Dugundji's Theorem Revisited

Marcelo E. Coniglio[a] and Newton M. Peron[b]

[a] *Department of Philosophy, IFCH, and*

Centre for Logic, Epistemology and The History of Science (CLE)

State University of Campinas (UNICAMP), Campinas, SP, Brazil

`coniglio@cle.unicamp.br`

[b] *Federal University of South Frontier (UFFS)*

Chapecó, SC, Brazil

newton.peron@uffs.edu.br

Abstract

In 1940 Dugundji proved that no system between **S1** and **S5** can be characterized by finite matrices. Dugundji's result forced the development of alternative semantics, in particular Kripke's relational semantics. The success of this semantics allowed the creation of a huge family of modal systems. With few adaptations, this semantics can characterize almost the totality of the modal systems developed in the last five decades.

This semantics however has some limits. Two results of incompleteness (for the systems **KH** and **VB**) showed that not every modal logic can be characterized by Kripke frames. Besides, the creation of non-classical modal logics puts the problem of characterization of finite matrices very far away from the original scope of Dugundji's result.

In this sense, we will show how to update Dugundji's result in order to make precise the scope and the limits of many-valued matrices as semantic of modal systems. A brief comparison with the useful Chagrov and Zakharyaschev's criterion of tabularity for modal logics is provided.

1 Introduction

The birth of symbolic modal logic seems to have a date: in general it is postulated[1] that Lewis inaugurated in 1912 this large family of logics. Aiming to create a new implication, the *strict implication*, the author proposes in 1918 the system **S3**.

Shortly thereafter, in 1920, Łukasiewicz presents a set of matrices for a 3-valued logic $\mathbf{Ł}_3$ in order to modelize the new modal concept of *possibly* true.[2] In 1932, Lewis ppoposes the hierarchy **S1-S5**. Thus, a question arises: is it possible that finite logical matrices can characterize the systems **S1-S5**?

[1] According to [1].

[2] See [18].

This question was resolved by Dugundji eight years later: it was shown that not only Łukasiewicz's matrices, but no finite matrix can be a complete semantics for any system between **S1** and **S5**.

It is worth noting that the system **S5** has the particular property of being a limit to the many-valued modal semantics. As shown by S. Scroogs in his 1951's article [22], every regular extension of **S5** (that is, every set of formulas in the language of **S5** extending its theorems and closed under substitutions, *modus ponens* and the necessitation rule) can be characterized by finite matrices. Besides, these extensions can be axiomatized by adding to **S5** instances of the formula used by Dugundji in order to prove his incompleteness result.

A natural question is to find another modal system with the feature of being a limit in the above sense. This problem was solved in 1977 by L. Esakia and V. Meskhi (see [10]), showing that there exist exactly four regular extensions of **S4** different to **S5** – the systems **K1.2**, **K2.2**, **K3.1** and **K3.2**[3] – having the same property of **S5**, namely: every regular extension of them can be characterized by finite matrices.

Observe that the four systems above are outside the Lewis hierarchy and so they are outside the scope of Dugundji's theorem. There is also a large list of important modal systems based on Propositional Classical Logic **CPL** that are also outside the scope of Dugundji's result. Among them it is worth mentioning: **K**, **D**, **T**, **B**, **GL**, **VB**, **KH**, **S0.5**, and others.

It is not clear, on the other hand, that Dugundji's argument holds for modal logics whose non-modal propositional fragment is not classical (such as implicative, positive, paraconsistent or paracomplete modal logics).

Perhaps one of the most interesting among all these fragments was proposed by Henkin[4] in 1949. Henkin's system has only the implication $\supset$ as operator, which preserves convenient properties such as the Deduction Metatheorem. We will see that a big family of modal systems whose non-modal propositional substratum is between Henkin's Implicative Calculus and Propositional Classical Logic cannot be characterized by finite matrices.

What we demonstrate here[5] is that the original result of Dugundji can be extended in two different senses: by embracing many modal systems developed from the forties until today, on the one hand, and by considering some modal logics whose non-modal fragment is not classical, on the other.

2 The Lewis' Systems S1-S5

The system **S1**, the first one of the five systems of the Lewis' hierarchy, is defined as an extension of Propositional Classical Logic **PC** (presented in the propositional language just containing $\neg$, $\wedge$), by adding an unary operator $\Diamond$.

[3]The systems **K1.1**, **K2.2** and **K3.1** were proposed by Sobociński in [23], while **K3.2** was formulated by Zeman in [24], pp.253.

[4]In [14].

[5]Although it was considered the original Dugundji's article [9], we preferred to follow the clearer proof of it given in [5].

By defining the *strict implication* $\strictif$ as $\alpha \strictif \beta = \neg\Diamond(\alpha \wedge \neg\beta)$, the system **S1** adds to **PC** the following:[6]

Axioms

(B1)	$(p \wedge q) \strictif (q \wedge p)$
(B2)	$(p \wedge q) \strictif p$
(B3)	$p \strictif (p \wedge p)$
(B4)	$(p \wedge (q \wedge r)) \strictif ((p \wedge q) \wedge r)$
(B6)	$((p \strictif q) \wedge (q \strictif r)) \strictif (p \strictif r)$
(B7)	$(p \wedge (p \strictif q)) \strictif q$

Inference Rules

Uniform Substitution	A valid formula remains valid if a formula is uniformly substituted in it for a propositional variable
Substitution of Strict Equivalents	Two strictly equivalent formulas are intersubstitutable, where α and β are strictly equivalent if $\alpha \strictif \beta$ and $\beta \strictif \alpha$ are both valid.
Adjunction	$\alpha \wedge \beta$ follows from α and β
Strict Inference	β follows from α and $\alpha \strictif \beta$

It can be proven that $\strictif$ enjoys some useful properties of an implication connective. For instance, $p \strictif p$ is a theorem of **S1** (notation: $\vdash_{\mathbf{S1}} p \strictif p$):

1.	$p \strictif (p \wedge p)$	[(B3)]
2.	$(p \wedge p) \strictif p$	[*Uniform Substitution* in (B2)]
3.	$(p \strictif (p \wedge p)) \wedge ((p \wedge p) \strictif p)$	[*Adjunction* in 1 and 2]
4.	$((p \strictif (p \wedge p)) \wedge ((p \wedge p) \strictif p)) \strictif (p \strictif p)$	[*Uniform Substitution* in (B6)]
5.	$p \strictif p$	[*Strict Inference* in 3 and 4]

Now, consider the following list of axioms:

(B8)	$\Diamond(p \wedge q) \strictif \Diamond p$
(A8)	$(p \strictif q) \strictif (\neg\Diamond q \strictif \neg\Diamond p)$
(C10)	$\neg\Diamond\neg p \strictif \neg\Diamond\neg\neg\Diamond\neg p$
(C11)	$\Diamond p \strictif \neg\Diamond\neg\Diamond p$

Then the Lewis hierarchy is defined as follows:

- **S2** $=$ **S1** $\cup \{$(B8)$\}$
- **S3** $=$ **S1** $\cup \{$(A8)$\}$

[6] The axioms and rules below can be found in [15] and [1]. We exclude axiom (B5) $p \strictif \neg\neg p$, since McKinsey proves in [20] that it can be deduced from the others.

- **S4** = **S1** ∪ {(C10)}
- **S5** = **S3** ∪ {(C11)}

Lewis shown that **S1** ⊂ **S2** ⊂ **S3** ⊂ **S4** ⊂ **S5**, by means of the following strategy: he introduces four-valued truth-tables for the connectives of these logics with some designated truth-values, which satisfy all the axioms of **S1**, unless (B8), such that the inferences rules take designated values into designated ones; thus, **S1** ⊂ **S2**. The same method is used to prove the other (strict) inclusions.

Up to now, we just introduce axioms and rules for $\Diamond$ and $\prec$. Gödel introduced, for the first time, the operator $\Box$ defined as $\neg\Diamond\neg$, with the aim of showing that the intuitionistic propositional calculus **IPC** can be interpreted in **S4**. In order to do this, he proposed the following version of **S4**:[7]

(K) $\Box(p \to q) \to (\Box p \to \Box q)$
(T) $\Box p \to p$
(4) $\Box p \to \Box\Box p$

Necessitation Rule If α is a theorem then $\Box\alpha$ is also a theorem

In order to obtain **S5** it is sufficient to add to **S4** the following:

(5) $\Diamond p \to \Box\Diamond p$

3 Dugundji's Theorem

In a short paper (having just two pages) Dugundji showed that no finite matrix can characterize any modal logic between **S1** and **S5**.

The procedure is simple: by defining a family of formulas Σ_n of the modal language (for $n \geq 1$), it is shown that any matrix with n truth-values which models **S1** (or any extension of it) also validates Σ_n. On the other hand, it is shown that there exist an infinite matrix which models **S5** but it does not validade any Σ_n. But then, if any system between **S1** and **S5** could be characterized by a finite matrix with, say, n elements, then Σ_n would be a theorem of that system (because of the first result and by completeness) and so Σ_n would be a theorem of **S5**, contradicting the second result.

In order to formalize the argument above, it is necessary to recall some notions about matrix semantics.

Definition 3.1 *Let $\mathbb{L}$ be a propositional language. A matrix $\mathcal{M}$ over $\mathbb{L}$ is a triple $\mathcal{M} = \langle M, D, O\rangle$ in which:*

(i) $M \neq \emptyset$ is a set of truth-values

(ii) $D \subseteq M$ is a set of designated truth-values

[7] This axiomatic, as well as the translation of **IPC** into **S4**, can be found in [11].

(iii) O is a set of operations over M interpreting the connectives of $\mathbb{L}$. □

Definition 3.2 *A matrix $\mathcal{M}$ characterizes a propositional logical system **L** if O contains an operation for each connective of the language of **L**, of the same arity, such that all theorems of **L** and only them receive designated values when **L** is interpreted in $\mathcal{M}$.*[8] *A matrix $\mathcal{M}$ is a model of a logical system **L** if all theorems of **L** (but not necessarily only them) receive designated values when **L** in interpreted in $\mathcal{M}$* □

It is worth noting that the formulas proposed by Dugundji are adapted from the formulas used by Gödel in order to prove the incompleteness of **IPC** (Intuitionistic Propositional Calculus) by finite matrices, by substituting the material bi-implication by the strict bi-implication.[9] Dugundji's strategy will be followed here, by proposing the formulas below:

Definition 3.3 *For each natural number n, the adapted Dugundji's formula D_n is defined in the following way:*

$$D_n =_{def} \bigvee_{i \neq j} (p_i \twoheadrightarrow p_j)$$

in which $1 \leq i, j \leq n+1$ and $p_i \twoheadrightarrow p_j$ means $\Box(p_i \supset p_i) \supset \Box(p_j \supset p_i)$. □

Proposition 3.4 *Any finite matrix with n truth-values which is a model of an extension of **S1** validates the formula D_n.*

Proof:

Suppose that there exists a matrix $\mathcal{M}$ with n truth-values which models an extension of **S1**. Let v be a valuation over $\mathcal{M}$. Since there are only n values for $n+1$ variables, there exists i and j such that the values assigned by v to p_i and p_j will coincide, and so the values assigned by v to $(p_i \twoheadrightarrow p_j)$ and $(p_i \twoheadrightarrow p_i)$ will also coincide. On the other hand $\vdash_{\mathbf{S1}} p \to p$ and so, by **PC**, (K) and *Necessitation Rule*, it follows that $(p_i \twoheadrightarrow p_i)$ is a theorem of **S1**. By **PC**, it follows that $(p_i \twoheadrightarrow p_i) \vee \alpha$ and $\alpha \vee (p_i \twoheadrightarrow p_i)$ are also theorems of **S1**, for every α. Therefore the valuation v satisfies $\alpha \vee (p_i \twoheadrightarrow p_i) \vee \beta$ for every α and β, and so it satisfies D_n. Thus, $\mathcal{M}$ validates Dugundji's adapted formula D_n. ■

The second part of Dugundji's argument shows that there exists an infinite matrix which models **S5** but it does not validate any formula D_n of Dugundji.

Definition 3.5 *Let $\mathcal{M}_\infty$ be the infinite matrix such that*

- *$M = \wp(\mathbb{N})$, that is, the powerset of the set $\mathbb{N}$ of natural numbers;*

[8] Technically, such interpretations are given by means of homomorphisms (valuations) from the algebra of formulas of **L** into the algebra $\langle M, O \rangle$.

[9] The original proof of Gödel's result can be found in [12].

- $D = \{\mathbb{N}\}$;
- $O = \{\cup, \cap, \overline{(\cdot)}, \blacksquare, \blacklozenge\}$,

where $\cup, \cap$ *and* $\overline{(\cdot)}$ *denote the usual set-theoretic operations of union, intersection and complement, respectively, and where* $\blacksquare$ *and* $\blacklozenge$ *are defined as follows:*

$$\blacksquare X = \begin{cases} \mathbb{N} & \text{if } X = \mathbb{N} \\ \emptyset & \text{otherwise} \end{cases}$$

$$\blacklozenge X = \begin{cases} \mathbb{N} & \text{if } X \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}.$$

Proposition 3.6 *The infinite matrix* $\mathcal{M}_\infty$ *models* **S5***, but does not validate any formula* D_n*.*

Proof: We will prove that every axiom of **S5** is valid in the matrix $\mathcal{M}_\infty$, and that every valuation over $\mathcal{M}_\infty$ which satisfies the premises of an inference rule of **S5** also satisfies the conclusion of that rule. Let $v : Var \longrightarrow M$ be a valuation over $\mathcal{M}_\infty$. Thus, v is a function which assigns an element of $\wp(\mathbb{N})$ to any propositional variable of the language of **S5** and, for every formulas α and β:

- $v(\neg\alpha) = \overline{v(\alpha)}$;
- $v(\alpha \to \beta) = \overline{v(\alpha)} \cup v(\beta)$;
- $v(\Box\alpha) = \blacksquare(v(\alpha))$;
- $v(\Diamond\alpha) = \blacklozenge(v(\alpha))$.

With respect to the axioms of classical logic and the rule of *Modus Ponens*, they are satisfied by v because of the algebraic completeness of classical logic with respect to Boolean algebra. Concerning the modal axioms:

(K) $v(\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)) = \overline{\blacksquare(\overline{v(\alpha)} \cup v(\beta))} \cup \overline{\blacksquare v(\alpha)} \cup \blacksquare v(\beta)$

- if $v(\alpha) \neq \mathbb{N}$, then $\overline{\blacksquare v(\alpha)} = \mathbb{N}$
- if $v(\alpha) = \mathbb{N}$, then
  - if $v(\beta) = \mathbb{N}$, then $\blacksquare v(\beta) = \mathbb{N}$
  - if $v(\beta) \neq \mathbb{N}$, then $\overline{v(\alpha)} \cup v(\beta) \neq \mathbb{N}$ and $\overline{\blacksquare(\overline{v(\alpha)} \cup v(\beta))} = \mathbb{N}$

(T) $v(\Box\alpha \to \alpha) = \overline{\blacksquare v(\alpha)} \cup v(\alpha)$

- if $v(\alpha) = \mathbb{N}$, then $\overline{\blacksquare v(\alpha)} \cup v(\alpha) = \mathbb{N}$
- if $v(\alpha) \neq \mathbb{N}$, then $\overline{\blacksquare v(\alpha)} = \mathbb{N}$

(4) $v(\Box\alpha \to \Box\Box\alpha) = \overline{\blacksquare v(\alpha)} \cup \blacksquare\blacksquare v(\alpha)$

- if $v(\alpha) = \mathbb{N}$, then $\blacksquare v(\alpha) = \mathbb{N}$ and $\blacksquare\blacksquare v(\alpha) = \mathbb{N}$
- if $v(\alpha) \neq \mathbb{N}$, then $\blacksquare v(\alpha) = \emptyset$ and $\overline{\blacksquare v(\alpha)} = \mathbb{N}$

(5) $v(\Diamond\alpha \to \Box\Diamond\alpha) = \overline{\blacklozenge v(\alpha)} \cup \blacksquare\blacklozenge v(\alpha)$

- if $v(\alpha) = \emptyset$, then $\blacklozenge v(\alpha) = \emptyset$ and $\overline{\blacklozenge v(\alpha)} = \mathbb{N}$
- if $v(\alpha) \neq \emptyset$, then $\blacklozenge v(\alpha) = \mathbb{N}$ and $\blacksquare\blacklozenge v(\alpha) = \mathbb{N}$

Concerning the *Necessitation Rule*, observe that if $v(\alpha) = \mathbb{N}$ then $\blacksquare v(\alpha) = \mathbb{N}$. From this, it follows (by induction on the length of derivations) that $\mathcal{M}_\infty$ validates any theorem of **S5** and so it is a model of **S5**.

Finally, observe that every formula D_n can be falsified in $\mathcal{M}_\infty$. In fact, consider the valuation v over $\mathcal{M}_\infty$ which assigns to each propositional variable p_i the singleton $X_i = \{i\} \in \wp(\mathbb{N})$. Note that, if $i \neq j$, then $\overline{X_i} \cup X_j \neq \mathbb{N}$ and $\overline{X_j} \cup X_i \neq \mathbb{N}$. Thus:

$$v(p_i \twoheadrightarrow p_j) = \blacksquare(\overline{X_i} \cup X_i) \cap \blacksquare(\overline{X_j} \cup X_i) = \emptyset \cap \emptyset = \emptyset.$$

Therefore, Dugundji's formula D_n takes the non-designated value $\emptyset$ under the valuation v in the matrix $\mathcal{M}_\infty$. ■

Theorem 3.7 (Dugundji's Theorem) *No modal system* **L** *between* **S1** *and* **S5** *can be characterized by a finite matrix.*

Proof: Suppose that some system **L** between **S1** and **S5** can be characterized by a matrix with n elements. Then, by Proposition 3.4, the formula D_n would be validated by such matrix and so D_n would be a theorem of **L**. Then, D_n would be a theorem of **S5** and so $\mathcal{M}_\infty$ would validate D_n, which is an absurd, by Proposition 3.6. ■

4 Other modal systems

Consider the following axiom schemas and rules of inference, where α, β and γ are (meta)variables ranging over formulas:

(A1)	$\alpha \supset (\beta \supset \alpha)$
(A2)	$(\alpha \supset \beta) \supset ((\alpha \supset (\beta \supset \gamma)) \supset (\alpha \supset \gamma))$
(A3)	$(\alpha \supset \gamma) \supset (((\alpha \supset \beta) \supset \gamma) \supset \gamma)$
(GL)	$\Box(\Box\alpha \supset \alpha) \supset \Box\alpha$
(MP)	from α and $\alpha \supset \beta$ infer β
(N)	if $\vdash \alpha$ then $\vdash \Box\alpha$
(N')	if $\vdash \alpha$ and α is a $\mathbf{PC}^{\supset}$-tautology, then $\vdash \Box\alpha$
(N*)	if $\vdash \alpha \supset \beta$ then $\vdash \Box\alpha \supset \Box\beta$

Definition 4.1

(i) $\mathbf{PC}^{\supset} = \{$(A1), (A2), (A3),(MP)$\}$[10]

(ii) $\mathbf{S0.5}^{0,\supset} = \mathbf{PC}^{\supset} \cup \{$(K),(N')$\}$

(iii) $\mathbf{C2}^{\supset} = \mathbf{PC}^{\supset} \cup \{$(K),(N*)$\}$[11]

(iv) $\mathbf{K}^{\supset} = \mathbf{PC}^{\supset} \cup \{$(K),(N)$\}$

(v) $\mathbf{GL} = \mathbf{K} \cup \{$(4),(GL)$\}$

Since we are considering weaker versions of **S5** whose non-modal fragment does not have a classical negation, it will be convenient to consider the following definition of disjunction in $\mathbf{PC}^{\supset}$ in terms of the material implication:

$$\alpha \vee \beta =_{def} (\alpha \supset \beta) \supset \beta.$$

Theorem 4.2

(i) $\vdash_{\mathbf{PC}^{\supset}} \alpha \supset \alpha$

(ii) $\vdash_{\mathbf{PC}^{\supset}} \alpha \supset (\alpha \vee \beta)$

(iii) $\vdash_{\mathbf{PC}^{\supset}} \alpha \supset (\beta \vee \alpha)$

(iv) $\vdash_{\mathbf{PC}^{\supset}} (\alpha \vee (\beta \vee \gamma)) \supset ((\alpha \vee \beta) \vee \gamma)$

(v) $\vdash_{\mathbf{PC}^{\supset}} ((\alpha \vee \beta) \vee \gamma) \supset (\alpha \vee (\beta \vee \gamma))$

Proof: See [4], p. 27-28.[12] ■

Will will prove that any modal logic between $\mathbf{S0.5}^0$ and **S5** or between **C2** and **S5** whose non-modal fragment is between $\mathbf{PC}^{\supset}$ and **PC** can not be characterized by finite matrices. Then, we will show an analous result for systems between **K** and **GL**.

5 Enlarging the scope of Dugundji's Theorem

In this section, three results will be obtained (theorems 5.3, 5.4 and 5.8) which enlarge the original scope of Dugundji's Theorem.

Proposition 5.1 *Any finite matrix with n truth-values that is a model of an extension of* $\mathbf{S0.5}^{0,\supset}$ *validates D_n.*

[10] This constitutes the original axiomatization of Henkin's system $\mathbf{PC}^{\supset}$ in [14].

[11] The systems **S0.5** and **C2** were considered by Lemmon in [17] and [16] as being modally minimal. $\mathbf{S0.5}^0$ is obtained by removing the axiom **(T)** from **S0.5**, check [8] p. 207. As the reader may have noticed, the notation $^{\supset}$ means that the non-modal propositional base is $\mathbf{PC}^{\supset}$.

[12] The defined disjunction satisfies additional interesting properties; however, the above features are enough for our purposes.

Proof: Analogous to that for Proposition 3.4, replacing (N) by (N') and **PC** by $\mathbf{PC}^{\supset}$. ■

Proposition 5.2 *Any finite matrix with n truth-values that is a model of an extension of* $\mathbf{C2}^{\supset}$ *validates* D_n.

Proof: Analogous to that for Proposition 3.4, replacing (N) by (N*) and **PC** by $\mathbf{PC}^{\supset}$. ■

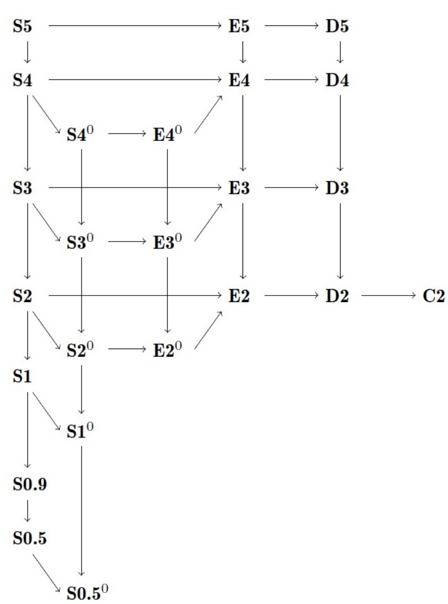
Theorem 5.3 *No modal system* **L** *between* $\mathbf{S0.5}^0$ *and* **S5** *whose non-modal fragment is between* $\mathbf{PC}^{\supset}$ *and* **PC** *can be characterized by finite matrices.*

Proof: Based on Proposition 3.6, the proof is analogous to that for Theorem 3.7, but now using Proposition 5.1 instead of Proposition 3.4. ■

Theorem 5.4 *No modal system* **L** *between* **C2** *and* **S5** *whose non-modal fragment is between* $\mathbf{PC}^{\supset}$ *and* **PC** *can be characterized by a finite matrix.*

Proof: The proof is similar to the previous one, but now using Proposition 5.2. ■

From theorems 5.3 and 5.4, the scope of the new versions of Dugundji's Theorem includes now the systems displayed in the diagram below.[13]



[13]In the diagram, $\mathbf{L_1} \longrightarrow \mathbf{L_2}$ means that $\mathbf{L_1}$ is an proper extension of **L2**. See [22].

Proposition 5.5 *Any finite matrix of n truth-values that is a model of $\mathbf{K}^{\supset}$ validates also the formula D_n.*

Proof: The argument is analogous to that of Proposition 3.4. ■

Definition 5.6 *Let $\mathcal{M}'_\infty$ be the infinite matrix such that*

- $M = \wp(\mathbb{N})$
- $D = \{\mathbb{N}\}$
- $O = \{\cup, \cap, \overline{(\cdot)}, \boxtimes\}$, *in which $\cup, \cap$ and $\overline{(\cdot)}$ denote the usual set-theoretic operations of union, intersection and complement, respectively, while the operator $\boxtimes$ is defined as follows:*[14]

$$\boxtimes X = \begin{cases} \mathbb{N} & \text{if } X \text{ is cofinite} \\ \mathbb{N} - \{0\} & \text{otherwise} \end{cases} .$$

Consider valuations over $\mathcal{M}'_\infty$ as mappings v which assign an element of $\wp(\mathbb{N})$ to each formula of **GL** *in the following way:*

- $v(\neg\alpha) = \overline{v(\alpha)}$;
- $v(\alpha \to \beta) = \overline{v(\alpha)} \cup v(\beta)$;
- $v(\Box\alpha) = \boxtimes(v(\alpha))$.

Proposition 5.7 *The infinite matrix $\mathcal{M}'_\infty$ is a model of* **GL**.

Proof: Let us see that every modal axiom is valid in $\mathcal{M}'_\infty$:

(K) $v(\mathrm{K}) = \overline{\boxtimes(\overline{v(\alpha)} \cup v(\beta))} \cup \overline{\boxtimes v(\alpha)} \cup \boxtimes v(\beta)$

- if $v(\beta)$ is cofinite, then $\boxtimes v(\beta) = \mathbb{N}$ and so $v(\mathrm{K}) = \mathbb{N}$.
- if $v(\beta)$ is not cofinite, then $\boxtimes v(\beta) = \mathbb{N} - \{0\}$.
  - if $v(\alpha)$ is not cofinite, then $\boxtimes v(\alpha) = \mathbb{N} - \{0\}$. Then $\overline{\boxtimes v(\alpha)} = \{0\}$ and $\overline{\boxtimes v(\alpha)} \cup \boxtimes v(\beta) = \mathbb{N}$. Therefore $v(\mathrm{K}) = \mathbb{N}$.
  - if $v(\alpha)$ is cofinite, then $\overline{v(\alpha)}$ is finite. Thus, $\overline{v(\alpha)} \cup v(\beta)$ is not cofinite. Then, $\boxtimes(\overline{v(\alpha)} \cup v(\beta)) = \mathbb{N} - \{0\}$ and so $\overline{\boxtimes(\overline{v(\alpha)} \cup v(\beta))} = \{0\}$. Therefore $v(\mathrm{K}) = \mathbb{N}$.

(4) $v(4) = \overline{\boxtimes v(\alpha)} \cup \boxtimes \boxtimes v(\alpha)$

- if $v(\alpha)$ is cofinite, then $\boxtimes v(\alpha) = \mathbb{N}$. Since $\mathbb{N}$ is cofinite, it follows that $\boxtimes \boxtimes v(\alpha) = \mathbb{N}$ and so $v(4) = \mathbb{N}$.

[14]The function that calculates $\boxtimes$ was inspired in [21], as an example of a modal operator of diagonalizable algebras.

- if $v(\alpha)$ is not cofinite, then $\boxtimes v(\alpha) = \mathbb{N} - \{0\}$. Since $\mathbb{N} - \{0\}$ is cofinite, then $\boxtimes \boxtimes v(\alpha) = \mathbb{N}$ and so $v(4) = \mathbb{N}$.

(GL) $v(\mathrm{GL}) = \overline{\boxtimes(\overline{\boxtimes v(\alpha)} \cup v(\alpha))} \cup \boxtimes v(\alpha)$

- if $v(\alpha)$ is cofinite, then $\boxtimes v(\alpha) = \mathbb{N}$ and $v(\mathrm{GL}) = \mathbb{N}$.
- if $v(\alpha)$ is not cofinite, then $\boxtimes v(\alpha) = \mathbb{N} - \{0\}$. Thus $\overline{\boxtimes v(\alpha)} = \{0\}$ and so $\overline{\boxtimes v(\alpha)} \cup v(\alpha)$ is not cofinite. Then, $\boxtimes(\overline{\boxtimes v(\alpha)} \cup v(\alpha)) = \mathbb{N} - \{0\}$ and $\overline{\boxtimes(\overline{\boxtimes v(\alpha)} \cup v(\alpha))} = \{0\}$. Therefore $v(\mathrm{GL}) = \{0\} \cup (\mathbb{N} - \{0\}) = \mathbb{N}$.

Finally, if $v(\alpha) = \mathbb{N}$, then $v(\alpha)$ is cofinite and $\boxtimes v(\alpha) = \mathbb{N}$. So, $\mathcal{M}'_\infty$ preserves (N) and all the **GL** axioms. ■

Theorem 5.8 *No system* **L** *between* **K** *and* **GL** *whose non-modal fragment is between* $\mathbf{PC}^{\supset}$ *and* **PC** *can be characterized by finite matrices.*

Proof: Given $n \geq 1$, consider the Dugundji's formula D_n and the matrix $\mathcal{M}'_\infty$ of Proposition 5.7. Let v be the valuation over $\mathcal{M}'_\infty$ which associates to each propositional variable p_i (for $1 \leq i \leq n+1$) the set

$$X_i = \{x \ : \ x = (n+1) \cdot k + (i-1) \text{ for some } k \in \mathbb{N}\}.$$

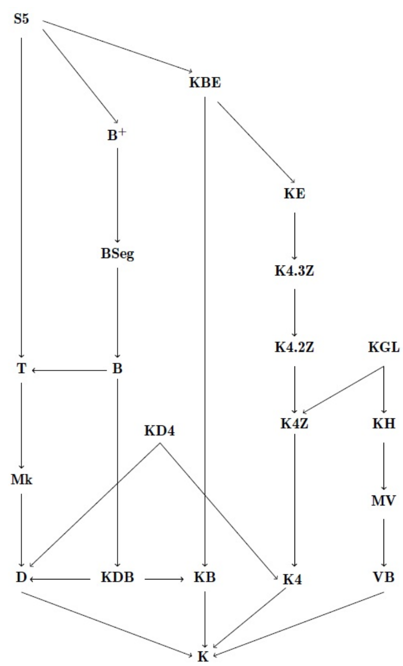
Let $1 \leq i, j \leq n+1$ such that $i \neq j$. Then, $X_j \cap X_i = \emptyset$ and so $\overline{X_j} \cup X_i = \overline{X_j}$ such that $\overline{X_j}$ is not cofinite. From this,

$$\overline{\boxtimes(\overline{X_i} \cup X_i)} \cup \boxtimes(\overline{X_j} \cup X_i) = \overline{\boxtimes \mathbb{N}} \cup \boxtimes \overline{X_j} = \emptyset \cup (\mathbb{N} - \{0\}) = \mathbb{N} - \{0\}$$

and then v assigns to D_n the non-designated value $\mathbb{N} - \{0\}$.

Suppose now that there is an n-valued matrix $\mathcal{M}'_n$ which characterizes a system **L** between **K** and **GL**. Then, by Proposition 5.5, the formula D_n would be a theorem of **L** and so a theorem of **GL**. But then, by Proposition 5.7, D_n would receive the designated value $\mathbb{N}$ through the valuation v in $\mathcal{M}'_\infty$, a contradiction. ■

The diagram below displays some well-known modal systems that lies within the scope of Theorem 5.8, another new version of Dugundji's Theorem.



6 Chagrov and Zakharyaschev's criterion of tabularity

The question of finding a finite matrix semantics for a logic (modal or not) is related to decidability of that logic. Indeed, a logic characterized by a finite matrix is decidable, and this is why Dugundji-like theorems are so relevant. A particular case of decidability of a modal system **L** is obtained by the so-called *tabularity* property:

Definition 6.1 *A modal system is* tabular *if it can be characterized by a finite Kripke frame. That is, there exists a Kripke frame $\mathcal{F} = \langle W, R \rangle$ where W is finite such that, for every formula α of the language of* **L***: $\langle \mathcal{F}, V \rangle, w \Vdash \alpha$ for every valuation V over $\mathcal{F}$ and every $w \in W$ (or for every distinguished w, if* **L** *is not normal), iff $\vdash_{\mathbf{L}} \alpha$.*

It should be observed that tabularity is a particular case of characterizability by a finite matrix. Indeed, if a modal system **L** is characterized by a finite frame $\mathcal{F} = \langle W, R \rangle$ then it is characterized by a finite matrix $\mathcal{M}_{\mathcal{F}} = \langle \wp(W), D, O \rangle$ such that $D = \{Z\}$, where Z is W, if **L** is normal, and Z is the set of distinguished worlds, otherwise. The interpretation O of the connectives is defined as expected, namely $O(\wedge)(X, Y) = X \cap Y$, $O(\vee)(X, Y) = X \cup Y$, $O(\supset)(X, Y) = (W \setminus X) \cup Y$, $O(\neg)(X) = W \setminus X$, and

$$O(\Box)(X) =_{def} \{w \in W \ : \ R[w] \subseteq X\}$$

$$O(\Diamond)(X) =_{def} \{w \in W \ : \ R[w] \cap X \neq \emptyset\}$$

for every $X, Y \in \wp(W)$, where $R[w] = \{w' \in W \ : \ wRw'\}$. Thus, a tabular modal logic is a modal logic characterizable by a finite matrix having a single designated truth-value.

In [6] a criterion of tabularity was obtained. In fact, they show that a modal logic which extends **K** can be characterized by a finite Kripke frame iff certain formulas are derivable (see [6], Theorem 12.1 (i)). More precisely:

Theorem 6.2 *Fix $n \geq 1$. Let $p_1, \ldots, p_n$ be the first n propositional variables in Var, and let φ_i be the formula $p_1 \wedge p_2 \wedge \ldots \wedge p_{i-1} \wedge \neg p_i \wedge p_{i+1} \wedge \ldots \wedge p_n$, for $1 \leq i \leq n$. Consider the formulas α_n and β_n defined as follows:*

$$\alpha_n =_{def} \neg(\varphi_1 \wedge \Diamond(\varphi_2 \wedge \Diamond(\varphi_3 \wedge \ldots \wedge \Diamond\varphi_n)\ldots))$$

$$\beta_n =_{def} \bigwedge_{m=0}^{n-1} \neg\Diamond^m(\Diamond\varphi_1 \wedge \ldots \wedge \Diamond\varphi_n).$$

Then, an extension **L** *of* **K** *is tabular iff $\vdash_{\mathbf{L}} (\alpha_n \wedge \beta_n)$ for some $n \geq 1$.*

Being so, an extension **L** of **K** is tabular iff there is some $n \geq 1$ such that the canonical model of **L** has the following properties:

1. if $w_1 R w_2 R \ldots R w_k$ (for k distinct worlds) then $k < n$ (w_1 must be distinguished, if **L** is not normal);

2. in every chain as above, w_k is of branching $\leq n - 1$, that is: if $w_k R w'_1$, $\ldots$, $w_k R w'_m$ (for m distinct worlds) then $m < n$.

It is worth noting that Theorem 5.8 is related to Theorem 6.2, in the following sense: if **L** is an extension of **K** contained in **GL**, since it can be proved that the canonical model of **GL** refutes α_n or β_n for every $n \geq 1$, then **L** is not tabular, by Theorem 6.2. Moreover, in (a restricted version of) Theorem 5.8 the system **GL** could be changed to **GL.3**, the system obtained from **GL** by adding the linearity axiom

(L) $\quad \Box(\Box p \supset q) \vee \Box(\Box^+ q \supset p)$

where $\Box^+\alpha$ denotes $(\alpha \wedge \Box\alpha)$. This is an easy consequence of Theorem 6.2 and the fact that **GL.3** is characterized by the frame $\langle \mathbb{N}, > \rangle$. Analogously, **GL** could be changed to **Grz.3** in (a restricted version of) Theorem 5.8, recalling that **Grz.3** is obtained from **K** by adding axioms

(grz) $\quad \Box(\Box(p \supset \Box p) \supset p) \supset p$
(sc) $\quad \Box(\Box p \supset q) \vee \Box(\Box q \supset p)$

Since **Grz.3** is characterized by the frame $\langle \mathbb{N}, \geq \rangle$, the result follows again by Theorem 6.2.

However, Theorem 5.8 is not a particular case of Theorem 6.2, by two reasons. Firstly, the former shows (when restricted to modal logics based on **PC**) that some class of modal logics cannot be characterized by finite matrices, while the latter guarantees that the same class of logics cannot be characterized by finite matrices with just one designated truth-value. Moreover, Theorem 5.8 also applies to modal logics whose non-modal fragment is between $\mathbf{PC}^{\supset}$ and **PC**. In contrast, Theorem 6.2 only applies to modal logics based on **PC**.

7 Conclusion

In this paper we show that the scope of Dugundji's Theorem can be enlarged not only to modal systems **S1 - S5**, but for a large class of well-known normal and non-normal modal systems whose non-modal fragment lies between Henkin's implicative calculus and Propositional Classical Logic.

However, this list is not exhaustive. Among the non-normal modal systems, certain extensions of **S5** such as **S6**, **S7**, **S8** and **S9** were not considered. With respect to normal modal systems, we can mention the K-systems proposed by Sobociński, such as **K1**, **K1.1**, **K1.2**, **K2**, **K2.1**, **K2.2**, **K3.1** and **K4**. Concerning the latter, it is worth mentioning that, as a consequence of Esakia and Meskhi's result, we know that **K4** can be characterized by finite matrices. An open question is to determine its characteristic matrices. Another question is to determine if there exist some other modal systems (normal or not) different from **K1.2**, **K2.2**, **K3.1**, **K3.2** and **S5** such that all of their extensions can be characterized by finite matrices.

Matrix semantics is not just an alternative to Kripke semantics for modal logics. Besides being extremely intuitive, a suitable matrix semantics could reduce the algorithmic complexity with respect to the relational semantics, increasing the potential applications of modal logic to Computer Science.

We do not present here a matrix semantics alternative to the usual Kripke models. Our results are, in a sense, negative. However, they intend to make a contribution to the question of determine the class of systems which can be characterizable by a finite matrix semantics, besides the extremely useful Chagrov and Zakharyaschev's criterion of tabularity discussed in Section 6.

It is worth noting that both Gödel's incompleteness theorem for intuitionistic logic and Dugundji's Theorem (and its generalizations) use the fact that there are infinite propositional variables in the language. The argument, however, is no longer valid in a language with a finite number of variables. It suggests that a modal logic defined in a language with finite variables could be characterized by finite matrices, and so this kind of modal logics would be interesting from the point of view of applications.

We hope that all the questions mentioned here can contribute to the current resumption of matrix semantics for modal logic, that was marginalized after the incredible success of Kripke semantics, as observed in [3].

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