On the ordered Dedekind real numbers in toposes

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Dedicated to Luiz Carlos Pereira on the occasion of his 60th birthday

Abstract

In 1996, W. Veldman and F. Waaldijk presents a constructive (intuitionistic) proof for the homogeneity of the ordered structure of the Cauchy real numbers, and so this result holds in any topos with natural number object. However, it is well known that the real numbers objects obtained by the traditional constructions of Cauchy sequences and Dedekind cuts are not necessarily isomorphic in an arbitrary topos with natural numbers object. Consequently, Veldman and Waaldijk's result does not apply to the ordered structure of Dedekind real numbers in toposes. The main result to be proved in the present paper is that the ordered structure of the Dedekind real numbers object is homogeneous, in any topos with natural numbers object. This result is obtained within the framework of local set theory.

1 Introduction: From Intuitionism to Toposes

As it is well-known, Intuitionism was introduced by L.E.J. Brouwer in 1907 ([4]) as a philosophy of mathematics based on the idea that mathematics is a creation of the mind. From this perspective, the truth of mathematical statements is stated by means of *mental constructions*. In particular, the principle of the excluded middle ($A \vee \neg A$) is rejected under the intuitionistic point of view because of the lack of *constructivistic* character.

The main logical principles of Intuitionism were formalized by A. Heyting through the so-called Intuitionistic logic, in which, by the reasons pointed out above, the third-excluded law is no longer valid. Intuitionistic logic was adopted as the logical framework underlying the *constructive mathematics*, including the constructive analysis of E. Bishop and the recursive analysis developed in Russia.

Some decades after the introduction of Intuitionism, S. Eilenberg and S. MacLane proposed in 1945 the basis of a structuralistic approach to mathematics via the theory of *categories and functors* ([7]). The main feature of category theory is that the concepts of structure ('object') and morphism ('arrow') are taken as primitive.

Category theory meets Intuitionism through the notion of *topos*. Toposes are particular categories endowed with enough structure which allows to consider them as a kind of *mathematical universes* or *universes of sets*. They constitute models of a higher-order intuitionistic type theory, where the subobject classifier Ω corresponds to a type of (intuitionistic) truth values. Being so, it is usually claimed that the internal logic of toposes is intuitionistic. This is rigorously supported by the Mitchell-Bénabou language defined on toposes, together with the associated Kripke-Joyal semantics (see, for instance, [12], Chapter VI).

In 1988, J.L. Bell introduced *Local Set Theory* (see [2]) as another formal (or logical) counterpart of the notion of toposes, formulated in a typed intuitionistic logic. It can be considered as a generalization of classical set theory such that the category of sets can always be obtained, and shown to be a topos (as in the 'classical', set-theoretic framework). Moreover, any topos can be obtained as the category of sets within a suitable local set theory, which exposes to what extent toposes can be considered as a generalization of the categories of sets.

The development of (set-theoretic) Model Theory for Intuitionistic logic was already considered in the literature (see, for instance, [8], [6], [16] and more recently [5]). Given the close relationship between toposes and intuitionistic logic pointed out above, it seems natural to analyze intuitionistic model theory by using the framework of local set theory (see [15] and [14]).

This paper, continuing our previous research, deals with the real numbers object obtained in a topos with natural number object, constructed by means of a generalization of the Dedekind techniques. The intuitive meaning of the Dedekind real numbers defined, in particular, in the emblematic topos $\mathbf{Sh}(X)$ of sheaves over a topological space X, compared with the usual construction in the topos **Set** of sets, was clearly described by Bell:

> *Consider, for example, the concept of 'real-valued continuous function on a topological space X'. Any such function may be regarded as a real number (or quantity)* varying continuously *over X. Now consider the topos $\mathbf{Sh}(X)$ of sheaves on X. Here* everything *is varying (continuously) over X, so shifting from* **Set** *to* $\mathbf{Sh}(X)$ *essentially amounts to*

placing oneself in a framework which is, so to speak, itself 'co-moving' with the variation over X of a given variable real number. This causes its variation not to be 'noticed' in $\mathbf{Sh}(X)$*; it is accordingly regarded as being a* constant *real number. In this way the concept of 'real-valued continuous function on X' is transformed into the concept of 'real number' when interpreted in* $\mathbf{Sh}(X)$*. [...] Putting in the other way round, the concept 'real number', interpreted in* $\mathbf{Sh}(X)$ *corresponds to the concept 'real-valued continuous function on X' interpreted in* $\mathbf{Set}$.

J. L. Bell, [2], pp. 240.

The idea of quantities varying continuously can be connected to the model-theoretic concept of *homogeneity*. The classical model-theoretic notion of homogeneity of a structure was introduced by B. Jónsson in 1960 ([10]). It can be formulated as follows: If κ is an infinite cardinal, a set-based structure $\mathfrak{A}$ is said to be κ-homogeneous if, for every partial isomorphism of $\mathfrak{A}$ of cardinal $\kappa' < \kappa$, there exists an automorphism of $\mathfrak{A}$ which extends it (see [3] for details). In this paper we only deal with finite partial isomorphisms, which means that our homogeneous ordered structures are the $\aleph_0$-homogeneous ones.

Within a set-theoretic approach, [16] presents a constructive (intuitionistic) proof for the homogeneity of the ordered structure of the Cauchy real numbers, and so this result holds in any topos with natural number object. It is well known that, in an arbitrary topos with natural numbers object, the real numbers objects obtained by the traditional constructions of Cauchy sequences and Dedekind cuts are not necessarily isomorphic (see, for instance, [13] and [9]), and so as for the corresponding ordered structures. Consequently, the result obtained in [16] does not apply to the ordered structure of Dedekind real numbers in toposes. The main result to be proved in the present paper is that the ordered structure $\langle \mathbb{R}_d, < \rangle$ of the Dedekind real numbers object is homogeneous, in any topos with natural numbers object (cf. Theorem 5.2 below).

In [15] we introduced the concept of effectively homogeneous ordered structures, for which there is an effective procedure which extends every finite partial isomorphism to an automorphism. The present paper follows closely the formal treatment developed in [15], based on the logical framework of local set theory (cf. [2, 1]). As a consequence of this constructive approach, it will be shown that the structure $\langle \mathbb{R}_d, < \rangle$ is effectively homogeneous. The full details of this constructions can be found in [14].

Recall that local set theory is a typed set theory, presented through a sequent calculus S, whose underlying logic is (many-sorted) higher-order intuitionistic logic; in this manner, the primitive notion of set is replaced by that of terms of *power* types. The resulting local sets (or S-sets) and arrows (or S-maps) set up a category $\mathbf{C}(S)$ that can be shown to be a topos, called a *linguistic topos*. It can to show that every topos $\mathbf{E}$ is equivalent to a linguistic topos, namely $\mathbf{C}(\mathrm{T}(\mathbf{E}))$, where $\mathrm{T}(\mathbf{E})$ is the local set theory whose axioms are those which are valid in the canonical interpretation of the internal language of $\mathbf{E}$ into $\mathbf{E}$ itself. Thus, the categorial machinery of a topos can be translated into a logical one and so we may develop all the technical constructions within the environment of local set theory.

The layout of the paper is the following: In Section 2 we summarize some basic definitions and results previously introduced in [15], mostly concerned with local set theory and ordered structures; in Section 3 we define Dedekind cuts and establish the necessary background for the further proof of homogeneity; in Section 4 we briefly recall the main results concerning natural numbers and finite sequences which was exhibited in [15]; finally, it is proved in Section 5 the main result of this paper (Theorem 5.2): the ordered structure of the Dedekind real numbers is (effectively) homogeneous in any topos with natural numbers object.

2 Preliminaries on Local Set Theories

As mentioned in the previous section, the framework we adopt here is that of local set theories. For that reason, these preliminaries will serve specifically as a notational guide for the other sections. We address the reader to [2] (see also [1]) for a more detailed exposition of the subject of local set theories.

We begin by a *local language* $\mathcal{L}$, which is a higher-order language consisting of types, variables and terms defined as usual, with the following relevant features: if $\mathsf{A}_1, \ldots, \mathsf{A}_n$ and A are types then $\mathsf{A}_1 \times \cdots \times \mathsf{A}_n$ and $P\mathsf{A}$ are types (the *product* and the *power* types, respectively). If $n = 1$ then $\mathsf{A}_1 \times \cdots \times \mathsf{A}_n$ is A_1. On the other hand, if $n = 0$ then the empty product $\mathsf{A}_1 \times \cdots \times \mathsf{A}_n$ is denoted by $\mathbf{1}$ (the *unity* type). There is just one more distinguished type, the *truth-value* type, denoted by Ω. For every type A we have a denumerable set $x^1_{\mathsf{A}}, x^2_{\mathsf{A}}, \ldots$ of *variables* of type A. The set of *terms* of a given type A is defined recursively over the set of variables. The details of the construction can be found in the book [2] (see also [1]). The terms of type Ω are called *formulas* and denoted by α, β etc. A formula *in context* is an expression $\vec{x}.\alpha$, where $\vec{x}$ is a list $x_0, \ldots, x_{n-1}$ of distinct variables and α is a formula such that all its free variables are in $\vec{x}$.

Local sets are defined as being the closed terms of power type. They will be denoted by capital letters A, B, X, Y etc. Recall from [2, 1] that, if x is a variable of type A and t is a term of type $P\mathsf{A}$ then $(x \in t)$ is a formula.[1] If α is a formula then $\{x \ : \ \alpha\}$ is a term of type $P\mathsf{A}$ in which every occurrence of the variable x (of type A) is not free. This means that, in a local language, there is a binding operator $\{ : \}$ for every type A. As in [15], we will represent elements of a local set with the same letter of the local set to which they belong, though in low case letters (possibly indexed). For example: $a, a', a_0 \in A$; $b', b'', b_1 \in B$; $x, x_1, x_2 \in X$ and so on. In a local language $\mathcal{L}$, all the customary logical symbols (including connectives and other binding operators, such as quantifiers) can be *defined* using the primitive symbols of the language. Two special local sets deserve a mention: $U_{\mathsf{A}} = \{x_{\mathsf{A}} \ : \ \top\}$ and $\emptyset_{\mathsf{A}} = \{x_{\mathsf{A}} \ : \ \bot\}$, for every type A.

A *local set theory* is a sequent calculus S over a local language $\mathcal{L}$ satisfying specific rules for higher-order intuitionistic logic.[2] The notation and terminology we will use in the sequel is standard and does not differ essentially from [2]. For instance, we will write $\Gamma \mathrel{|\!\!\sim_S} \alpha$ to denote that the sequent $\Gamma \Rightarrow \alpha$ is derived from the collection of sequents S. Now we recall some elementary facts about local set theories. An S-set X is said to be *inhabited* if $\mathrel{|\!\!\sim_S} \textsc{Inh}(X)$, where $\textsc{Inh}(X) :\Leftrightarrow \exists x. x \in X$. We also say that a formula in context $\vec{z}.\alpha$ is *decidable in* the S-set $\prod_{i<n} X_i$, where $X_i \subseteq U_{\mathsf{A}_i}$ (for $i = 0, \ldots, n-1$), if

$$\mathrel{|\!\!\sim_S} \forall x_0 \in X_0, \ldots, x_{n-1} \in X_{n-1}[\alpha \vee \neg\alpha].$$

Equivalently, $\vec{z}.\alpha$ is decidable in $\prod_{i<n} X_i$ if the S-set

$$\left\{ \langle x_0, \ldots, x_{n-1} \rangle \in \prod_{i<n} X_i \ : \ \alpha \right\}$$

is a complemented element in the Heyting algebra of the S-subsets of $\prod_{i<n} U_{\mathsf{A}_i}$. In particular, $z.\alpha$ is decidable in $X \subseteq U_{\mathsf{A}}$ if $\mathrel{|\!\!\sim_S} \forall x \in X[\alpha \vee \neg\alpha]$. Moreover, we say that $\vec{z}.\alpha$ is *decidable* if it is decidable in $\prod_{i<n} U_{\mathsf{A}_i}$. If $\vec{z}.\alpha$ and $\vec{z}.\beta$ are decidable in $\prod_{i<n} X_i$, then so are $\vec{z}.(\alpha \wedge \beta)$, $\vec{z}.(\alpha \vee \beta)$ and $\vec{z}.(\alpha \to \beta)$ (see [15]).

Next we enunciate some basic definitions concerning ordered structures in a local set theory, taken from [15]. Recall that $\mathbf{C}(S)$ is the topos constructed from the local set theory S. A (*partially*) *ordered* $\mathbf{C}(S)$-*structure*

[1] To be strict, there is a symbol $\in_{\mathsf{A}}$ for every type A.

[2] We assume here that the reader is acquainted with the basic axioms and inference rules of higher-order intuitionistic logic.

(in short, "an order") is a pair $\langle A, <\rangle$, where A is an S-set and $< \subseteq A \times A$ is a relation satisfying

$$\mid\!\sim_S \neg(a < a) \;\text{ and }\; (a' < a''),\, (a'' < a''') \mid\!\sim_S (a' < a''').$$

A *homomorphism* f from $\langle A, <\rangle$ into $\langle B, <\rangle$ is defined to be an S-map $f : A \to B$ which preserves the order. Ordered $\mathbf{C}(S)$-structures and homomorphisms form a category denoted by $\mathrm{Ord}[\mathbf{C}(S)]$. We say that an order $\langle A, <\rangle$ is *linear* if it satisfies

$$\mid\!\sim_S \;(a' < a'') \;\vee\; (a' = a'') \;\vee\; (a'' < a').$$

An order $\langle A, <\rangle$ is said to be *dense in* $\langle B, <\rangle$ if there exists a monomorphism $i : \langle A, <\rangle \to \langle B, <\rangle$ in $\mathrm{Ord}[\mathbf{C}(S)]$ such that

$$b' < b'' \mid\!\sim_S \exists a. b' < i(a) < b''.$$

The monomorphism i will be clear from the context. For example, in **Set** this monomorphism is represented in most cases by the inclusion $A \subseteq B$. An order $\langle A, <\rangle$ is *dense* if it is dense in itself by means of id_A.

An order $\langle A, <\rangle$ is *persistent in* an order $\langle B, <\rangle$ if it is dense in $\langle B, <\rangle$ and

$$\mid\!\sim_S \forall b \exists a', a''. i(a') < b < i(a''),$$

where i is the monomorphism mentioned in the definition of density. Finally, an order $\langle A, <\rangle$ is *persistent* if it is persistent in itself. Intuitively, $\langle A, <\rangle$ is persistent if it is dense and does not have endpoints.

3 Dedekind cuts in Local Set Theories

In classical set theory, Dedekind cuts are presented as constructions performed exclusively over the set of rational numbers. This approach is supported by Cantor's back and forth theorem, which also holds in any topos with natural numbers object (cf. [15]). Nevertheless, the construction can be generalized to arbitrary linearly ordered structures in order to embrace pure toposes, not necessarily containing a natural numbers object.

As in the classical case, a *Dedekind cut over* a linearly ordered $\mathbf{C}(S)$-structure $\mathfrak{A} = \langle A, <\rangle$ is a pair $\langle X, Y\rangle \in PA \times PA$ satisfying

$$\mid\!\sim_S \mathrm{D{\small ED}}_{\mathfrak{A}}(\langle X, Y\rangle),$$

where

$$
\begin{aligned}
\textsc{Ded}_{\mathfrak{A}}(\langle X, Y\rangle) :\Leftrightarrow\ & \textsc{Inh}(X) \ \wedge\ \textsc{Inh}(Y) \ \wedge\ X \cap Y = \emptyset \\
& \wedge \forall a'(a' \in X \leftrightarrow \exists a'' \in X. a' < a'') \\
& \wedge \forall a'(a' \in Y \leftrightarrow \exists a'' \in Y. a'' < a') \\
& \wedge \forall a', a''(a' < a'' \to a' \in X \ \vee\ a'' \in Y).
\end{aligned} \tag{1}
$$

We represent the S-set of all the Dedekind cuts of $\mathfrak{A}$ by $\mathfrak{d}A$. Thus:

$$\mathfrak{d}A := \{u \in PA \times PA \ :\ \textsc{Ded}_{\mathfrak{A}}(u)\}.$$

By defining a strict order over $\mathfrak{d}A$ by

$$u < v :\Leftrightarrow u, v \in \mathfrak{d}A \ \wedge\ \exists a. a \in \pi'(v) \cap \pi''(u), \tag{2}$$

we obtain a $\mathbf{C}(S)$-structure $\langle \mathfrak{d}A, < \rangle$, which will be denoted by $\mathfrak{d}\mathfrak{A}$. (In the expression above, π' and π'' denote the canonical projections.) The next lemma assures that the construction by Dedekind cuts preserves isomorphisms between $\mathbf{C}(S)$-structures.

Lemma 3.1. *If $\mathfrak{A} = \langle A, < \rangle$ and $\mathfrak{A} = \langle B, < \rangle$ are isomorphic linearly ordered $\mathbf{C}(S)$-structures, then $\mathfrak{d}\mathfrak{A}$ and $\mathfrak{d}\mathfrak{B}$ are isomorphic.*

Proof. Let $h : \mathfrak{A} \to \mathfrak{B}$ be such an isomorphism. First observe that

$$\mid\!\sim_S \forall u \in \mathfrak{d}A. \textsc{Ded}_{\mathfrak{B}}(\langle h \circ \pi'(u), h \circ \pi''(u)\rangle).$$

Now define $\tilde{h} : \mathfrak{d}A \to \mathfrak{d}B$ by

$$u \mapsto \langle h \circ \pi'(u), h \circ \pi''(u)\rangle.$$

Since h is bijective, we have

$$\mid\!\sim_S \forall V \in PB\, \exists! U \in PA. h(U) = V$$

and consequently

$$\mid\!\sim_S \forall v \in \mathfrak{d}B\, \exists! u \in \mathfrak{d}A[h \circ \pi'(u) = \pi'(v) \ \wedge\ h \circ \pi''(u) = \pi''(v)],$$

which shows that $\tilde{h}$ is a bijection. It remains to show that $\tilde{h}$ preserves the relation $<$. In fact:

$$
\begin{aligned}
u, v \in \mathfrak{d}A \mid\!\sim_S u < v \quad & \leftrightarrow \exists a. a \in \pi'(v) \cap \pi''(u) \\
& \leftrightarrow \exists a. h(a) \in h \circ \pi'(v) \cap h \circ \pi''(u) \\
& \leftrightarrow \exists b. b \in h \circ \pi'(v) \cap h \circ \pi''(u) \\
& \leftrightarrow \tilde{h}(u) < \tilde{h}(v),
\end{aligned}
$$

where the first and second lines follow from the definition of $<$, see (2) above. $\square$

The following lemma guarantees that, if the $\mathbf{C}(S)$-structure $\mathfrak{A}$ is persistent, then there exists a monomorphism $i_{\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{d}\mathfrak{A}$.

Lemma 3.2. *Let $\mathfrak{A} = \langle A, < \rangle$ be a linearly ordered, persistent and inhabited $\mathbf{C}(S)$-structure. Then*

$$\mathrel{|\!\sim}_S \forall a.\mathrm{DED}_{\mathfrak{A}}(\langle \{a' \ : \ a' < a\}, \{a' \ : \ a < a'\} \rangle)$$

and, moreover, the morphism $i_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{d}\mathfrak{A}$, defined by

$$a \mapsto \langle \{a' \ : \ a' < a\}, \{a' \ : \ a < a'\} \rangle,$$

is a monomorphism.

Proof. The first part follows directly from the definition of Dedekind cut, recall (1) above: both sets are inhabited (because $\mathfrak{A}$ does not have endpoints), disjoint, open (because $\mathfrak{A}$ is dense) and consecutive (because $\mathfrak{A}$ is linear). The second part follows from

$$\begin{aligned} i_{\mathfrak{A}}(a_0) = i_{\mathfrak{A}}(a_1) \mathrel{|\!\sim}_S \quad & \langle \{a' \ : \ a' < a_0\}, \{a' \ : \ a_0 < a'\} \rangle \\ & \qquad\qquad = \langle \{a' \ : \ a' < a_1\}, \{a' \ : \ a_1 < a'\} \rangle \\ & [a' < a_0 \leftrightarrow a' < a_1] \ \wedge \ [a_0 < a' \leftrightarrow a_1 < a'] \\ & a_0 = a_1, \end{aligned}$$

where the last line is a consequence of the linearity of $\mathfrak{A}$. □

We call $i_{\mathfrak{A}}$ the canonical monomorphism associated to $\mathfrak{A}$. This notation will be kept from now on, with possible omission of the index $\mathfrak{A}$.

Lemma 3.3. *Let $\mathfrak{A} = \langle A, < \rangle$ and $\mathfrak{B} = \langle B, < \rangle$ be two linearly ordered and persistent $\mathbf{C}(S)$-structures. Then, for every isomorphism $h : \mathfrak{A} \to \mathfrak{B}$, there exists an isomorphism $\tilde{h} : \mathfrak{d}\mathfrak{A} \to \mathfrak{d}\mathfrak{B}$ which extends h, that is, for which $\tilde{h} \circ i_{\mathfrak{A}} = i_{\mathfrak{B}} \circ h$.*

Proof. Repeating the steps of the proof of Lemma 3.1, we obtain from the isomorphism $h : \mathfrak{A} \to \mathfrak{B}$ an isomorphism $\tilde{h} : \mathfrak{d}\mathfrak{A} \to \mathfrak{d}\mathfrak{B}$ defined by

$$u \mapsto \langle h \circ \pi'(u), h \circ \pi''(u) \rangle.$$

It remains to show that $\tilde{h}$ extends h:

$$\begin{aligned} \mathrel{|\!\sim}_S \tilde{h} \circ i_{\mathfrak{A}}(a) \quad & = \langle h(\{a' \ : \ a' < a\}), h(\{a' \ : \ a < a'\}) \rangle \\ & = \langle \{h(a') \ : \ a' < a\}, \{h(a') \ : \ a < a'\} \rangle \\ & = \langle \{h(a') \ : \ h(a') < h(a)\}, \{h(a') \ : \ h(a) < h(a')\} \rangle \\ & = \langle \{b \ : \ b < h(a)\}, \{b \ : \ h(a) < b\} \rangle \\ & = i_{\mathfrak{B}} \circ h(a). \end{aligned}$$

□

Concerning uniform persistence, we obtain the following result.

Proposition 3.4. *If the linearly ordered* $\mathbf{C}(S)$*-structure* $\mathfrak{A} = \langle A, < \rangle$ *is persistent, then it is uniformly persistent in* $\mathfrak{d}\mathfrak{A}$.

Proof. First we must show that $\mathfrak{A}$ is persistent in $\mathfrak{d}\mathfrak{A}$. The following expression reveals that the former is dense in the latter (recall that i is the canonical monomorphism from Lemma 3.2):

$$\begin{aligned} u, v \in \mathfrak{d}A \mid\!\sim_S u < v \quad & \to \exists a . a \in \pi'(v) \cap \pi''(u) \\ & \to \exists a, a', a''[a' < a < a'' \quad \wedge \quad a', a'' \in \pi'(v) \cap \pi''(u)] \\ & \to \exists a . u < i(a) < v, \end{aligned}$$

where the first line follows from the definition of $<$ (see (2) above), the second from the openness of the cut, and the third again from the definition (2) of $<$. The persistence of $\mathfrak{A}$ in $\mathfrak{d}\mathfrak{A}$ is proved by:

$$\begin{aligned} u \in \mathfrak{d}A \mid\!\sim_S & \quad \exists a', a''[a' \in \pi'(u) \ \wedge \ a'' \in \pi''(u)] \\ \mid\!\sim_S & \quad \exists a', a''. i(a') < u < i(a''), \end{aligned}$$

in which we used the fact that "cuts are inhabited" (first line) and the definition (2) of $<$ (second line). Finally, since $\mathfrak{A}$ is persistent, we have as a particular case of Lemma 3.3 that every automorphism from $\mathfrak{A}$ into itself can be extended to an automorphism from $\mathfrak{d}\mathfrak{A}$ into itself. Hence we conclude that $\mathfrak{A}$ is uniformly persistent in $\mathfrak{d}\mathfrak{A}$. □

The proposition bellow prescribes a sufficient condition for a given substructure to reach its base-structure with respect to cuts.

Proposition 3.5. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are linearly ordered* $\mathbf{C}(S)$*-structures such that* $\mathfrak{B}$ *is dense in* $\mathfrak{A}$*, then* $\mathfrak{d}\mathfrak{B}$ *is isomorphic to* $\mathfrak{d}\mathfrak{A}$.

Proof. Let $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$ be a monomorphism in $\mathrm{Ord}[\mathbf{C}(S)]$ such that

$$a' < a'' \mid\!\sim_S \exists b . a' < i(b) < a''.$$

We want to construct an isomorphism $h : \mathfrak{d}\mathfrak{A} \to \mathfrak{d}\mathfrak{B}$ from i. Note that

$$u \in \mathfrak{d}A \mid\!\sim_S \langle \{b \ : \ i(b) \in \pi'(u)\}, \{b \ : \ i(b) \in \pi''(u)\} \rangle \in \mathfrak{d}B$$

because $\mathfrak{B}$ is dense in $\mathfrak{A}$. The S-function $h : \mathfrak{d}A \to \mathfrak{d}B$, defined by

$$u \mapsto \langle \{b \ : \ i(b) \in \pi'(u)\}, \{b \ : \ i(b) \in \pi''(u)\} \rangle,$$

becomes a natural choice. Now we verify that h is in fact a bijection. First we show that it is an injection. Note that

$$\begin{aligned} u, v \in \mathfrak{d}A \mathrel{|\!\!\sim_S} h(u) = h(v) &\to \{b \ : \ i(b) \in \pi'(u)\} = \{b \ : \ i(b) \in \pi'(v)\} \\ &\to \forall b[i(b) \in \pi'(u) \leftrightarrow i(b) \in \pi'(v)]. \end{aligned}$$

On the other hand,

$$\begin{aligned} \forall b[i(b) \in \pi'(u) \leftrightarrow i(b) \in \pi'(v)] \mathrel{|\!\!\sim_S} \ & a \in \pi'(u) \\ &\to \exists a'[a < a' \ \wedge \ a' \in \pi'(u)] \\ &\to \exists b'[a < i(b') < a' \ \wedge \ i(b') \in \pi'(u)] \\ &\to \exists b'[a < i(b') \ \wedge \ i(b') \in \pi'(v)] \\ &\to a \in \pi'(v). \end{aligned}$$

Similarly:

$$\forall b[i(b) \in \pi'(u) \leftrightarrow i(b) \in \pi'(v)] \mathrel{|\!\!\sim_S} a \in \pi'(v) \to a \in \pi'(u),$$

thus:

$$\begin{aligned} u, v \in \mathfrak{d}A \mathrel{|\!\!\sim_S} h(u) = h(v) &\to \forall a[a \in \pi'(u) \leftrightarrow a \in \pi'(v)] \\ &\to \pi'(u) = \pi'(v). \end{aligned}$$

Repeating the argument for π'' we obtain:

$$u, v \in \mathfrak{d}A \mathrel{|\!\!\sim_S} h(u) = h(v) \to \pi''(u) = \pi''(v).$$

Finally:

$$u, v \in \mathfrak{d}A \mathrel{|\!\!\sim_S} h(u) = h(v) \to u = v,$$

showing that h is an injection. Now, to prove that h is surjective, we observe that:

$$\begin{aligned} \mathrel{|\!\!\sim_S} \ & \pi' \circ h(\langle\{a \ : \ \exists b \in \pi'(v).a < i(b)\}, \{a \ : \ \exists b \in \pi''(v).i(b) < a\}\rangle) \\ &= \{b \ : \ i(b) \in \{a \ : \ \exists b \in \pi'(v).a < i(b)\}\} \\ &= \{b \ : \ \exists b' \in \pi'(v).i(b) < i(b')\} \\ &= \{b \ : \ \exists b' \in \pi'(v).b < b'\} \\ &= \pi'(v). \end{aligned}$$

Similarly:

$$\mathrel{|\!\!\sim_S} \pi'' \circ h(\langle\{a \ : \ \exists b \in \pi'(v).a < i(b)\}, \{a \ : \ \exists b \in \pi''(v).i(b) < a\}\rangle) = \pi''(v).$$

Thus:

$$\mathrel{|\!\!\sim_S} h(\langle\{a \ : \ \exists b \in \pi'(v).a < i(b)\}, \{a \ : \ \exists b \in \pi''(v).i(b) < a\}\rangle) = v.$$

Therefore we conclude that:

$$v \in \mathfrak{d}B \mathrel{|\!\sim_S} \exists u \in \mathfrak{d}A.h(u) = v.$$

It remains to show that h is an isomorphism. On the one hand:

$$\begin{aligned} u, v \in \mathfrak{d}A \mathrel{|\!\sim_S} u < v \quad & \to \exists a.a \in \pi'(v) \cap \ \pi''(u) \\ & \to \exists a, a'[a < a' \ \wedge \ a, a' \in \pi'(v) \cap \ \pi''(u)] \\ & \to \exists b.i(b) \in \pi'(v) \cap \ \pi''(u) \\ & \to \exists b.b \in \{b' \ : \ i(b') \in \pi'(v)\} \\ & \qquad\quad \cap \{b' \ : \ i(b') \in \pi''(u)\} \\ & \to \exists b.b \in \pi' \circ h(v) \cap \ \pi'' \circ h(u) \\ & \to h(u) < h(v) \end{aligned}$$

and, on the other hand:

$$\begin{aligned} u, v \in \mathfrak{d}A \mathrel{|\!\sim_S} h(u) < h(v) \quad & \to \exists b.b \in \pi' \circ h(v) \cap \ \pi'' \circ h(u) \\ & \to \exists b.b \in \{b' \ : \ i(b') \in \pi'(v)\} \\ & \qquad\quad \cap \{b' \ : \ i(b') \in \pi''(u)\} \\ & \to \exists b.i(b) \in \pi'(v) \cap \ \pi''(u) \\ & \to \exists a.a \in \pi'(v) \cap \ \pi''(u) \\ & \to u < v. \end{aligned}$$

□

An interesting consequence from Propositions 3.4 and 3.5 is that, if $\mathfrak{A}$ is a linearly ordered and persistent $\mathbf{C}(S)$-structure and $\mathfrak{d}\mathfrak{A}$ is also linear, then $\mathfrak{d}\mathfrak{d}\mathfrak{A}$ is isomorphic to $\mathfrak{d}\mathfrak{A}$. This means that the Dedekind cuts construction is idempotent, in some sense, when applied to persistent $\mathbf{C}(S)$-structures.

The next lemma sums up most of precedent results.

Lemma 3.6. *Consider the following conditions:* (i) $\mathfrak{A}$ *and* $\mathfrak{B}$ *are linearly ordered and persistent* $\mathbf{C}(S)$*-structures;* (ii) f *is a partial isomorphism from* $\mathfrak{A}$ *into* $\mathfrak{B}$*;* (iii) $\mathrm{dom}(f)$ *is dense in* $\mathfrak{A}$ *and* $\mathrm{cod}(f)$ *is dense in* $\mathfrak{B}$*. Then there exists an isomorphism* $h : \mathfrak{d}\mathfrak{A} \to \mathfrak{d}\mathfrak{B}$ *which extends* f*, that is, for which* $h \circ i_{\mathfrak{A}}|_{\mathrm{dom}(f)} = i_{\mathfrak{B}} \circ f$.

Proof. Let $\mathfrak{A}' = \langle \mathrm{dom}(f), <_{\mathfrak{A}} \rangle$ and $\mathfrak{B}' = \langle \mathrm{cod}(f), <_{\mathfrak{B}} \rangle$ be the respective (isomorphic) $\mathbf{C}(S)$-structures generated by $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$; we infer easily from the hypothesis that $\mathfrak{A}'$ and $\mathfrak{B}'$ are persistent. From Lemma 3.3, there exists an isomorphism $\tilde{f} : \mathfrak{d}\mathfrak{A}' \to \mathfrak{d}\mathfrak{B}'$ which extends f. Now, from Proposition 3.5, there exist isomorphisms $h' : \mathfrak{d}\mathfrak{A} \to \mathfrak{d}\mathfrak{A}'$ and $h'' : \mathfrak{d}\mathfrak{B} \to \mathfrak{d}\mathfrak{B}'$. Thus we may define $h : \mathfrak{d}\mathfrak{A} \to \mathfrak{d}\mathfrak{B}$ by $h = h''^{-1} \circ \tilde{f} \circ h'$. It is straightforward to show that h is an isomorphism which extends $\tilde{f}$; hence, it also extends f and the proof is complete. □

4 Natural numbers in Local Set Theories

The notion of natural numbers object in toposes has been widely studied in the literature. However, the original definition of natural numbers object, introduced by F.W. Lawvere in 1964 (see [11]), makes sense in any category with finite products.

In this section we briefly recall some elementary results on natural numbers in a local set theory, including a statement of the Cantor's back and forth theorem proved in [15].

A local set theory N is said to be *naturalized* if its language has a distinguished type N, a closed term 0 (zero) of type N and an N-function s (successor) of type $\mathsf{N} \times \mathsf{N}$ satisfying the usual (second-order) Peano axioms. The N-set $U_{\mathsf{N}} = \{x_{\mathsf{N}} \; : \; \top\}$ will be denoted by $\mathbb{N}$ and called N-set of natural numbers. The elements of $\mathbb{N}$ will be denoted by m, n, n' etc.

In a naturalized local theory N the following *primitive recursion principle* (PRP) is valid (see [2, 1] for details):

$$x \in X, g \in X^{X \times \mathbb{N}} \mathrel{|\!\!\sim}_N \exists! f \in X^{\mathbb{N}}[f(0) = x \;\wedge\; \forall n. f \circ s(n) = g(f(n), n)].$$

Consider the N-function $[\cdot] : \mathbb{N} \to P\mathbb{N}$ defined by PRP as follows:

$$\mathrel{|\!\!\sim}_N \exists! [\cdot] \in (P\mathbb{N})^{\mathbb{N}}[[0] = \emptyset \;\wedge\; [s(n)] = [n] \cup \{n\}].$$

The relation $m < n :\Leftrightarrow m \in [n]$ is a strict order in $\mathbb{N}$ such that, as expected, $\mathrel{|\!\!\sim}_N [n] = \{m \; : \; m < n\}$. Then we may show, as in the classical case, some basic facts about the natural numbers such as existence of minimal element (namely, 0), discretion, irreflexivity, transitivity, linearity and decidability, this last allowing us to define the linear partial order $\leq$ by $m \leq n :\Leftrightarrow (m = n) \vee (m < n)$. Using PRP it is straightforward to define addition, product and exponentiation as appropriate N-functions.

As usual, the integers object $\mathbb{Z}$ is defined as the coproduct of $\mathbb{N}$ with its image by s. Extending strict order, addition and product from the system $\mathbb{N}$, the resulting $\mathbf{C}(N)$-structure is a linearly ordered commutative ring. From this, the rational numbers object $\mathbb{Q}$ can be defined by reproducing the classical quotient construction.[3] Finally, extending strict order, addition and product from $\mathbb{Z}$, the resulting $\mathbf{C}(N)$-structure is again a linearly ordered commutative ring. Furthermore, the linearly ordered $\mathbf{C}(N)$-structure $\langle \mathbb{Q}, < \rangle$ is persistent (recall the end of Section 2).

[3] The reader interested in the details of this construction can consult, for instance, [12].

We recall from [15] the following notation: the finite conjunction of a formula α is the N-function $\bigwedge_{i<(\cdot)} \alpha : \mathbb{N} \to U_\Omega$ defined by PRP:

$$\mid\!\sim_N \bigwedge_{i<0} \alpha = \top \ \wedge \ \forall n \left[\bigwedge_{i<n+1} \alpha = \bigwedge_{i<n} \alpha \ \wedge \ \alpha(n) \right].$$

Analogously, it is defined the finite disjunction $\bigvee_{i<(\cdot)} \alpha : \mathbb{N} \to U_\Omega$. It can be easily shown that, if the formula α is decidable in $\mathbb{N}$, then so are $\bigwedge_{i<n} \alpha$ and $\bigvee_{i<n} \alpha$ for each n (cf. [15]).

The next useful result was proved in [15].

Proposition 4.1 (Minimum principle)**.** *If the formula α is decidable in $\mathbb{N}$, then*

$$\exists n.\alpha(n) \mid\!\sim_N \exists n[\alpha(n) \ \wedge \ \forall m(\alpha(m) \to n \leq m)].$$

Since $\leq$ is antisymmetric, the minimum will be unique. This unique minimum will be denoted by $\mu n.\alpha(n)$.

A given N-set X is said to be *denumerable* if it satisfies the axiom $\mid\!\sim_N \textsc{Den}(X)$, where

$$\textsc{Den}(X) :\Leftrightarrow X = \emptyset \ \vee \ \exists g \in X^{\mathbb{N}}.\textsc{Sur}(g)$$

and

$$\textsc{Sur}(g) :\Leftrightarrow \forall y \exists x.g(x) = y.$$

An N-map g such that $\textsc{Sur}(g)$ is said to be an *enumeration* of X, and will be frequently denoted by g_X. If X satisfies the property $\mid\!\sim_N X \simeq \mathbb{N}$, where $X \simeq Y$ is the obvious formula stating that there exists an isomorphism in $\mathbf{C}(N)$ between X and Y, we say that X is *completely denumerable* or simply *countable*. Clearly, if X and Y are denumerable (respectively countable) N-sets, then the product $X \times Y$ is denumerable (resp. countable). An N-set X is *finite* if satisfies $\mid\!\sim_N \textsc{Fin}(X)$, where $\textsc{Fin}(X) :\Leftrightarrow \exists n.[X \simeq [n]]$.

A *sequence on* X is an N-map $f : \mathbb{N} \to X + \{\sharp\}$, where $+$ denotes the coproduct in $\mathbf{C}(N)$ and $\sharp$ is any element, say 0. Given that $X + \{\sharp\}$ represents a disjoint union in N, it is tacitly assumed that $\mid\!\sim_N \sharp \notin X$. A *finite sequence on* X is defined to be a sequence on X satisfying the axiom $\mid\!\sim_N \textsc{Fsq}_X(f)$, where

$$\textsc{Fsq}_X(f) :\Leftrightarrow \exists n[f([n]) \subseteq X \ \wedge \ f(\mathbb{N} - [n]) = \{\sharp\}].$$

It is worth noting that the natural number n in the above definition is unique.

The N-set of all the finite sequences of X is given by:

$$X^* := \{f \in (X + \{\sharp\})^{\mathbb{N}} \ : \ \textsc{Fsq}_X(f)\}.$$

The elements of X^* will be denoted by $\vec{x}$, $\vec{x}_0$, $\vec{x}_1$ etc.[4] The element of X^* given by the constant N-map $n \mapsto \sharp$, will be denoted by $\sharp$, by abuse of notation. It was shown in [15] that, if the N-set X is denumerable (resp. countable), then X^* is denumerable (resp. countable).

We can associate, to each finite sequence $\vec{x}$ of X^*, a natural number which (intuitively) represents the length or number of relevant elements of $\vec{x}$. Thus, consider the N-map $\textsc{lgh}_X : X^* \to \mathbb{N}$ given by $\vec{x} \mapsto \mu n.[\vec{x}(n) = \sharp]$. Using this N-map we can collect all the finite sequences of a given length:

$$X^n := \{\vec{x} \in X^* \ : \ \textsc{lgh}_X(\vec{x}) = n\}.$$

The *effective image* of a finite sequence $\vec{x}$ is defined as follows:

$$\textsc{eim}_X(\vec{x}) := \vec{x}([\textsc{lgh}_X(\vec{x})]) = \{\vec{x}(n) \ : \ n < \textsc{lgh}_X(\vec{x})\}.$$

From now on, and when no confusion arises, we will write $\textsc{Fsq}$, $\textsc{lgh}$ and $\textsc{eim}$ instead of $\textsc{Fsq}_X$, $\textsc{lgh}_X$ e $\textsc{eim}_X$, respectively.

The following topos version of Cantor's back and forth theorem, as well as the technical Corollary 4.3, were proved in [15].

Theorem 4.2 (Cantor's back and forth theorem)**.** *Let $\langle A, < \rangle$ and $\langle B, < \rangle$ be linearly ordered, inhabited, linear, persistent and denumerable $\mathbf{C}(N)$-structures. Then $\langle A, < \rangle \simeq \langle B, < \rangle$.*

Corollary 4.3. *Suppose the following conditions:*

- *$\langle A, < \rangle$ and $\langle B, < \rangle$ are linearly ordered, inhabited, linear, persistent and denumerable $\mathbf{C}(N)$-structures;*
- *$\alpha(x, y)$ is a formula decidable in $A \times B$;*
- *$\{a \ : \ \exists b.\alpha(a, b)\}$ is persistent in B and $\{b \ : \ \exists a.\alpha(a, b)\}$ is persistent in A.*

Then there exists an isomorphism $h : \langle A, < \rangle \to \langle B, < \rangle$ such that $\mid\!\sim_N \forall a.\alpha(a, h(a))$.

[4] Note that this notation has already been used for contexts. However, this ambiguity will not lead to confusion.

5 Dedekind real numbers and homogeneity

The Dedekind real numbers object is the N-set of all the Dedekind cuts (recall Section 3) over the ordered structure $\mathfrak{Q} = \langle \mathbb{Q}, < \rangle$ of the rational numbers. Thus, the Dedekind real numbers object is given by:

$$\mathbb{R}_d := \mathfrak{d}\mathbb{Q};$$

and the corresponding ordered $\mathbf{C}(S)$-structure is obtained as:

$$\mathfrak{R}_{\mathfrak{d}} := \mathfrak{d}\mathfrak{Q}.$$

Then all the general results established in section 3 for Dedekind cuts can be applied in particular to $\mathfrak{R}_{\mathfrak{d}}$. Our aim in this section is to prove that this structure is homogeneous.

Recall from [15] that, if $\langle A, < \rangle$ is a (partially) ordered $\mathbf{C}(N)$-structure, a finite sequence $f : \mathbb{N} \to A \times A$ *preserves the pairing order* if $\mid\!\sim_N \textsc{Ppo}(f)$, where

$$\textsc{Ppo}(f) :\Leftrightarrow \bigwedge_{m<n<\mathrm{LGH}(f)} [\pi' \circ f(m) < \pi' \circ f(n) \leftrightarrow \pi'' \circ f(m) < \pi'' \circ f(n)].$$

An order $\langle A, < \rangle$ is said to be *homogeneous* if, for every finite sequence $f : \mathbb{N} \to A \times A$ which preserves the pairing order, there exists an automorphism of $\langle A, < \rangle$ which extends $\textsc{eim}(f)$. Intuitively, every finite partial isomorphism can be extended to an automorphism. The order $\langle A, < \rangle$ is said to be *effectively* homogeneous if there exists an effective (constructively defined) procedure for extending every f (see [15] and [14] for details).

It was proved in [15] that the structure $\mathfrak{Q}$ is effectively homogeneous. Moreover, it is the least effectively homogeneous structure (up to isomorphisms), in the sense that it can be constructed a monomorphism from $\mathfrak{Q}$ into every other effectively homogeneous structure.

Before the proof of the homogeneity of $\mathfrak{R}_{\mathfrak{d}}$ (Theorem 5.2) we need a technical lemma, which uses the N-map $* : X^* \times X \to X^*$ defined by:

$$\begin{aligned} \mid\!\sim_N \quad & \forall n[(n = \textsc{lgh}(\vec{x}) \to (\vec{x} * x)(n) = x) \\ & \wedge\ (n \neq \textsc{lgh}(\vec{x}) \to (\vec{x} * x)(n) = \vec{x}(n))]. \end{aligned}$$

Intuitively, $*$ adds a new element of X to any sequence in X^*. The usual relation of *apartness* can be defined in a given partial order $\langle A, < \rangle$ as follows:

$$a' \# a'' :\Leftrightarrow (a' < a'')\ \vee\ (a'' < a').$$

Using $*$ and $\#$, we obtain the following:

Lemma 5.1. *If $\vec{r}$ is a finite sequence in $\mathbb{R}_d$, then the N-set $\mathbb{Q}_{\vec{r}}$, defined by*

$$\mathbb{Q}_{\vec{r}} := \left\{ q \ : \ \bigwedge_{n<\mathrm{LGH}(\vec{r})} \vec{r}(n) \# i(q) \right\},$$

is dense in $\mathbb{Q}$ and denumerable.

Proof. First we show, by induction on the length of $\vec{r}$, that

$$\begin{aligned} \mid\!\sim_N \forall \vec{q} \in \mathbb{Q}^* \quad [\ (\mathrm{LGH}(\vec{q}) = \mathrm{LGH}(\vec{r}) + 1 \quad \wedge \quad \textstyle\bigwedge_{n<m<\mathrm{LGH}(\vec{q})} \vec{q}(n) \neq \vec{q}(m)) \\ \to \ \textstyle\bigvee_{n<\mathrm{LGH}(\vec{q})} \ \vec{q}(n) \in \mathbb{Q}_{\vec{r}}\ ] \end{aligned} \tag{3}$$

(intuitively, every finite sequence of rational numbers with $\mathrm{LGH}(\vec{r})+1$ different elements has at least one element in $\mathbb{Q}_{\vec{r}}$). If $\mathrm{LGH}(\vec{r}) = 0$, then $\mathbb{Q}_{\vec{r}} = \mathbb{Q}$ and the checking is immediate. Now let $\mathrm{LGH}(\vec{r}) = n+1$ (so that $\mathrm{LGH}(\vec{q}) = n+2$): in this case, if we take (the unique) $\vec{r}_0$ such that $\vec{r} = \vec{r}_0 * \vec{r}(\mathrm{LGH}(\vec{r}) - 1)$, by induction hypothesis we have:

$$\mid\!\sim_N \bigvee_{n,m<\mathrm{LGH}(\vec{q})} [\ \vec{q}(n) \neq \vec{q}(m) \ \wedge \ \vec{q}(n), \vec{q}(m) \in \mathbb{Q}_{\vec{r}_0}\ ].$$

On the other hand, from the definition of Dedekind cut (recall the expression (1) at the beginning of Section 3) it follows that

$$\mid\!\sim_N \ (q(n) \in \pi'(\vec{r}(\mathrm{LGH}(\vec{r}) - 1))) \quad \vee \quad (q(m) \in \pi''(\vec{r}(\mathrm{LGH}(\vec{r}) - 1))).$$

Thus:

$$\mid\!\sim_N \bigvee_{n,m<\mathrm{LGH}(\vec{q})} [\ \vec{q}(n) \neq \vec{q}(m) \quad \wedge \quad (\vec{q}(n) \in \mathbb{Q}_{\vec{r}} \ \vee \ \vec{q}(m) \in \mathbb{Q}_{\vec{r}})\ ]$$

and the expression (3) is proved. So we can take the element $q_0 \in \mathbb{Q}_{\vec{r}}$ defined by:

$$q_0 := \min\{q \ : \ g_{\mathbb{Q}}(q) < \mathrm{LGH}(\vec{r} + 1)\},$$

where the above minimum has the obvious meaning, and construct a surjective map $g : \mathbb{Q}^{\mathrm{LGH}(\vec{r})+1} \twoheadrightarrow \mathbb{Q}_{\vec{r}}$ by:

$$\begin{aligned} \mid\!\sim_N \ & \left[\textstyle\bigwedge_{n<m<\mathrm{LGH}(\vec{q})} \vec{q}(n) \neq \vec{q}(m) \ \to \ g(\vec{q}) = \min(\vec{q})\right] \\ & \wedge \left[\neg \textstyle\bigwedge_{n<m<\mathrm{LGH}(\vec{q})} \vec{q}(n) \neq \vec{q}(m) \ \to \ g(\vec{q}) = q_0\right]. \end{aligned}$$

Since $\mathbb{Q}^{\mathrm{LGH}(\vec{r})+1}$ is totally denumerable, we conclude that $\mathbb{Q}_{\vec{r}}$ is denumerable. The density of $\mathbb{Q}_{\vec{r}}$ in $\mathbb{Q}$ follows from the expression bellow, which says intuitively that, given two rational numbers, there are arbitrarily many others between them:

$$\mathrel{|\!\sim}_N \forall q', q'' \exists \vec{q} \left[\vec{q}(0) = \frac{q' + q''}{2} \quad \wedge \bigwedge_{n < \mathrm{LGH}(\vec{r})+1} \vec{q}(n+1) = \frac{\vec{q}(n) + q''}{2} \right].$$

Hence, it suffices to apply expression (3). □

Theorem 5.2. *The* $\mathbf{C}(N)$*-structure* $\mathfrak{R}_{\mathfrak{d}} = \langle \mathbb{R}_d, < \rangle$ *of the ordered Dedekind real numbers object is (effectively) homogeneous.*

Proof. Let $f : \mathbb{N} \to \mathbb{R}_d \times \mathbb{R}_d$ be a finite sequence preserving the pairing order. By Lemma 5.1, the N-sets $\mathbb{Q}_{\pi' \circ f}$ and $\mathbb{Q}_{\pi'' \circ f}$ are denumerable and dense in $\mathbb{Q}$, and hence they are also persistent. Now let α be the formula defined by:

$$\alpha(q', q'') :\Leftrightarrow \bigwedge_{n < \mathrm{LGH}(f)} \left[\pi' \circ f(n) < q' \leftrightarrow \pi'' \circ f(n) < q'' \right].$$

Observe that α is decidable in $\mathbb{Q}_{\pi' \circ f} \times \mathbb{Q}_{\pi'' \circ f}$ and, moreover,

$$\mathrel{|\!\sim}_N \{ q' \in \mathbb{Q}_{\pi' \circ f} \ : \ \exists q'' \in \mathbb{Q}_{\pi'' \circ f}. \alpha(q', q'') \} = \mathbb{Q}_{\pi' \circ f},$$

as well as

$$\mathrel{|\!\sim}_N \{ q'' \in \mathbb{Q}_{\pi'' \circ f} \ : \ \exists q' \in \mathbb{Q}_{\pi' \circ f}. \alpha(q', q'') \} = \mathbb{Q}_{\pi'' \circ f}.$$

In effect, by induction on the length of f we easily prove that

$$\mathrel{|\!\sim}_N \forall q' \in \mathbb{Q}_{\pi' \circ f} \exists q'' \in \mathbb{Q}_{\pi'' \circ f}. \alpha(q', q'')$$

and also the converse. So, by Corollary 4.3 there exists an isomorphism $h : \langle \mathbb{Q}_{\pi' \circ f}, < \rangle \to \langle \mathbb{Q}_{\pi'' \circ f}, < \rangle$ such that $\mathrel{|\!\sim}_N \forall q \in \mathbb{Q}_{\pi' \circ f}. \alpha(q, h(q))$ (that is, h extends $\textsc{eim}(f)$). Now, by Lemma 3.6, there exists an automorphism $\tilde{h}$ of $\mathfrak{R}_{\mathfrak{d}}$ which extends h and therefore also extends $\textsc{eim}(f)$. □

Finally, it should be observed that, with slight modifications, the proof above can be adapted to the Cauchy real numbers defined in any topos with natural numbers object. That is:

Theorem 5.3. *The* $\mathbf{C}(N)$*-structure* $\mathfrak{R}_{\mathfrak{c}} = \langle \mathbb{R}_c, < \rangle$ *of the ordered Cauchy real numbers object is (effectively) homogeneous.*

A detailed construction of the Cauchy real numbers defined in local set theories, as well as several missing details of the proof of Veldman and Waaldijk's result, can be found in [14].

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