$ (having a single negation which is simultaneously paraconsistent in the sense of P^k and weakly-intuitionistic in the sense of I^n), introduced in [13], deserve further research.

We conclude this section by observing that both plain fibring and unrestricted plain fibring are operations defined over matrix logics characterized by a single matrix, but the result is a matrix logic characterized, in general, by a *set* of matrices (not necessarily being a singleton). We can easily correct this asymmetry by generalizing the operation of plain fibring to matrix logics in general.

Definition 16. *Let* $\mathcal{L}_i = \langle C_i, \mathcal{K}_i \rangle$ *(with* $i = 1, 2$*) be two matrix logics. The* plain fibring *of* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *is the matrix logic* $\mathcal{L}_1 \odot \mathcal{L}_2 = \langle C_1 \uplus C_2, \mathcal{K}_1 \odot \mathcal{K}_2 \rangle$ *such that* $\mathcal{K}_1 \odot \mathcal{K}_2$ *is the class of* $C_1 \uplus C_2$*-matrices*

$$\{(M_1)_g + (M_2)_f \ : \ M_1 \in \mathcal{K}_1,\ M_2 \in \mathcal{K}_2 \ \textit{ and } \ (f, g) \in A_2^{A_1} \times A_1^{A_2} \textit{ is admissible}\}.$$

Proposition 12. *Suppose that, for every* $M_1 \in \mathcal{K}_1$ *and every* $M_2 \in \mathcal{K}_2$*, the logics* $\langle C_1, M_1 \rangle$ *and* $\langle C_2, M_2 \rangle$ *are compatible. Then* $\mathcal{L}_1 \odot \mathcal{L}_2$ *is a conservative extension of both* $\mathcal{L}_1$ *and* $\mathcal{L}_2$*.*

Proof. Let $\Gamma \cup \{\varphi\} \subseteq L(C_1)$. Then $\Gamma \vdash_{\mathcal{L}_1} \varphi$ iff, for every $M_1 \in \mathcal{K}_1$ and every $M_2 \in \mathcal{K}_2$, $\Gamma \vdash_{M_1 \odot M_2} \varphi$, by adapting the proof of Proposition 10, iff $\Gamma \vdash_{\mathcal{K}_1 \odot \mathcal{K}_2} \varphi$. The proof for $\mathcal{L}_2$ is analogous. The details are left to the reader. □

A matrix $M = \langle A, D \rangle$ is said to be *trivial* if $D = \emptyset$ or $D = A$. The following result is a direct consequence of the proposition above.

Corollary 1. *Let* $\mathcal{L}_i = \langle C_i, \mathcal{K}_i \rangle$ *(with* $i = 1, 2$*) be two matrix logics. Suppose that* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *do not contain trivial matrices. Then* $\mathcal{L}_1 \odot \mathcal{L}_2$ *is a conservative extension of both* $\mathcal{L}_1$ *and* $\mathcal{L}_2$*.* □

Of course an analogous generalization can be obtained for unrestricted plain fibring. In this case, the unrestricted plain fibring of two matrix logics $\mathcal{L}_i = \langle C_i, \mathcal{K}_i \rangle$ (with $i = 1, 2$), such that $\mathcal{K}_1$ and $\mathcal{K}_2$ do not contain trivial matrices, is a weak conservative extension of the given logics. The details of this construction are left to the reader.

8 Concluding Remarks

In this article we introduce some operations for combining matrix logics, in the same spirit that the original formulation of fibring. In this sense, this paper should be seen as a first step in the direction of obtaining mechanisms for combining matrix logics. The unrestricted plain fibring constitutes a simple generalization of the fibring of modal logics, which produces a weak conservative extension of the given logics. In order to obtain a conservative extension of the given logics (as usually is expected for a "good" combination technique) it is necessary to restrict the extension mappings which define the class of matrix models. This originates the so-called plain fibring of logics. On the other hand, when the given logics are characterized by single matrices sharing the set of truth-values as well as the designated values, they can be easily combined by putting together both matrices, given the so-called direct union of the logics. An application of this operation to fuzzy logics defined by t-norms was shown.

The study of the general properties of the plain fibring and the direct union of logics deserves future research. In particular, the relationship between these operations and the categorical fibring (in the sense of [19]), as well as the connections with *cryptofibring* (see [5]) should be analyzed. On the other hand, if the given logics have also a proof-theoretic presentation (such as Hilbert calculi) it would be interesting to give a proof-theoretic presentation of the logics obtained by plain fibring and direct union. Another topic to be studied is the possibility of sharing connectives through the combination process, in analogy to the *constrained fibring* (see [19]).

Finally, we believe that the techniques here introduced can be applied to obtain new interesting logics by composition (that is, by splicing logics), as it was done in Example 6, as well as to better understand a given logic by means of decomposition of the logic into fragments of it (that is, by splitting logics), as it was done in Examples 2 and 5.

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References

1. W. Blok and D. Pigozzi. *Algebraizable Logics*, volume 77 (396) of *Memoirs of the American Mathematical Society.* AMS, Providence, Rhode Island, 1989.

2. C. Caleiro. *Combining Logics*. PhD thesis, IST-Lisboa, Portugal, 2000.
3. C. Caleiro, W. Carnielli, J. Rasga, and C. Sernadas. Fibring of Logics as a Universal Construction. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 13. Kluwer Academic Publishers, 2005.
4. C. Caleiro, W. A. Carnielli, M. E. Coniglio, A. Sernadas, and C. Sernadas. Fibring Non-Truth-Functional Logics: Completeness Preservation. *Journal of Logic, Language and Information*, 12(2):183–211, 2003.
5. C. Caleiro and J. ramos. Cryptofibring. In W.A. Carnielli, F.M. Dionísio, and P. Mateus, editors, *Proceedings of CombLog'04 - Workshop on Combination of Logics: Theory and Applications*, pages 87–92, Lisboa, Portugal, 2004. Departamento de Matemática, Instituto Superior Técnico.
6. W.A. Carnielli and M.E. Coniglio. A categorial approach to the combination of logics. *Manuscrito*, 22(2):69–94, 1999.
7. M. E. Coniglio and V. L. Fernández. Categorial fibring of algebraizable logics. To appear, 2005.
8. M.E. Coniglio. The meta-fibring environment: Preservation of meta-properties by fibring. *CLE e-Prints*, 5(4), 2005.
URL = `http://www.cle.unicamp.br/e-prints/vol_5,n_4,2005.htm`.
9. M.E. Coniglio, A. Sernadas, and C. Sernadas. Fibring logics with topos semantics. *Journal of Logic and Computation*, 13(4):595–624, 2003.
10. J. Czelakowski. *Protoalgebraic Logics*, volume 10 of *Trends in Logic, Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2001.
11. V. L. Fernández. *Fibrilação de lógicas na Hierarquia de Leibniz (Fibring Logics in the Leibniz Hierarchy, in Portuguese)*. PhD thesis, IFCH – State University of Campinas, Brazil, 2005.
12. V. L. Fernández and M. E. Coniglio. Syntactic fibring of algebraizable logics. *XIII Brazilian Logic Conference – Book of Abstracts*, pages 61–62, 2003.
13. V.L. Fernández. Semântica de sociedades para lógicas n-valentes (Society semantics for n-valued logics, in Portuguese). Master's thesis, IFCH – State University of Campinas, Brazil, 2001.
14. V.L. Fernández and M.E. Coniglio. Fibring algebraizable consequence systems. In W.A. Carnielli, F.M. Dionísio, and P. Mateus, editors, *Proceedings of CombLog'04 - Workshop on Combination of Logics: Theory and Applications*, pages 93–98, Lisboa, Portugal, 2004. Departamento de Matemática, Instituto Superior Técnico.
15. D. Gabbay. Fibred semantics and the weaving of logics: Part 1. *The Journal of Symbolic Logic*, 61(4):1057–1120, 1996.
16. D. Gabbay. *Fibring Logics*. Clarendon Press - Oxford, 1999.
17. P. Hájek. *Metamathematics of Fuzzy Logic*. Trends in Logic - Studia Logica Library. Kluwer Academic Publishers, Dordrecht, 1998.
18. J. Łoś and R. Suszko. Remarks on sentential logics. *Indagationes Mathematicae*, 20:177–183, 1958.
19. A. Sernadas, C. Sernadas, and C. Caleiro. Fibring of logics as a categorical construction. *Journal of Logic and Computation*, 9 (2):149–179, 1999.
20. A. M. Sette. On the propositional calculus P^1. *Mathematica Japonicae*, 18:173–180, 1973.
21. A. M. Sette and W. A. Carnielli. Maximal weakly-intuitionistic logics. *Studia Logica*, 55:181–203, 1995.
22. R. Wójcicki. Matrix approach in methodology of sentential calculi. *Studia Logica*, 32:7 – 37, 1973.
23. A. Zanardo, A. Sernadas, and C. Sernadas. Fibring: Completeness preservation. *The Journal of Symbolic Logic*, 66(1):414–439, 2001.