On a Four-Valued Modal Logic with Deductive Implication[1]

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Abstract

In this paper we propose to enrich the four-valued modal logic associated to Monteiro's Tetravalent modal algebras (TMAs) with a deductive implication, that is, such that the Deduction Meta–theorem holds in the resulting logic. All this lead us to establish some new connections between TMAs, symmetric (or involutive) Boolean algebras, and modal algebras for extensions of **S5**, as well as their logical counterparts.

1 Introduction

The logic $\mathcal{TML}$ was introduced in [6] as a four-valued modal logic naturally associated to Monteiro's last algebras, namely, tetravalent modal algebras (TMAs). TMAs were first considered by Antonio Monteiro and mainly studied by I. Loureiro (see [8]) and A. V. Figallo et al. (see [5]). A Gentzen style system for $\mathcal{TML}$ was presented in [6] in the original signature $\{\wedge, \vee, \neg, \Box, \bot\}$.

In [5], it was shown that it is possible to define an implication operator $\succ$ in TMAs in terms of the original operators. This new connective was named *contrapositive implication* and it was shown that it has some "good" properties. Taking profit of the features of this implication, a Hilbert style system for $\mathcal{TML}$ was presented in [4] in the signature $\{\succ, \bot\}$ (since the other connectives can be defined from this). But, the Deduction Meta–theorem does not hold in this logic with respect to the contrapositive implication. Moreover, it was proved that $\mathcal{TML}$ is not functionally complete and, as a consequence, that in $\mathcal{TML}$ is not possible to define an implication connective such that the deduction meta–theorem holds.

In this paper, we propose to enrich the four-valued modal logic $\mathcal{TML}$ with a deductive implication in such a way that in the resulting logic the Deduction Meta–theorem holds. From this, we will find some new connections between TMAs, symmetric (or involutive) Boolean algebras, and modal algebras for extensions of **S5**, as well as their logical counterparts. Specifically, we will prove that tetravalent modal algebras expanded with a deductive implication are the same as Boolean algebras plus a De Morgan negation, that is, symmetric (or involutive) Boolean algebras. On the other hand, we also shown that these structures are modal algebras for modal logic **S5** satisfying additional equations such that they are generated by the Henle algebra H_2.

[1] This paper constitutes a revised and improved version from the preprint [3].

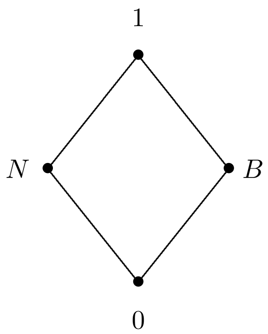
2 Preliminaries

A structure $\langle A, \wedge, \vee, \sim, T, 0\rangle$ of type $(2, 2, 1, 1, 0)$ is a *symmetric Boolean algebra* if the reduct $\langle A, \wedge, \vee, \sim, 0\rangle$ is a Boolean algebra and T is an involutive automorphism of A (cf. [10, 11]), that is, $T : A \to A$ is an automorphism such that $T(T(x)) = x$ for every $x \in A$. On the other hand, an *involutive Boolean algebra* is a structure $\langle A, \wedge, \vee, \sim, \neg, 0\rangle$ of type $(2, 2, 1, 1, 0)$ where $\langle A, \wedge, \vee, \sim, 0\rangle$ is a Boolean algebra and $\neg$ is an involutive dual automorphism of A ([12]), that is, $\neg : A \to A^d$ is an isomorphism (where A^d denotes the dual of A) such that $\neg\neg x = x$ for every $x \in A$. It is equivalent to say that $\neg$ is a De Morgan negation, that is, a negation $\neg$ defined on a distributive bounded lattice such that $\neg(x \vee y) = \neg x \wedge \neg y$ and $\neg\neg x = x$ (and so $\neg 0 = 1$).

As observed in [12], both kinds of structures coincide. Besides, the class of these algebras is a variety which is generated by the two–atoms Symmetric Boolean algebra.

Finally, a *tetravalent modal algebra* is an algebra $\mathfrak{A} = \langle A, \wedge, \vee, \neg, \Box, 0\rangle$ of type $(2, 2, 1, 1, 0)$ such that its non-modal reduct $\mathfrak{A}^{\sim} = \langle A, \wedge, \vee, \neg, 0\rangle$ is a De Morgan algebra and the unary operation $\Box$ satisfies, for all $x \in A$, the following: $\Box x \wedge \neg x = 0$, and $\neg\Box x \wedge x = \neg x \wedge x$ (cf. [6]).

The class of all tetravalent modal algebras constitute a variety which is denoted by **TMA**. Besides, **TMA** is generated by the four-element De Morgan algebra $\mathfrak{M}_{4m} = \langle M_4, \wedge, \vee, \neg, \Box, 0\rangle$ where $M_4 = \{1, N, B, 0\}$ is



such that $\neg N = N$, $\neg B = B$, $\neg 0 = 1$ and $\neg 1 = 0$, and the unary operator $\Box$ is defined as follows: $\Box 1 = 1$, and $\Box x = 0$ otherwise. This algebra has two proper prime filters, namely, $F_N = \{1, N\}$ and $F_B = \{1, B\}$.

The *Tetravalent Modal Logic*, $\mathcal{TML}$ is the logic defined by the matrix $\mathcal{M} = \langle \mathfrak{M}_{4m}, F_N\rangle$ or equivalently by $\mathcal{M} = \langle \mathfrak{M}_{4m}, F_B\rangle$ (see [4]).

3 A deductive implication on $\mathfrak{M}_{4m}$

From a semantic point of view, what we are looking for is a binary operator $\Rightarrow$ defined on $\mathfrak{M}_{4m}$ in such a way that the following condition holds, for every $x, y, z \in \mathfrak{M}_{4m}$

$$z \wedge x \leq y \text{ iff } z \leq x \Rightarrow y, \tag{1}$$

It is easy to see that if an implication as above can be defined in a lattice, it must be unique. A binary operator satisfying (1) is called the *residual* of $\wedge$. As we prove in [4], the contrapositive implication (see [5]) is not the residual of $\wedge$. Moreover, there is no term definable binary operation in $\mathfrak{M}_{4m}$ satisfying condition (1).

Proposition 3.1 *There is no term definable binary operation $\Rightarrow$ in the algebra $\mathfrak{M}_{4m}$ such that condition (1) holds.*

Proof. Since $N \wedge B \leq 0$ then $N \leq B \Rightarrow 0$, from (1), and so $B \Rightarrow 0 \in \{1, N\}$. But then $B \Rightarrow 0 = 1$, since $\{0, 1, B\}$ is a **TMA**-subalgebra of $\mathfrak{M}_{4m}$, and so $B \leq B \Rightarrow 0$. This implies, by (1), that $B = B \wedge B \leq 0$, a contradiction. ■

Since $\mathfrak{M}_{4m}$ is a finite distributive lattice, there exists a unique binary operation $\Rightarrow$ satisfying (1). However, in the light of Proposition 3.1, $\Rightarrow$ is not term definable. It is worth noting that the unary operation $\sim$ given by $\sim x = x \Rightarrow 0$ for every x is a Boolean complement in $\mathfrak{M}_{4m}$, that is: $x \wedge \sim x = 0$ and $x \vee \sim x = 1$ for every x. On the other hand, if the Boolean complement $\sim$ is added to $\mathfrak{M}_{4m}$ then $\Rightarrow$ can be defined as $x \Rightarrow y = \sim x \vee y$ for every x, y. It is easy to see, by a combinatory argument, that the truth-tables for both operators must be the following:

$\Rightarrow$	0	N	B	1
0	1	1	1	1
N	B	1	B	1
B	N	N	1	1
1	0	N	B	1

x	$\sim x$
0	1
N	B
B	N
1	0

By the considerations above, it is easy to prove the following:

Proposition 3.2 *Adding to $\mathfrak{M}_{4m}$ the Boolean complement $\sim$ is equivalent to adding to $\mathfrak{M}_{4m}$ the deductive implication $\Rightarrow$.*

Let $\mathfrak{M}_{4m}^{\sim}$ be the algebra $\langle M_4, \wedge, \vee, \sim, \neg, \Box, 0\rangle$ of type $(2, 2, 1, 1, 1, 0)$ where the reduct $\langle M_4, \wedge, \vee, \neg, \Box, 0\rangle$ is the four-element **TMA**-algebra $\mathfrak{M}_{4m}$ and $\sim$ is defined as in the table above.

Proposition 3.3 *In $\mathfrak{M}_{4m}^{\sim}$ the following identities hold, for all $x \in M_4$:*

(i) $\neg{\sim} x = {\sim}\neg x$,

(ii) $\Box x = x \wedge \neg{\sim} x$,

(iii) $\Diamond x = x \vee \neg{\sim} x$, *where* $\Diamond x =_{def} \neg\Box\neg x$,

(iv) $\neg\Box x = {\sim}\Box x$,

(v) $\Box\neg x = \Box{\sim}x$.

Because of Corollary 3.2, the envisaged extension of the four-valued modal logic $\mathfrak{M}_{4m}$ by a deductive implication is equivalent to the extension by a classical negation. Being so, and by reasons that will be clear in the following sections, we will focus on the logic obtained from $\mathfrak{M}_{4m}$ by adding the classical negation $\sim$. As we shall see, the algebraic models will be symmetric (or involutive) Boolean algebras.

4 De Morgan algebras and Tetravalent modal algebras extended by a Boolean complement

From now on, an operator $\sim : A \to A$ in a bounded lattice A such that $x \wedge {\sim}x = 0$ and $x \vee {\sim}x = 1$ for every $x \in A$ will be called a *Boolean complement* in A. It is easy to see that, if A is also distributive, the Boolean complement ${\sim}x$ of an element x of A (if it exists) is unique.

In this section we will see that if we begin by a De Morgan algebra instead of a tetravalent modal algebra and we extend it by a Boolean complement, the resulting structure can define a unique modality $\Box$ which satisfies the properties of a tetravalent modal algebra. Because of the uniqueness of $\Box$ it follows that tetravalent modal algebras plus a Boolean complement are the same as De Morgan algebras plus a Boolean complement. Additionally, as we will see, these structures coincide with the symmetric (or involutive) Boolean algebras.

Proposition 4.1 *Let $\langle A, \wedge, \vee, \neg, 0\rangle$ be a De Morgan algebra, and consider a Boolean complement $\sim$ in A, where $1 = \neg 0$ is the top element of A. Let $\Box x =_{def} x \wedge \neg{\sim}x$ for every $x \in A$. Then $\langle A, \wedge, \vee, \neg, \Box, 0\rangle$ is a tetravalent modal algebra.*

Proof. Let $x \in A$. Then

$$\begin{aligned} x \vee \neg\Box x &= x \vee \neg(x \wedge \neg{\sim}x) \\ &= x \vee (\neg x \vee \neg\neg{\sim}x) \\ &= x \vee (\neg x \vee {\sim}x) = 1. \end{aligned}$$

On the other hand:

$$\begin{aligned} x \wedge \neg\Box x &= x \wedge \neg(x \wedge \neg{\sim}x) \\ &= x \wedge (\neg x \vee {\sim}x) \\ &= (x \wedge \neg x) \vee (x \wedge {\sim}x) \\ &= (x \wedge \neg x) \vee 0 = x \wedge \neg x. \end{aligned}$$

■

Lemma 4.2 *([12]) Let $\langle A, \wedge, \vee, \neg, 0\rangle$ be a De Morgan algebra, and consider a Boolean complement $\sim$ in A. Then $\neg{\sim}x = {\sim}\neg x$ for every $x \in A$.*

Proof. Let $x \in A$. Then

$$\begin{aligned} \neg x \wedge \neg{\sim}x &= \neg(x \vee {\sim}x) \\ &= \neg 1 = 0, \end{aligned}$$

and, dually,

$$\begin{aligned} \neg x \vee \neg{\sim}x &= \neg(x \wedge {\sim}x) \\ &= \neg 0 = 1. \end{aligned}$$

From this it follows that $\neg{\sim}x$ is the Boolean complement of $\neg x$, that is: $\neg{\sim}x = {\sim}\neg x$, for every $x \in A$, by the uniqueness of $\sim$. ■

The following result relates, as we shall see, tetravalent modal algebras extended by a Boolean complement with symmetric (or involutive) Boolean algebras.

Proposition 4.3 *Let $\langle A, \wedge, \vee, \neg, \Box, 0\rangle$ be a tetravalent modal algebra, and consider a Boolean complement $\sim$ in A. Then $\Box x = x \wedge \neg{\sim}x$ for every $x \in A$.*

Proof. Let $x \in A$. Then $\Box x \leq x$, by definition of tetravalent modal algebras. On the other hand $x \vee \neg\Box x = 1$ and so

$$\begin{aligned} {\sim}x &= {\sim}x \wedge 1 = {\sim}x \wedge (x \vee \neg\Box x) \\ &= ({\sim}x \wedge x) \vee ({\sim}x \wedge \neg\Box x) \\ &= {\sim}x \wedge \neg\Box x \end{aligned}$$

therefore ${\sim}x \leq \neg\Box x$. From this $\Box x \leq \neg{\sim}x$ and so $\Box x \leq x \wedge \neg{\sim}x$. Let $y \in A$ such that $y \leq x$ and $y \leq \neg{\sim}x$. Then $y \leq {\sim}\neg x$, by Lemma 4.2. Therefore $y = y \wedge x$ and $y \wedge \neg x = 0$. From this, and recalling that $x \vee \neg x = \neg x \vee \Box x$,

$$\begin{aligned} y &= y \vee 0 = (y \wedge x) \vee (y \wedge \neg x) = y \wedge (x \vee \neg x) \\ &= y \wedge (\neg x \vee \Box x) = (y \wedge \neg x) \vee (y \wedge \Box x) \\ &= 0 \vee (y \wedge \Box x) = y \wedge \Box x. \end{aligned}$$

That is, $y \leq \Box x$. This means that $\Box x = x \wedge \neg{\sim}x$. ■

Corollary 4.4 *Let $\langle A, \wedge, \vee, \neg, 0\rangle$ be a De Morgan algebra equiped additionally with a Boolean complement $\sim$. Then, the operator $\Box$ defined by $\Box x = x \wedge \neg{\sim}x$ for every $x \in A$ is the unique unary operator over A such that $\langle A, \wedge, \vee, \neg, \Box, 0\rangle$ is a tetravalent modal algebra. Thus, De Morgan algebras plus a Boolean complement are definitionally equivalent to tetravalent modal algebras plus a Boolean complement.*

It is here where involutive (or symmetric) Boolean algebras appear. Corollary 4.4 can be recast as follows:

Corollary 4.5 *Let $\langle A, \wedge, \vee, \sim, \neg, 0\rangle$ be an involutive Boolean algebra. Then, the operator $\Box$ defined by $\Box x = x \wedge \neg{\sim}x$ for every $x \in A$ is the unique unary operator over A such that $\langle A, \wedge, \vee, \neg, \Box, 0\rangle$ is a tetravalent modal algebra. Thus, involutive (or symmetric) Boolean algebras are definitionally equivalent to tetravalent modal algebras plus a Boolean complement.*

From the considerations above, the variety of involutive (or symmetric) Boolean algebras can be defined in several ways. By convenience, we shall adopt the perspective of seeing such algebras as Boolean algebras extended with a De Morgan negation. Clearly this variety is generated by the reduct of $\mathfrak{M}_{4m}^{\sim}$ over the language $\wedge, \vee, \sim, \neg, \bot$, by Proposition 3.3(ii).

Recall that the implication operator $\Rightarrow$ is defined in any Boolean algebra as follows: $x \Rightarrow y = {\sim}x \vee y$. Since $\wedge$, $\vee$ and 0 can be defined in terms of $\Rightarrow$ and $\sim$

then we can consider, from now on, the absolutely free algebra $\mathfrak{Fm} = \langle Fm, \Rightarrow, \sim, \neg\rangle$ of type (2,1,1) generated by some denumerable set Var of variables as the formal language for the variety **IBA** of symmetric (or involutive) Boolean algebras. A logic preserving degrees of truth (see [2]) associated to **IBA** can be naturally defined as follows:

Definition 4.6 The logic of involutive Boolean algebras defined over $\mathfrak{Fm}$ is the propositional logic $IBA = \langle Fm, \models_{IBA}\rangle$ given as follows: for every set $\Gamma \cup \{\alpha\} \subseteq Fm$, $\Gamma \models_{IBA} \alpha$ if and only if there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that, for every $\mathfrak{U} \in$ **IBA** and $h \in Hom(\mathfrak{Fm}, \mathfrak{U})$, $\bigwedge\{h(\gamma) \ : \ \gamma \in \Gamma_0\} \leq h(\alpha)$. In particular, $\emptyset \models_{IBA} \alpha$ if and only if $h(\alpha) = 1$ for every $h \in Hom(\mathfrak{Fm}, \mathfrak{U})$.

Remark 4.7 There exist several approaches in the literature to the study of logics preserving degrees of truth associated to a class of agebras. An interesting precedent can be found in [7], in which R. Jansana introduces and studies the so-called *semilattice-based logics* associated to semilattices.

A logic is naturally associated to the algebra $\mathfrak{M}^{\sim}_{4m}$ as in the previous definition:

Definition 4.8 The four-valued modal logic with classical negation $M^{\sim}_{4m}$ defined over $\mathfrak{Fm}$ is the propositional logic $M^{\sim}_{4m} = \langle Fm, \models_{M^{\sim}_{4m}}\rangle$ given as follows: for every $\Gamma \cup \{\alpha\} \subseteq Fm$, $\Gamma \models_{M^{\sim}_{4m}} \alpha$ iff there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that, for every $h \in Hom(\mathfrak{Fm}, \mathfrak{M}^{\sim}_{4m})$, $\bigwedge\{h(\gamma) \ : \ \gamma \in \Gamma_0\} \leq h(\alpha)$. In particular, $\emptyset \models_{M^{\sim}_{4m}} \alpha$ if and only if $h(\alpha) = 1$ for every $h \in Hom(\mathfrak{Fm}, \mathfrak{M}^{\sim}_{4m})$.

Since **IBA** is generated by $\mathfrak{M}^{\sim}_{4m}$, the following result is immediate:

Proposition 4.9 *The logics IBA and $M^{\sim}_{4m}$ coincide, that is: for every $\Gamma \cup \{\alpha\} \subseteq Fm$, $\Gamma \models_{IBA} \alpha$ iff $\Gamma \models_{M^{\sim}_{4m}} \alpha$.*

Proposition 4.10 *$M^{\sim}_{4m}$ is a strong conservative extension of propositional classical logic* **CPL***: $\Gamma \models_{M^{\sim}_{4m}} \beta$ iff $\Gamma \models \beta$ in* **CPL***, for every $\Gamma \cup \{\beta\} \subseteq Fm$ without occurrences of $\neg$*

Proof. It is immediate, from the definition of $M^{\sim}_{4m}$. ■

Remark 4.11 $M^{\sim}_{4m}$ satisfies the Deduction Metatheorem: $\Gamma, \alpha \models_{M^{\sim}_{4m}} \beta$ iff $\Gamma \models_{M^{\sim}_{4m}} (\alpha \Rightarrow \beta)$, for every $\Gamma \cup \{\alpha, \beta\} \subseteq Fm$.

The next result shows that $M^{\sim}_{4m}$ can be seen as a matrix logic. This follows from the definition of the four-valued modal logic $\mathcal{TML}$ (see the end of Section 2). This fact will be useful in the sequel.

Proposition 4.12 *Let $\mathcal{M}^{\sim}_{N} = \langle \mathfrak{M}^{\sim}_{4m}, \{N, 1\}\rangle$ and $\mathcal{M}^{\sim}_{B} = \langle \mathfrak{M}^{\sim}_{4m}, \{B, 1\}\rangle$ be the logical matrices obtained from the algebra $\mathfrak{M}^{\sim}_{4m}$. Then*

(i) $\models_{M^{\sim}_{4m}} = \models_{\mathcal{M}^{\sim}_{N}}$,

(ii) $\models_{M^{\sim}_{4m}} = \models_{\mathcal{M}^{\sim}_{B}}$.

Therefore, the logic $M^{\sim}_{4m}$ for **IBA** *can be characterized by a single logical matrix.*

Proof. The proof is analogous to that for M_{4m} found in [4]. ■

5 A Hilbert-style presentation for $M_{4m}^{\sim}$

In this section we define a Hilbert calculus for the logic $M_{4m}^{\sim}$ of $\mathfrak{M}_{4m}^{\sim}$ which, by Proposition 4.9, can be considered as the logic preserving degrees of truth for the variety **IBA** of involutive (or symmetric) Boolean algebras. Observe that in the language Fm the equations characterizing a De Morgan negation $\neg$ are the following: $x = \neg\neg x$, and $\neg({\sim}x \Rightarrow y) = {\sim}(\neg x \Rightarrow {\sim}\neg y)$.

Definition 5.1 Denote by $\mathcal{H}_{4m}^{\sim} = \langle Fm, \vdash_{\mathcal{H}_{4m}^{\sim}} \rangle$ the propositional logic defined through the following Hilbert calculus, where α, β, $\gamma \in Fm$.

Axioms

(A1) $\alpha \Rightarrow (\beta \Rightarrow \alpha)$

(A2) $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$

(A3) $({\sim}\beta \Rightarrow {\sim}\alpha) \Rightarrow (({\sim}\beta \Rightarrow \alpha) \Rightarrow \beta)$

(A4) $\neg\neg\alpha \Rightarrow \alpha$

(A5) $\alpha \Rightarrow \neg\neg\alpha$

(A6) $\neg({\sim}\alpha \Rightarrow \beta) \Rightarrow {\sim}(\neg\alpha \Rightarrow {\sim}\neg\beta)$

(A7) ${\sim}(\neg\alpha \Rightarrow {\sim}\neg\beta) \Rightarrow \neg({\sim}\alpha \Rightarrow \beta)$

Inference Rules

(MP) $\dfrac{\alpha \qquad \alpha \Rightarrow \beta}{\beta}$ (CP) $\dfrac{\alpha \Rightarrow \beta}{\neg\beta \Rightarrow \neg\alpha}$

Remark 5.2 Axioms (A1)-(A3) plus (MP) constitute a sound and complete axiomatization of propositional classical logic in the language generated by $\Rightarrow$ and $\sim$ (cf. [9]). On the other hand, axioms (A4)-(A7) describe the conditions required for a De Morgan negation in the given language, as mentioned above. Finally, the rule (CP) is required in order to guarantee a crucial property of a De Morgan negation, namely: $a \leq b$ implies that $\neg b \leq \neg a$. Additionally, this axiom guarantees that the De Morgan negation $\neg$ preserves logical equivalences.

Definition 5.3 (1) A derivation of a formula α in $\mathcal{H}_{4m}^{\sim}$ is a finite sequence of formulas $\alpha_1 \dots \alpha_n$ such that α_n is α and every α_i is either an instance of an axiom, or α_i is the consequence of α_j and $\alpha_k = (\alpha_j \Rightarrow \alpha_i)$ by (MP) for some $j, k \leq i-1$, or $\alpha_i = (\neg\beta \Rightarrow \neg\gamma)$ is the consequence of $\alpha_j = (\gamma \Rightarrow \beta)$ by (CP) for some $j \leq i-1$. We say that α is derivable in $\mathcal{H}_{4m}^{\sim}$, and we write $\vdash_{\mathcal{H}_{4m}^{\sim}} \alpha$, if there exists a derivation of it in $\mathcal{H}_{4m}^{\sim}$.
(2) Let $\Gamma \cup \{\alpha\}$ be a set of formulas in Fm. We say that α is derivable in $\mathcal{H}_{4m}^{\sim}$ from Γ, and we write $\Gamma \vdash_{\mathcal{H}_{4m}^{\sim}} \alpha$, if either α is derivable in $\mathcal{H}_{4m}^{\sim}$, or there exists a finite, non-empty subset $\{\gamma_1, \dots, \gamma_n\}$ of Γ such that $(\gamma_1 \Rightarrow (\dots(\gamma_n \Rightarrow \alpha))\dots)$ is derivable in $\mathcal{H}_{4m}^{\sim}$.

Remark 5.4 $\mathcal{H}^{\sim}_{4m}$ satisfies, by the very definition, the Deduction Metatheorem: $\Gamma, \alpha \vdash_{\mathcal{H}^{\sim}_{4m}} \beta$ iff $\Gamma \vdash_{\mathcal{H}^{\sim}_{4m}} (\alpha \Rightarrow \beta)$, for every $\Gamma \cup \{\alpha, \beta\} \subseteq Fm$. Additionally, $\vdash_{\mathcal{H}^{\sim}_{4m}} \alpha$ iff $\emptyset \vdash_{\mathcal{H}^{\sim}_{4m}} \alpha$.

Let $\equiv \subseteq Fm \times Fm$ be the relation defined by

$$\equiv =_{def} \{(\alpha, \beta) \ : \ \vdash_{\mathcal{H}^{\sim}_{4m}} \alpha \Rightarrow \beta \text{ and } \vdash_{\mathcal{H}^{\sim}_{4m}} \beta \Rightarrow \alpha\}.$$

Lemma 5.5 *The relation $\equiv$ is a congruence on $\mathfrak{Fm}$.*

Proof. Clearly, $\equiv$ is compatible with $\Rightarrow$ and $\sim$ since (A1)–(A3) plus (MP) is an axiomatization of CPL. That is, $\equiv$ is a Boolean congruence. On the other hand, by rule (CP), it is immediate that $\equiv$ is compatible with $\neg$. ■

Theorem 5.6 *The Lindenbaum algebra $\mathfrak{Fm}/\equiv$ of $\mathcal{H}^{\sim}_{4m}$ is an involutive Boolean algebra with: $|\alpha| \Rightarrow |\beta| =_{def} |\alpha \Rightarrow \beta|$, $\sim|\alpha| =_{def} |\sim\alpha|$ and $\neg|\alpha| =_{def} |\neg\alpha|$, where $|\gamma|$ denotes the equivalence class of the formula γ.*

Proof. The operations are well-defined, by Lemma 5.5. It is clear that $\mathfrak{Fm}/\equiv$ is a Boolean algebra with an additional operation $\neg$. On the other hand, by axioms (A4)–(A7) and by rule (CP), we have that $\neg$ is a De Morgan negation. ■

Theorem 5.7 *(Soundness and Completeness of $\mathcal{H}^{\sim}_{4m}$) The following conditions are equivalent, for every subset $\Gamma \cup \{\beta\}$ of Fm:*

(i) $\Gamma \vdash_{\mathcal{H}^{\sim}_{4m}} \beta$,

(ii) $\Gamma \models_{M^{\sim}_{4m}} \beta$.

Proof. (i) $\Rightarrow$ (ii) (Soundness): It is easy to see that every axiom of $\mathcal{H}^{\sim}_{4m}$ is valid in $M^{\sim}_{4m}$. On the other hand, if an instance of the premises of (MP) is valid in $M^{\sim}_{4m}$ then the respective conclusion is also valid in $M^{\sim}_{4m}$. The same holds for (CP). From this it follows that the theorems of $\mathcal{H}^{\sim}_{4m}$ are valid in $M^{\sim}_{4m}$.

Suppose now that $\Gamma \vdash_{\mathcal{H}^{\sim}_{4m}} \beta$. If $\vdash_{\mathcal{H}^{\sim}_{4m}} \beta$ then β is valid in $M^{\sim}_{4}$, by the observation above. Thus, $\Gamma \models_{M^{\sim}_{4m}} \beta$. Otherwise, there exists a finite, non-empty set $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ such that $\vdash_{\mathcal{H}^{\sim}_{4m}} (\gamma_1 \Rightarrow (\ldots \Rightarrow (\gamma_n \Rightarrow \beta)\ldots))$. Then $\models_{M^{\sim}_{4m}} (\gamma_1 \Rightarrow (\ldots \Rightarrow (\gamma_n \Rightarrow \beta)\ldots))$ (as observed above) and so $\{\gamma_1, \ldots, \gamma_n\} \models_{\mathcal{H}^{\sim}_{4m}} \beta$. From this, $\Gamma \models_{M^{\sim}_{4m}} \beta$.
(ii) $\Rightarrow$ (i) (Completeness): Suppose that $\Gamma \models_{M^{\sim}_{4m}} \beta$. By Definition 4.6, there exists a finite subset $\{\gamma_1, \ldots, \gamma_n\}$ of Γ such that $\bigwedge\{h(\gamma) \ : \ \gamma \in \Gamma_0\} \leq h(\beta)$. If $n = 0$, that is, if $\models_{M^{\sim}_{4m}} \beta$, then $h(\beta) = 1$ for every $h \in Hom(\mathfrak{Fm}, \mathfrak{U})$ for all $\mathfrak{U} \in$ **IBA**, by Definition 4.6. In particular, $h(\beta) = 1$ for every $h \in Hom(\mathfrak{Fm}, \mathfrak{Fm}/\equiv)$, by Theorem 5.6. Let $h : \mathfrak{Fm} \to \mathfrak{Fm}/\equiv$ be the canonical map given by $h(\delta) = |\delta|$, for every δ. Then $h \in Hom(\mathfrak{Fm}, \mathfrak{Fm}/\equiv)$ and so $|\beta| = 1$. It follows that $\vdash_{\mathcal{H}^{\sim}_{4m}} \beta$. Thus, $\Gamma \vdash_{\mathcal{H}^{\sim}_{4m}} \beta$. On the other hand, if $n > 0$, $\gamma_1, \ldots, \gamma_n \models_{M^{\sim}_{4m}} \beta$. Then, by Remark 4.11, $\models_{M^{\sim}_{4m}} (\gamma_1 \Rightarrow (\ldots(\gamma_n \Rightarrow \beta))\ldots)$. By the former case, $\vdash_{\mathcal{H}^{\sim}_{4m}} (\gamma_1 \Rightarrow (\ldots(\gamma_n \Rightarrow \beta))\ldots)$. From this it follows that $\Gamma \vdash_{\mathcal{H}^{\sim}_{4m}} \beta$ as desired. ■

Corollary 5.8 *The consequence relation $\vdash_{\mathcal{H}^{\sim}_{4m}}$ introduced in Definition 5.3(2) is Tarskian, finitary and structural.*

6 $M_{4m}^{\sim}$ as a normal extension of S5

In this section, we will prove that there is still one more perspective to understand involutive Boolean algebras: they are modal algebras for $S5$ satisfying additional modal axioms and so they are generated by an algebra belonging to the hierarchy of Henle algebras (cf. [13]).

In [12] it was observed that, given an involutive Boolean algebra A, the operator $\exists x = x \vee \neg{\sim}x$ defines a possibility operator on A, in the sense of [10, 11]. It was also obtained a characterization of involutive Boolean algebras as a particular case of monadic Boolean algebras. Moreover, it was proved that the De Morgan negation $\neg$ can be defined in terms of $\exists$ and the other operators of A as

$$\neg x = (x \wedge \exists{\sim}x) \vee {\sim}\exists x. \tag{2}$$

It should be noted that Monteiro's definition of the possibility operator $\exists$ (or, in the notation of modal algebras, $\Diamond$, see Proposition 3.3(iii)) coincides, up to duality, with the necessity operator $\Box$ we found in Proposition 4.1 which, by Proposition 4.3, is unique. By adapting equation (2), it follows that

$$\neg x = (x \wedge {\sim}\Box x) \vee \Box{\sim}x. \tag{3}$$

From this, we can enrich a Boolean algebra with a modal operator $\Box$ satisfying certain properties, instead of adding a De Morgan negation. In both cases it is obtained the same class of structures, namely the involutive Boolean algebras, since the De Morgan negation can then be defined by equation (3). The key is to translate the equations which characterize $\neg$ into the language of Boolean algebras plus $\Box$. Thus:

Proposition 6.1 *Involutive Boolean algebras are definitionally equivalent to Boolean algebras equipped with a unary operator $\Box$ satisfying the following equations:*

$$\neg'(x \vee y) = \neg' x \wedge \neg' y \tag{4}$$

$$\neg'\neg' x = x \tag{5}$$

where $\neg' x$ is an abbreviation for $(x \wedge {\sim}\Box x) \vee \Box{\sim}x$, for every x.

Proof. As it was shown, given an involutive Boolean algebra then, by defining $\Box x = x \wedge \neg{\sim}x$, the obtained structure is a tetravalent modal algebra. Now, define $\neg' x = (x \wedge {\sim}\Box x) \vee \Box{\sim}x$. Then it is easy to see that $\neg' x = \neg x$ for every x. Since $\neg$ is a De Morgan negation and $\neg' x = \neg x$ for every x, the equations (4) and (5) hold.

Conversely, if a Boolean algebra is equipped with a monadic operator $\Box$ satisfying the equations (4) and (5) for $\neg' x$ given by $(x \wedge {\sim}\Box x) \vee \Box{\sim}x$ then clearly $\neg' x$ is a De Morgan negation and so the induced structure is an involutive Boolean algebra. ■

Proposition 6.2 *Let $\langle A, \wedge, \vee, \sim, \Box, 0\rangle$ be an involutive Boolean algebra seen as a Boolean algebra together with an operator $\Box$ as in Proposition 6.1. Then, the operator $\neg'$ defined by $\neg' x = (x \wedge \sim\Box x) \vee \Box\sim x$, for every $x \in A$, is the unique unary operator over A such that $\langle A, \wedge, \vee, \neg', \Box, 0\rangle$ is a tetravalent modal algebra.*

Proof. Clearly $\neg'$ is a De Morgan negation and so $\langle A, \wedge, \vee, \neg', \sim, 0\rangle$ is a De Morgan algebra enriched by a Boolean complement $\sim$. By Corollary 4.4, the operator $\Box' x = x \wedge \neg'\sim x$ is such that $\langle A, \wedge, \vee, \neg', \Box', 0\rangle$ is a tetravalent modal algebra. It is easy to prove that $\Box' x = \Box x$ and then it follows that $\langle A, \wedge, \vee, \neg', \Box, 0\rangle$ is a tetravalent modal algebra.

Now, suppose that $\neg$ is a De Morgan negation in A such that the structure $\langle A, \wedge, \vee, \neg, \Box, 0\rangle$ is a tetravalent modal algebra. By Proposition 4.3, $\Box x = x \wedge \neg\sim x$. From this, it follows easily that $\neg' x = (x \wedge \sim\Box x) \vee \Box\sim x = \neg x$ for every x. ■

Recall that a *modal algebra* is a structure $\langle A, \wedge, \vee, \sim, \Box, 0, 1\rangle$ of type $(2, 2, 1, 1, 0, 0)$ where the reduct $\langle A, \wedge, \vee, \sim, 0, 1\rangle$ is a Boolean algebra and $\Box$ is an unary operator on A such that $\Box 1 = 1$ and $\Box(x \wedge y) = \Box x \wedge \Box y$ (cf. [1]). By its turn, a *modal algebra for* **S5** is a modal algebra satisfying, additionally, the following relations, for every x:

(T) $\Box x \leq x$;

(4) $\Box x \leq \Box\Box x$;

(B) $x \leq \Box\sim\Box\sim x$.

We thus obtain the following result:

Proposition 6.3 *Involutive Boolean algebras are definitionally equivalent to modal algebras for modal logic* **S5** *satisfying additionally the equations (4) and (5) of Proposition 6.1.*

Proof. Let A be an involutive algebra A seen as a tetravalent modal algebra plus a Boolean complement $\sim$, by Corollary 4.5. Then, A is a modal algebra which satisfies the properties (T), (4), (B) above (cf. [6]). On the other hand, by Proposition 6.1 the equations (4) and (5) also hold. The converse is a consequence of Proposition 6.1. ■

Recall that a sentencial calculus L' is a (weak) *extension* of another sentencial calculus L if they are defined in the same language and every formula provable in L is provable in L'. In [13], it was introduced the notion of normal extension of **S5** as any extension S of **S5** that is closed under substitution, modus ponens and, if α is provable in S, then $\Box\alpha$ is provable in S.

On the other hand, recall that a *Henle algebra* is an algebra $\langle A, \wedge, \vee, \sim, i, 0, 1\rangle$ of type $(2, 2, 1, 1, 0, 0)$ such that its reduct $\langle A, \wedge, \vee, \sim, 0, 1\rangle$ is a Boolean Algebra, $i(1) = 1$ and $i(x) = 0$ for all $x \in A$, $x \neq 1$ (cf. [13]). A *Henle matrix* is the logical matrix $\langle H, \{1\}\rangle$ such that H is a Henle algebra. H_n is the Henle algebra with n atoms (2^n elements). Besides, if H is a finite Henle matrix, then there exists $n \in \mathbb{N}$ such that H and H_n are isomorphic.

Theorem 6.4 *([13]) The following conditions are equivalent:*

(i) *S is a proper normal extension of* **S5***;*

(ii) *S is the logic given by the matrix $\langle H_n, \{1\}\rangle$, for some $n \in \mathbb{N}$.*

By the results of Section 4, it is clear that $\mathfrak{M}^{\sim}_{4m}$ is a Henle algebra when it is presented in the language $\wedge, \vee, \sim, 0, 1$ and $\Box$. Then, it is immediate that $\mathfrak{M}^{\sim}_{4m} = H_2$.

Taking into account that $M^{\sim}_{4m}$ can be seen as a modal logic and that modal logics are mainly concerned with theorems (or valid formulas), we may consider the matrix logic given by the matrix $\mathcal{M}_1 = \langle \mathfrak{M}^{\sim}_{4m}, \{1\}\rangle$ over the signature $\wedge, \vee, \sim, 0, 1, \Box$. That is: $\Gamma \models_{\mathcal{M}_1} \alpha$ iff there exists a finite subset Γ_0 of Γ such that, for every homomorphism h over $\mathfrak{M}^{\sim}_{4m}$, if $h(\gamma) = 1$ for every $\gamma \in \Gamma_0$ then $h(\alpha) = 1$. Then, we have that $\mathcal{M}_1 = \langle H_2, \{1\}\rangle$ and so, by Theorem 6.4:

Theorem 6.5 *The matrix logic given by $\mathcal{M}_1$ is a proper normal extension of* **S5***.*

Remark 6.6 Observe that, in the terminology of [2], $\mathcal{M}_1$ is the truth-preserving logic associated to IBAs. The fact that the (global) logic of involutive Boolean algebras is an extension of **S5** is an interesting discovery: it means that the variety generated by the Henle algebra H_2 is **IBA**.

Finally, a Hilbert-style axiomatization for $\mathcal{M}_1$ in the language Fm can be easily obtained from $\mathcal{H}^{\sim}_{4m}$, following the technique used in [4] for the logics of TMAs. Indeed, let $(\mathcal{H}^{\sim}_{4m})^N$ be the Hilbert system identical to $\mathcal{H}^{\sim}_{4m}$, but where the notion of derivation is the usual one in Hilbert calculi. That is, instead of Definition 5.3 we consider the following:

Definition 6.7 Let $\Gamma \cup \{\alpha\}$ be a set of formulas in Fm. We say that α is derivable in $(\mathcal{H}^{\sim}_{4m})^N$ from Γ, and we write $\Gamma \vdash_{(\mathcal{H}^{\sim}_{4m})^N} \alpha$, if there exists a finite sequence of formulas $\alpha_1 \dots \alpha_n$ such that α_n is α and every α_i is either an instance of an axiom, or $\alpha_i \in \Gamma$, or α_i is the consequence of α_j and $\alpha_k = (\alpha_j \Rightarrow \alpha_i)$ by (MP) for some $j, k \leq i-1$, or $\alpha_i = (\neg\beta \Rightarrow \neg\gamma)$ is the consequence of $\alpha_j = (\gamma \Rightarrow \beta)$ by (CP) for some $j \leq i-1$.

Theorem 6.8 *(Soundness) If $\Gamma \vdash_{(\mathcal{H}^{\sim}_{4m})^N} \alpha$ then $\Gamma \models_{\mathcal{M}_1} \alpha$.*

Proof. Every axiom is valid, and the inference rules preserve validity. ■

In order to prove completeness, it will be useful to consider the modality $\Box\alpha =_{def} \alpha \wedge \neg{\sim}\alpha$, where $\alpha \wedge \beta =_{def} {\sim}(\alpha \Rightarrow {\sim}\beta)$ (recall Section 4). By adapting a result by [6], it is immediate to show the following:

Proposition 6.9 $\Gamma \models_{\mathcal{M}_1} \alpha$ *iff* $\Box\Gamma \models_{M^{\sim}_{4m}} \alpha$.

On the other hand, it is easy to prove the following:

Proposition 6.10 *For every formula α in Fm: $\alpha \vdash_{(\mathcal{H}^{\sim}_{4m})^N} \Box\alpha$.*

Proof. Let γ be a theorem of $(\mathcal{H}_{4m}^{\sim})^N$. Then $\vdash_{(\mathcal{H}_{4m}^{\sim})^N} \neg\neg\gamma$, by (A5) and (MP). Consider now the following sketch of derivation in $(\mathcal{H}_{4m}^{\sim})^N$:

1. α (Hyp.)
2. $\alpha \Rightarrow ({\sim}\alpha \Rightarrow \neg\gamma)$ (by classical logic)
3. ${\sim}\alpha \Rightarrow \neg\gamma$ (MP, 1,2)
4. $\neg\neg\gamma \Rightarrow \neg{\sim}\alpha$ (CP, 3)
5. $\neg{\sim}\alpha$ (MP with 4 and the theorem $\neg\neg\gamma$)
6. $\alpha \wedge \neg{\sim}\alpha$ (by 1,5 and classical logic)

■

Proposition 6.11 *If* $\Box\Gamma \vdash_{\mathcal{H}_{4m}^{\sim}} \alpha$ *then* $\Gamma \vdash_{(\mathcal{H}_{4m}^{\sim})^N} \alpha$.

Proof. Assume that $\Box\Gamma \vdash_{\mathcal{H}_{4m}^{\sim}} \alpha$. If $\vdash_{\mathcal{H}_{4m}^{\sim}} \alpha$ then the result is obvious, since both calculi have the same theorems. If $\nvdash_{\mathcal{H}_{4m}^{\sim}} \alpha$, there exists a finite, non-empty subset $\{\gamma_1, \ldots, \gamma_n\}$ of Γ such that $\vdash_{\mathcal{H}_{4m}^{\sim}} (\Box\gamma_1 \Rightarrow (\ldots(\Box\gamma_n \Rightarrow \alpha))\ldots)$, and so $\vdash_{(\mathcal{H}_{4m}^{\sim})^N} (\Box\gamma_1 \Rightarrow (\ldots(\Box\gamma_n \Rightarrow \alpha))\ldots)$. Consider now a derivation in $(\mathcal{H}_{4m}^{\sim})^N$ as follows: from the hypothesis $\gamma_1, \ldots, \gamma_n$ derive $\Box\gamma_1, \ldots, \Box\gamma_n$, by Proposition 6.10. Using (MP) n times with the theorem $(\Box\gamma_1 \Rightarrow (\ldots(\Box\gamma_n \Rightarrow \alpha))\ldots)$ it follows α. This shows that $\{\gamma_1, \ldots, \gamma_n\} \vdash_{(\mathcal{H}_{4m}^{\sim})^N} \alpha$, and so $\Gamma \vdash_{(\mathcal{H}_{4m}^{\sim})^N} \alpha$. ■

Theorem 6.12 *(Completeness) If* $\Gamma \models_{\mathcal{M}_1} \alpha$ *then* $\Gamma \vdash_{(\mathcal{H}_{4m}^{\sim})^N} \alpha$.

Proof. If $\Gamma \models_{\mathcal{M}_1} \alpha$ then $\Box\Gamma \models_{M_{4m}^{\sim}} \alpha$, by Proposition 6.9. Using Theorem 5.7 it follows that $\Box\Gamma \vdash_{\mathcal{H}_{4m}^{\sim}} \alpha$ and so $\Gamma \vdash_{(\mathcal{H}_{4m}^{\sim})^N} \alpha$, by Proposition 6.11. ■

7 Concluding Remarks

This paper shows that tetravalent modal algebras expanded with a deductive implication (or, equivalently, with a Boolean complement) are definitionally equivalent to De Morgan algebras expanded with a Boolean complement, that is, Boolean algebras plus a De Morgan negation. The latter are known as symmetric (or involutive) Boolean algebras. On the other hand, we also shown that these structures are modal algebras for modal logic **S5** satisfying additional equations such that they are generated by the Henle algebra H_2. Thus, one of the two logics that can be naturally associated to these algebraic structures is a proper normal extension of **S5**. Finally, a simple Hilbert presentation for these two logics was also proposed, proving the corresponding soundness and completeness theorems. In particular, the Hilbert calculus we propose for the normal extension of **S5** is interesting, since it does not include any axiom or inference rule using the modal operators.

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