Finite non-deterministic semantics for some modal systems

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Abstract

Trying to overcome Dugundji's result on uncharacterizability of modal logics by finite logical matrices, J. Kearns (in 1981) and J. Ivlev (in 1988) propose, independently, a characterization of some modal systems by means of four-valued multivalued truth-functions (by restricting the valuations using *level valuations*, in Kearns' approach), as an alternative to Kripke semantics. This constitutes an antecedent of the non-deterministic matrices introduced in 2001 by A. Avron and I. Lev. In this paper we propose a reconstruction of Kearns and Ivlev's results (which did not have the dissemination or impact they deserved) in an uniform way, obtaining an extension to another modal systems. The first part of the paper is devoted to four-valued Nmatrices, including Kearns and Ivlev's. Besides proving with full details Kearns' results for **T**, **S4** and **S5**, we also obtain a characterization of the system **B** by four-valued Nmatrices with level valuations. Concerning Ivlev's results, two new modal systems are introduced and characterized by Nmatrices. In the second part of this paper, six-valued Nmatrices are introduced which characterize a variant of the eight systems studied in the first part, by replacing axiom (T) by axiom (D). As a by-product, novel decision procedures for **T**, **S4**, **S5**, **D**, **KDB**, **KD4** and **KD45** are obtained, which open interesting possibilities in the study of the complexity of modal logics and, in particular, of Intuitionistic propositional logic (**IPC**), taking into account the Gödel-McKinsey-Tarski translation between **IPC** and **S4**.

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Introduction

In 1940 J. Dugundji proves that no logic between modal systems **S1** and **S5** can be characterized by a single logical matrix (see [10]). This negative result, that can be extended to a vast class of modal systems (see, for instance, [8] and [9]) forces the development of different semantics for modal systems. Kripke's possible-worlds semantics (see [21] and [22]) constitute, without doubts, the most notorious advance in the area. Trying to overcome Dugundji's results, and as an interesting alternative to Kripke semantics, J. Kearns (in 1981, see [19]) and J. Ivlev (in 1988, see [15]) propose, independently, a characterization of some modal systems by means of four-valued multivalued truth-functions. This constitutes an antecedent of the non-deterministic matrices (a.k.a. Nmatrices) introduced in 2001 by A. Avron and I. Lev, see [2]). [1]

Kearns proposes four-valued Nmatrices with a restriction on the valuations (called *level valuations*) for the systems **T**, **S4** and **S5**. Ivlev proposes four-valued Nmatrices for several weak modal systems which do not have the necessitation rule, constituting weaker versions of **T** and **S5**.

Besides its importance, Kearns and Ivlev's results did not have the dissemination or impact they deserved (for other developments concerning quasi-matrices see [17], [18] and [20]). In this paper we present a reconstruction of Kearns and Ivlev's results in an uniform way, obtaining an extension to another modal systems. The first part of the paper is devoted to four-valued Nmatrices, including Kearns and Ivlev's. Besides proving with full details Kearns' results for **T**, **S4** and **S5**, we also obtain a characterization of system **B** by four-valued Nmatrices with level valuations. Concerning Ivlev's results, the characterization by four-valued Nmatrices of the weaker versions (without the necessitation rule) of **T** and **S5** given by Ivlev is now extended to two new systems which represent the weaker versions (without necessitation) of **B** and **S4**. In the second part of this paper we propose six-valued Nmatrices which characterize a variant of the eight systems studied in the first part, by replacing axiom (T) by axiom (D).

The present research intends to contribute to the development of alternatives to Kripke semantics for modal logics, with interesting properties. In particular, and as a by-product, novel decision procedures for **T**, **S4**, **S5**, **D**, **KDB**, **KD4** and **KD45** are obtained in this paper, which open interesting possibilities in the study of the complexity of modal logics and, in

[1] Kearns uses the term "matrix" while Ivlev explains that his semantics is "an interpretation of modal statement in terms of necessary and possible truths as well as in terms of necessary and possible falsity". Afterwards, the term "quasi-matrix" was used to refer to Ivlev's Nmatrices, see [17], [18] and [20].

particular, of Intuitionistic propositional logic (**IPC**), taking into account the Gödel-McKinsey-Tarski translation between **IPC** and **S4**.

1 Ivlev systems and Nmatrices

In this section, the characterization of several weak modal systems by means of four-valued non-deterministic matrices due to J. Ivlev will be reviewed and expanded to two new systems. Firstly, let us recall the notion of Nmatrices.

In 2001, A. Avron and I. Lev propose in [2] a generalization of matrix semantics called *non-deterministic semantics* or *Nmatrices* (see also [3]). In formal terms:

Definition 1.1 *Let Σ be a propositional signature. An Nmatrix over Σ is a triple $\mathcal{M} = \langle M, D, \mathcal{O}\rangle$ such that M is a non-emptyset (of truth-values), $D \subseteq M$ is the set of designated values, and $\mathcal{O}$ is an interpretation mapping which assigns, to each n-ary connective c of Σ, a multivalued function $\mathcal{O}(c) : M^n \longrightarrow \wp(M) - \{\emptyset\}$. An $\mathcal{M}$-*valuation *is a mapping $v : For_\Sigma \longrightarrow M$ (where For_Σ is the algebra of formulas over Σ) such that $v(c(\alpha_1, \ldots, \alpha_n)) \in \mathcal{O}(c)(v(\alpha_1), \ldots, v(\alpha_n))$ if c is an n-ary connective of Σ. Given a subset $\Gamma \cup \{\alpha\}$ of For_Σ, α is a consequence of Γ in $\mathcal{M}$ if $v(\alpha) \in D$ for every $\mathcal{M}$-valuation v such that $v[\Gamma] \subseteq D$.*

Non-deterministic semantics is a useful alternative for logics which are not truth-functional or congruential (that is, without non-trivial congruences). Nmatrices were studied by A. Avron and his collaborators in order to give a satisfactory semantic account of certain logics, including the paraconsistent logics known as *logics of formal inconsistency* (**LFI**s, see [7]). A closely related antecedent of Nmatrices is constituted by the *possible-translations semantics*, introduced by W. Carnielli in [4], which can also be applied to **LFI**s and paraconsistent logics in general (see, for instance, [5] and [23]). It can be proved that Nmatrices constitute a particular case of possible-translation semantics (see [6]).

Several years ago, and previous to all these developments, J. Ivlev had defined in 1988 (see [15] and [16]) a semantics of four-valued Nmatrices (called *quasi-matrices*) in order to characterize a hierarchy of weak modal logics in which the necessitation rule is not valid. This weakness is compensated, at the proof-theoretic level, by the inclusion of additional modal axioms and an specific replacement rule. Semantically, the four-valued Nmatrices considered by Ivlev are defined as follows: the truth-values are t^n (necessarily true), t^c (possibly true), f^c (possibly false) and f^i (impossible). Let $t = \{t^n, t^c\}$ and $f = \{f^c, f^i\}$ be the sets of designated values and of non-designated values, respectively. These concepts were already studied in [11] in order to define a resolution principle for a fragment of S5. Consider now a signature Σ composed by $\neg$ (negation), $\Box$ (necessity) and $\to$ (implication), and consider the following multivalued functions interpreting the connectives of Σ:[2]

$\mathcal{N}_{\Box}^{\mathbf{T}}$

p	$\Box p$
t^n	t
t^c	f
f^c	f
f^i	f

$\mathcal{M}_{\neg}$

p	$\neg p$
t^n	f^i
t^c	f^c
f^c	t^c
f^i	t^n

$\mathcal{N}_{\to}$

$\to$	t^n	t^c	f^c	f^i
t^n	t^n	t^c	f^c	f^i
t^c	t^n	t	f^c	f^c
f^c	t^n	t	t	t^c
f^i	t^n	t^n	t^n	t^n

The disjunction operator $p \vee q = \neg p \to q$ and the possibility operator $\Diamond p = \neg\Box\neg p$ are defined in a classical way, producing the following multivalued functions:

$\mathcal{N}_{\vee}$

$\vee$	t^n	t^c	f^c	f^i
t^n	t^n	t^n	t^n	t^n
t^c	t^n	t	t	t^c
f^c	t^n	t	f^c	f^c
f^i	t^n	t^c	f^c	f^i

$\mathcal{N}_{\Diamond}^{\mathbf{T}}$

p	$\Diamond p$
t^n	t
t^c	t
f^c	t
f^i	f

Let **PC** be any sound and complete (w.r.t. the usual semantics) Hilbert calculus for classical propositional logic over the signature $\{\neg, \to\}$,[3] and consider the following modal axioms over the signature Σ (where $\vee$ and $\Diamond$ are abbreviations):

(K) $\Box(p \to q) \to (\Box p \to q)$

(K1) $\Box(p \to q) \to (\Diamond p \to \Diamond q)$

(K2) $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$

(M1) $\neg\Diamond p \to \Box(p \to q)$

(M2) $\Box q \to \Box(p \to q)$

(M3) $\Diamond q \to \Diamond(p \to q)$

(M4) $\Diamond\neg p \to \Diamond(p \to q)$

[2]In order to simplify the exposition, given a truth-value m of an Nmatrix, we will write m instead of $\{m\}$.

[3]We assume that the rule of Modus Ponens, denoted by (MP), is available in **PC**.

(T) $\Box p \to p$

and the following inference rule:

(DN) $\dfrac{\alpha[\neg\neg\beta]}{\alpha[\beta]}$ $\dfrac{\alpha[\beta]}{\alpha[\neg\neg\beta]}$

The rule (DN) allows to substitute each occurrence of the formula $\neg\neg\beta$ inside a formula α by β, and vice versa.

Now, the system **Tm** is obtained from **PC** by adding the axioms and rules above, namely:[4]

- $\mathbf{Tm} = \mathbf{PC} \cup \{(K), (K1), (K2), (M1), (M2), (M3), (M4), (T), (DN)\}$

A formula α is a tautology in **Tm** (notation: $\vDash_{\mathbf{Tm}} \alpha$) iff it only receives the designated values t^n or t^c when interpreted by means of valuations in the Nmatrix formed by the multivalued functions $\mathcal{N}_{\Box}^{\mathbf{T}}$, $\mathcal{M}_{\neg}$ and $\mathcal{M}_{\to}$, according to Definition 1.1.

Theorem 1.2 (Soundness of Tm) *Let α be a formula of* **Tm**. *Then:*

$$\vdash_{\mathbf{Tm}} \alpha \textit{ implies } \vDash_{\mathbf{Tm}} \alpha$$

Proof. The only axiom to be analyzed is (T). Thus, suppose that v is a valuation such that $v(p) = t^n$. Then, $v(T) = t^n$. On the other hand, if $v(p) \neq t^n$ then $v(\Box p) \in f$ and so $v(T) \in t$. ■

The completeness of system **Tm** will be proved now by defining a canonical valuation by means of a maximal consistent set of formulas. Recall that, given a modal system **L** based on classical logic, a set of formulas Δ in the language of **L** is a *closed theory* if it contains every theorem of **L**, and the following holds: for every formula α, if there exists a finite non-empty set $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Gamma$ such that the formula $\alpha_1 \to (\alpha_2 \to (\ldots \to (\alpha_n \to \alpha) \ldots))$ is a theorem of **L**, then $\alpha \in \Delta$. A closed theory Δ is *maximal consistent in* **L** if it does not contain any contradiction, but any extension of it is contradictory in **L**. This means that, if $\alpha \notin \Delta$, then $\alpha \to \beta \in \Delta$ for every formula β.[5]

Definition 1.3 *Let Δ be a set of fomulas which is maximal consistent in* **Tm**. *The canonical valuation $\mathbb{V}_{\mathbf{Tm}}^{\Delta} : For_{\mathbf{Tm}} \longrightarrow \{t^n, t^c, f^c, f^i\}$ associated to Δ is defined as follows:*

[4] In fact, Ivlev defines a system called $\mathbf{Sa}^+$ in which the operators $\Box$ and $\Diamond$ are treated independently. The system **Tm** proposed here is equivalent to $\mathbf{Sa}^+$, but defined in a simpler language.

[5] In other words, a maximal consistent set in **L** is a maximal non-trivial closed theory.

(i) $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\alpha) = t^n$ *iff* $\Box\alpha \in \Delta$ *(and* $\alpha \in \Delta$*)*

(ii) $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\alpha) = t^c$ *iff* $\neg\Box\alpha \in \Delta$ *and* $\alpha \in \Delta$

(iii) $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\alpha) = f^c$ *iff* $\Diamond\alpha \in \Delta$ *and* $\neg\alpha \in \Delta$

(iv) $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\alpha) = f^i$ *iff* $\neg\Diamond\alpha \in \Delta$ *(and* $\neg\alpha \in \Delta$*)* $\Box$

Lemma 1.4 $\mathbb{V}^{\Delta}_{\mathbf{Tm}}$ *is a* **Tm***-valuation.*

Proof. It suffices to analyze the following cases:

CASE 1: α is $\neg\beta$.

(i) If $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta) = t^n$ then $\Box\beta \in \Delta$. By definition of $\Diamond$ and by (DN), $\neg\Diamond\neg\beta \in \Delta$. Then, $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\neg\beta) = f^i$.

(ii) If $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta) = t^c$ then $\neg\Box\beta \in \Delta$ and $\beta \in \Delta$. Thus, $\Diamond\neg\beta \in \Delta$, $\neg\neg\beta \in \Delta$ and so $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\neg\beta) = f^c$.

(iii) If $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta) = f^c$, then $\Diamond\beta \in \Delta$ and $\neg\beta \in \Delta$. Thus, $\neg\Box\neg\beta \in \Delta$ and so $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\neg\beta) = t^c$.

(iv) If $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta) = f^i$ then $\neg\Diamond\beta \in \Delta$. By definition of $\Diamond$ and by (DN), $\Box\neg\beta \in \Delta$. Then $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\neg\beta) = t^n$.

CASE 2: α is $\Box\beta$.

(i) If $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta) = t^n$ then $\Box\beta \in \Delta$ and so $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\Box\beta) \in t$.

(ii) If $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta) = t^c$ then $\neg\Box\beta \in \Delta$ and so $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\Box\beta) \in f$.

(iii) If $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta) \in f$ then $\neg\beta \in \Delta$ and so $\neg\Box\beta \in \Delta$, by (T). Hence $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\Box\beta) \in f$.

CASE 3: α is $\beta \to \gamma$.

(i) If $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\gamma) = t^n$ then $\Box\gamma \in \Delta$. Hence, $\Box(\beta \to \gamma) \in \Delta$, by $(M2)$, and then $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta \to \gamma) = t^n$.

(ii) If $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta) = f^i$ then $\neg\Diamond\beta \in \Delta$. Hence, $\Box(\beta \to \gamma) \in \Delta$, by $(M1)$. Therefore $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta \to \gamma) = t^n$.

(iii) If $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\gamma) = t^c$ then $\gamma, \neg\Box\gamma \in \Delta$ whence $(\beta \to \gamma) \in \Delta$, by $(Ax1)$. In addition, if $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta) = t^n$ then $\Box\beta \in \Delta$. But $\neg\Box\gamma \in \Delta$ and so $(\Box\beta \wedge \neg\Box\gamma) \in \Delta$, hence $\neg(\Box\beta \to \Box\gamma) \in \Delta$. From this it follows that $\neg\Box(\beta \to \gamma) \in \Delta$, by (K), and so $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta \to \gamma) = t^c$.

(iv) If $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta) = t^n$ and $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\gamma) \in f$ then $\Box\beta, \beta, \neg\gamma \in \Delta$. From this, $(\beta \wedge \neg\gamma) \in \Delta$ and so $\neg(\beta \to \gamma) \in \Delta$, whence $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta \to \gamma) \in f$. If $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\gamma) = f^c$ then $\Diamond\gamma \in \Delta$ and so $\Diamond(\beta \to \gamma) \in \Delta$, by $(M3)$. This means that $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta \to \gamma) = f^c$. On the other hand, if $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\gamma) = f^i$ then $\neg\Diamond\gamma \in \Delta$ whence $(\Box\beta \wedge \neg\Diamond\gamma) \in \Delta$. Thus $\neg(\Diamond\neg\beta \vee \Diamond\gamma) \in \Delta$ and so $\neg\Diamond(\neg\beta \vee \gamma) \in \Delta$, por $(K2)$.

Therefore it follows that $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\neg\beta \vee \gamma) = \mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta \to \gamma) = f^i$.
(v) If $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta) = t^c$ and $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\gamma) \in f$ then $\neg\Box\beta, \beta, \neg\gamma \in \Delta$. Therefore $\neg(\beta \to \gamma) \in \Delta$, as it was proved above. Moreover, $\Diamond\neg\beta \in \Delta$ and so $\Diamond(\beta \to \gamma) \in \Delta$, by $(M4)$. Thus $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta \to \gamma) = f^c$.
(vi) If $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta) = f^c$ then $\Diamond\beta, \neg\beta \in \Delta$. Therefore $(\beta \to \gamma) \in \Delta$ and so $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta \to \gamma) \in t$. Moreover, if $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\gamma) = f^i$ then $\neg\Diamond\gamma \in \Delta$ and so $\neg(\Diamond\beta \to \Diamond\gamma) \in \Delta$. Thus $\neg\Box(\beta \to \gamma) \in \Delta$, by $(K1)$, and $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\beta \to \gamma) = t^c$. ■

Lemma 1.5 *Let Δ be a maximal consistent set in* **Tm**. *Then, there exists a* **Tm**-*valuation $\mathbb{V}$ such that, for every formula α:*

$$\mathbb{V}(\alpha) \in t \textit{ iff } \alpha \in \Delta$$

Proof. By Definition 1.3, $\mathbb{V}^{\Delta}_{\mathbf{Tm}}$ is a function such that $\mathbb{V}^{\Delta}_{\mathbf{Tm}}(\alpha) \in t$ iff $\alpha \in \Delta$. By Lemma 1.4, $\mathbb{V}^{\Delta}_{\mathbf{Tm}}$ is a **Tm**-valuation. ■

Theorem 1.6 (Completeness of Tm) *Let α be a formula of* **Tm**. *Then:*

$$\models_{\mathbf{Tm}} \alpha \quad \textit{implies} \quad \vdash_{\mathbf{Tm}} \alpha.$$

Proof. Suppose that $\nvdash_{\mathbf{Tm}} \alpha$. Then, by a classical result by Lindenbaum and Łos, there exists a a maximal consistent set of formulas in **Tm**, namely Δ, such that $\alpha \notin \Delta$. By Lemma 1.5, there exists a **Tm**-valuation $\mathbb{V}$ such that $\mathbb{V}(\alpha) \notin t$. Therefore $\nvDash_{\mathbf{Tm}} \alpha$ ■

Ivlev proposes from this a series of modal systems with the aim of expressing properties of his modal Nmatrices. Here, we only analyze the system based on axiom (T), proposing similar systems based on the following axioms:[6]

(B) $p \to \Box\Diamond p$

(4) $\Box p \to \Box\Box p$

(5) $\Diamond p \to \Box\Diamond p$

The modal systems associated to these axioms are the following:

Definition 1.7

$$\mathbf{TBm} = \mathbf{Tm} \cup \{(B)\}$$

[6]Besides **Sa**+, Ivlev proposes $\mathbf{S}\overline{a}$, **Sa**, $\mathbf{S}\overline{\delta}$, $\mathbf{S}\delta$, $\mathbf{S}\delta^+$, **Sb** and $\mathbf{Sb}^+$. To our purposes, just system $\mathbf{Sb}^+$ will be significant, which will be renamed as **T45m**.

$\mathbf{T4m} = \mathbf{Tm} \cup \{(4)\}$

$\mathbf{T45m} = \mathbf{T4m} \cup \{(5)\}$

It is easy to prove that $\mathbf{T45m} = \mathbf{TBm} \cup \mathbf{T5m}$. The semantics for these systems is given by the multivalued functions of **Tm** for $\neg$ and $\rightarrow$ by adding, respectively, the following:[7]

$\mathcal{N}_{\Box}^{\mathbf{B}}$

p	$\Box p$
t^n	t
t^c	f
f^c	f^i
f^i	f^i

$\mathcal{N}_{\Box}^{\mathbf{4}}$

p	$\Box p$
t^n	t^n
t^c	f
f^c	f
f^i	f

$\mathcal{M}_{\Box}^{\mathbf{5}}$

p	$\Box p$
t^n	t^n
t^c	f^i
f^c	f^i
f^i	f^i

TBm-valuations are defined according to Definition 1.1 by using the corresponding multivalued functions. **T4m**-valuations and **T45m**-valuations are defined in a similar way. The soundness and completeness theorems for these systems are obtained in the sequel.

Theorem 1.8 (Soundness of TBm) *Let α be a formula of* **TBm**. *Then:*

$$\vdash_{\mathbf{TBm}} \alpha \textit{ implies } \vDash_{\mathbf{TBm}} \alpha$$

Proof. It is enough to analyze axiom (B). Thus, if $v(p) \in f$ then $v(B) \in t$. And if $v(p) \notin f$ then $v(\Diamond p) = t^n$ and $v(B) \in t$ ■

Theorem 1.9 (Soundness of T4m) *Let α be a formula of* **T4m**. *Then:*

$$\vdash_{\mathbf{T4m}} \alpha \textit{ implies } \vDash_{\mathbf{T4m}} \alpha$$

Proof. Only axiom (4) willl be analyzed. Thus, if $v(\Box p) = t^n$ then $v(\Box\Box p) = t^n$ and $v(4) = t^n$. If $v(\Box p) \neq t^n$ then $v(\Box\Box p) \in f$ and $v(4) \in t$. ■

Theorem 1.10 (Soundness of T45m) *Let α be a formula of* **T45m**. *Then:*

$$\vdash_{\mathbf{T45m}} \alpha \textit{ implies } \vDash_{\mathbf{T45m}} \alpha$$

[7]The multivalued functions $\mathcal{N}_{\Box}^{\mathbf{T}}$, $\mathcal{M}_{\neg}$, $\mathcal{N}_{\rightarrow}$ and $\mathcal{M}_{\Box}^{\mathbf{5}}$ were proposed independently by Kearns in [19] and by Ivlev in [15]. The multivalued function $\mathcal{N}_{\Box}^{\mathbf{4}}$ was proposed by Kearns (but taking $\Box(f^i) = \{f^i\}$ instead of $\Box(f^i) = f$) in [19] in order to interpret **S4**, while $\mathcal{N}_{\Box}^{\mathbf{B}}$ is our proposal.

Proof. For the axioms and rules of **Tm** the proof is analogous to that for Theorem 1.2, by substituting $v(\alpha) \in t$ by $v(\alpha) = t^n$ and $v(\alpha) \in f$ by $v(\alpha) = f^-$ whenever α has the form $\Box\beta$ or $\Diamond\beta$. Concerning axiom (4), the proof is as the one for Theorem 1.9. For axiom (5), if $v(p) = f^i$ then $v(\Diamond p) = f^i$ and $v(5) = t^n$. If $v(p) \neq f^i$ then $v(\Diamond p) = t^n$, $v(\Box\Diamond p) = t^n$ and $v(5) = t^n$ ■

Definition 1.11 *Let Δ be a a maximal consistent set in* **Lm**, *where* $\mathbf{Lm} \in \{\mathbf{TBm}, \mathbf{T4m}, \mathbf{T45m}\}$. *The canonical valuation associated to Δ in* **Lm** *is the function $\mathbb{V}^{\Delta}_{\mathbf{Lm}} : For_{\mathbf{Lm}} \longrightarrow \{t^n, t^c, f^c, f^i\}$ defined as follows:*

(i) $\mathbb{V}^{\Delta}_{\mathbf{Lm}}(\alpha) = t^n$ *iff* $\Box\alpha \in \Delta$ *(and $\alpha \in \Delta$)*

(ii) $\mathbb{V}^{\Delta}_{\mathbf{Lm}}(\alpha) = t^c$ *iff* $\neg\Box\alpha \in \Delta$ *and* $\alpha \in \Delta$

(iii) $\mathbb{V}^{\Delta}_{\mathbf{Lm}}(\alpha) = f^c$ *iff* $\Diamond\alpha \in \Delta$ *and* $\neg\alpha \in \Delta$

(iv) $\mathbb{V}^{\Delta}_{\mathbf{Lm}}(\alpha) = f^i$ *iff* $\neg\Diamond\alpha \in \Delta$ *(and $\neg\alpha \in \Delta$)* □

Lemma 1.12 $\mathbb{V}^{\Delta}_{\mathbf{TBm}}$ *is a* **TBm***-valuation.*

Proof. **CASE 1** and **CASE 3** are proved as in Lemma 1.4. Concerning **CASE 2**, the same applies when $\mathbb{V}^{\Delta}_{\mathbf{TBm}}(\beta) = t^n$ or $\mathbb{V}^{\Delta}_{\mathbf{TBm}}(\beta) = t^c$. It remains to analyze just two situations:

CASE 2: α is $\Box\beta$.

(i) If $\mathbb{V}^{\Delta}_{\mathbf{TBm}}(\beta) = f^c$ then $\Diamond\beta \in \Delta$ and $\neg\beta \in \Delta$. Hence, by (B), it follows that $\Box\Diamond\neg\beta \in \Delta$ and, by definition of $\Diamond$ and (DN), $\Box\neg\Box\beta \in \Delta$ and $\neg\Diamond\Box\beta \in \Delta$. Therefore $\mathbb{V}^{\Delta}_{\mathbf{TBm}}(\Box\beta) = f^i$.

(ii) If $\mathbb{V}^{\Delta}_{\mathbf{TBm}}(\beta) = f^i$ then $\neg\Diamond\beta \in \Delta$. By (B) it can be inferred that $\neg\Box\Diamond\gamma \to \neg\gamma$ which implies $\Diamond\Box\neg\gamma \to \neg\gamma \in \Delta$. By substituting γ by $\neg\beta$ we conclude by (DN) that $\Diamond\Box\beta \to \beta \in \Delta$. But, from (T) it follows that $\beta \to \Diamond\beta \in \Delta$ which implies that $\Diamond\Box\beta \to \Diamond\beta \in \Delta$, and so $\neg\Diamond\beta \to \neg\Diamond\Box\beta \in \Delta$. Hence, by (MP) it follows that $\neg\Diamond\Box\beta \in \Delta$ and $\mathbb{V}^{\Delta}_{\mathbf{TBm}}(\Box\beta) = f^i$. ■

Lemma 1.13 *Let Δ be a maximal consistent set in* **TBm**. *Then, there exists a* **TBm***-valuation* $\mathbb{V}$ *such that, for every formula α:*

$$\mathbb{V}(\alpha) \in t \textit{ iff } \alpha \in \Delta.$$

Proof. Analogous to that of Lemma 1.5, by substituting Lemma 1.4 by Lemma 1.12. ■

Theorem 1.14 (Completeness of TBm) *Let* α *be a formula of* **TBm**. *Then:*

$$\vDash_{\mathbf{TBm}} \alpha \quad \textit{implies} \quad \vdash_{\mathbf{TBm}} \alpha.$$

Proof. Analogous to that of Theorem 1.6, by substituting Lemma 1.5 by Lemma 1.13. ■

Lemma 1.15 $\mathbb{V}^{\Delta}_{\mathbf{T4m}}$ *is a* **T4m**-*valuation.*

Proof. The proofs of **CASE 1** and **CASE 3** are identical to the ones of Lemma 1.4. Concerning **CASE 2**, the argument is the same when $\mathbb{V}^{\Delta}_{\mathbf{T4m}}(\beta) \neq t^n$. So, it remains just one case to be analyzed.

CASE 2: α is $\Box\beta$.

(i) If $\mathbb{V}^{\Delta}_{\mathbf{T4m}}(\beta) = t^n$ then $\Box\beta \in \Delta$ and, by (4), $\Box\Box\beta \in \Delta$. Being so, $\mathbb{V}^{\Delta}_{\mathbf{T4m}}(\Box\alpha) = t^n$. ■

Lemma 1.16 *Let* Δ *be a maximal consistent set in* **T4m**. *Then, there exists a* **T4m**-*valuation* $\mathbb{V}$ *such that, for every formula* α*:*

$$\mathbb{V}(\alpha) \in t \textit{ iff } \alpha \in \Delta.$$

Proof. Analogous to that of Lemma 1.5, by substituting Lemma 1.4 by Lemma 1.15. ■

Theorem 1.17 (Completeness of T4m) *Let* α *be a formula of* **T4m**. *Then:*

$$\vDash_{\mathbf{T4m}} \alpha \quad \textit{implies} \quad \vdash_{\mathbf{T4m}} \alpha.$$

Proof. Analogous to that of Theorem 1.6, by substituting Lemma 1.5 by Lemma 1.16. ■

Lemma 1.18 $\mathbb{V}^{\Delta}_{\mathbf{T45m}}$ *is a* **T45m**-*valuation.*

Proof. For **CASE 1** and **CASE 3** the proof is analogous to that of Lemma 1.4. With respect to **CASE 2**, given that (4) and (B) are theorems in **T45m**, the proof is analogous to that of Lema 1.15 when $\mathbb{V}^{\Delta}_{\mathbf{T4m}}(\beta) = t^n$; and to that of Lemma 1.12 when $\mathbb{V}^{\Delta}_{\mathbf{TBm}}(\beta) = f^c$ and $\mathbb{V}^{\Delta}_{\mathbf{TBm}}(\beta) = f^i$. So, it remains just one case to be analyzed.

CASE 2: α is $\Box\beta$.

(i) If $\mathbb{V}^{\Delta}_{\mathbf{T45m}}(\beta) = t^c$ then $\beta, \Diamond\neg\beta \in \Delta$ whence $\neg\Box\beta \in \Delta$. Then, $\Box\Diamond\neg\beta \in \Delta$, by (5). Hence $\Box\neg\Box\beta \in \Delta$ and $\neg\Diamond\Box\beta \in \Delta$, by definition of $\Diamond$ and (DN). Therefore $\mathbb{V}^{\Delta}_{\mathbf{T45m}}(\Box\beta) = f^i$. ■

Lemma 1.19 *Let Δ be a maximal consistent set in* **T45m**. *Then, there exists a* **T45m**-*valuation $\mathbb{V}$ such that, foe every formula α:*

$$\mathbb{V}(\alpha) \in t \textit{ iff } \alpha \in \Delta$$

Proof. Analogous to that of Lemma 1.5, by substituting Lemma 1.4 by Lemma 1.18. ■

Theorem 1.20 (Completeness of T45m) *Let α be a formula of* **T45m**. *Then:*

$$\vDash_{\mathbf{T45m}} \alpha \quad \textit{implies} \quad \vdash_{\mathbf{T45m}} \alpha.$$

Proof. Analogous to that of Theorem 1.6, by substituting Lemma 1.5 by Lemma 1.19. ■

2 Kearns' Nmatrices with level valuations

The four modal systems analyzed in the previous section, namely **Tm**, **TBm**, **T4m** and **T45m**, despite being characterized by finite Nmatrices, they are weak in the sense that they do not admit any form of the *Necessitation* rule, which allows to introduce the $\Box$ operator.

A. Avron argues in [1] that inference rules such as Necessitation are impure, in the sense that they cannot be represented by sequents of the form $\Gamma \vdash \Delta$ (for Γ and Δ sets of formulas). Thus, finding a complete semantics, by means of Nmatrices, of modal system with this kind of rules turns out to be a very difficult task. [8] This claim is supported by the early work by J. Kearns (see [19]), in which a complete semantics for some modal systems is given in terms of finite Nmatrices together with a refinement of the valuations called *level valuations*. The aim of this section is to reintroduce the main results of Kearns in an uniform way in the framework of Hilbert-style axiomatic systems. Additionally, and complementary to the systems presented in the previous section, some new completeness results of Nmatrices with level valuations will be given.

[8] In his words: "However, the necessitation rule, as it is used in modal logics, is *impure*: if $\vdash$ is supposed to be an extension of the classical consequence relation, then the necessitation rule cannot be translated into $\varphi \vdash \Box\varphi$. Indeed, in classical logic we have that $\Box\varphi \vdash \varphi \supset \Box\varphi$, and that $\vdash \varphi, \varphi \supset \Box\varphi$. Together with $\varphi \vdash \Box\varphi$ these facts entail $\vdash \varphi \supset \Box\varphi$ (using cuts). However, $\varphi \supset \Box\varphi$ is not valid in any interesting modal logic. It seems therefore that extra machinery, like the use of non-deterministic Kripke structures, should be added in order to handle rules of this sort."

Let us begin by considering the necessitation rule (Nec) in modal system $\mathbf{T}$. Because of this rule, $\vdash_{\mathbf{T}} \Box(\Box p \to p)$, by applying (Nec) to axiom (T). However, this theorem receives a non designated truth-value when interpreted in $\mathbf{Tm}$. The nondeterministic truth-table for such a theorem of $\mathbf{T}$ by using the Nmatrices of $\mathbf{Tm}$ is the following, in Ivlev's notation:

<table>
<tr><th>p</th><th>$\Box p$</th><th>$\Box p \to p$</th><th>$\Box(\Box p \to p)$</th></tr>
<tr><td rowspan="4">t^n</td><td rowspan="2">t^n</td><td rowspan="2">t^n</td><td>t^n</td></tr>
<tr><td>t^c</td></tr>
<tr><td rowspan="2">t^c</td><td rowspan="2">t^n</td><td>t^n</td></tr>
<tr><td>t^c</td></tr>
<tr><td rowspan="6">t^c</td><td rowspan="4">f^c</td><td rowspan="2">t^n</td><td>t^n</td></tr>
<tr><td>t^c</td></tr>
<tr><td rowspan="2">t^c</td><td>f^c</td></tr>
<tr><td>f^i</td></tr>
<tr><td rowspan="2">f^i</td><td rowspan="2">t^n</td><td>t^n</td></tr>
<tr><td>t^c</td></tr>
<tr><td rowspan="6">f^c</td><td rowspan="4">f^c</td><td rowspan="2">t^n</td><td>t^n</td></tr>
<tr><td>t^c</td></tr>
<tr><td rowspan="2">t^c</td><td>f^c</td></tr>
<tr><td>f^i</td></tr>
<tr><td rowspan="2">f^i</td><td rowspan="2">t^n</td><td>t^n</td></tr>
<tr><td>t^c</td></tr>
<tr><td rowspan="4">f^i</td><td rowspan="2">f^c</td><td rowspan="2">t^c</td><td>f^c</td></tr>
<tr><td>f^i</td></tr>
<tr><td rowspan="2">f^i</td><td rowspan="2">t^n</td><td>t^n</td></tr>
<tr><td>t^c</td></tr>
</table>

As it can be seen, $\Box(\Box p \to p)$ can assume the non-designated truth-values f^c or f^i when $v(p) \neq t^n$, invalidating the (Nec) rule, since $\Box p \to p$ always receive a designated truth-value. The level valuations are introduced in order to validate (Nec), by redefining (or restricting) the table above. The idea is that, when a column corresponding to a subformula is such that it only contains designated values, the value t^c is replaced by t^n in any row of such column, increasing by one the *level* of valuation. Let us denote by $\mathsf{Val}_k(\alpha)$

the set of kth-level valuations of the formula α, for $k \geq 0$. The sets $\mathsf{Val}_k(\alpha)$ can be represented by columns of a table. Then, according to the definition of level valuations, it is possible to associate, to each formula α, a sequence of sets (or columns of a table) $\mathsf{Val}_0(\alpha)$, $\mathsf{Val}_1(\alpha) \ldots \mathsf{Val}_k(\alpha)$ corresponding to the valuations of level $0, 1, \ldots, k$, respectively, with respect to α. Thus, the table above can be redefined as follows:

<table>
<tr><th>Val_0</th><th>Val_0</th><th>Val_0</th><th>Val_1</th><th>Val_1</th><th>Val_2</th></tr>
<tr><th>p</th><th>$\Box p$</th><th>$\Box p \to p$</th><th>$\Box p \to p$</th><th>$\Box(\Box p \to p)$</th><th>$\Box(\Box p \to p)$</th></tr>
<tr><td rowspan="4">t^n</td><td rowspan="2">t^n</td><td rowspan="2">t^n</td><td rowspan="2">t^n</td><td>t^n</td><td>t^n</td></tr>
<tr><td>t^c</td><td>t^n</td></tr>
<tr><td rowspan="2">t^c</td><td rowspan="2">t^n</td><td rowspan="2">t^n</td><td>t^n</td><td>t^n</td></tr>
<tr><td>t^c</td><td>t^n</td></tr>
<tr><td rowspan="6">t^c</td><td rowspan="4">f^c</td><td rowspan="2">t^n</td><td rowspan="2">t^n</td><td>t^n</td><td>t^n</td></tr>
<tr><td>t^c</td><td>t^n</td></tr>
<tr><td rowspan="2">t^c</td><td rowspan="2">t^n</td><td>t^n</td><td>t^n</td></tr>
<tr><td>t^c</td><td>t^n</td></tr>
<tr><td rowspan="2">f^i</td><td rowspan="2">t^n</td><td rowspan="2">t^n</td><td>t^n</td><td>t^n</td></tr>
<tr><td>t^c</td><td>t^n</td></tr>
<tr><td rowspan="6">f^c</td><td rowspan="4">f^c</td><td rowspan="2">t^n</td><td rowspan="2">t^n</td><td>t^n</td><td>t^n</td></tr>
<tr><td>t^c</td><td>t^n</td></tr>
<tr><td rowspan="2">t^c</td><td rowspan="2">t^n</td><td>t^n</td><td>t^n</td></tr>
<tr><td>t^c</td><td>t^n</td></tr>
<tr><td rowspan="2">f^i</td><td rowspan="2">t^n</td><td rowspan="2">t^n</td><td>t^n</td><td>t^n</td></tr>
<tr><td>t^c</td><td>t^n</td></tr>
<tr><td rowspan="4">f^i</td><td rowspan="2">f^c</td><td rowspan="2">t^c</td><td rowspan="2">t^n</td><td>t^n</td><td>t^n</td></tr>
<tr><td>t^c</td><td>t^n</td></tr>
<tr><td rowspan="2">f^i</td><td rowspan="2">t^n</td><td rowspan="2">t^n</td><td>t^n</td><td>t^n</td></tr>
<tr><td>t^c</td><td>t^n</td></tr>
</table>

The soundness of the method is clear, since we are in fact restricting in a suitable way the universe of valuations. In order to prove completeness, it will be convenient to formalize the notion of kth-level valuations. Recall that, in Ivlev's notation, $t = \{t^n, t^c\}$ and $f = \{f^c, f^i\}$ are the set of designated and non-designated truth-values, respectively.

Definition 2.1 *Let* $\mathsf{Val}^{\mathbf{T}}$ *be the set of* $\mathbf{Tm}$*-valuations (see Definition 1.1).*
(a) Let $(\mathsf{Val}_k^{\mathbf{T}})_{k\in\mathbb{N}}$ *be the sequence of subsets of* $\mathsf{Val}^{\mathbf{T}}$ *defined as follows:* $\mathsf{Val}_0^{\mathbf{T}} = \mathsf{Val}^{\mathbf{T}}$ *and, for every* $k \geq 0$,

$$\mathsf{Val}_{k+1}^{\mathbf{T}} = \{v \in \mathsf{Val}_k^{\mathbf{T}} \ : \ \textit{for every formula } \alpha, \ \mathsf{Val}_k^{\mathbf{T}}(\alpha) \subseteq t \textit{ implies } v(\alpha) = t^n\}$$

where $\mathsf{Val}_k^{\mathbf{T}}(\alpha) = \{v(\alpha) \ : \ v \in \mathsf{Val}_k^{\mathbf{T}}\}$*, for every* k *and* α*. Elements in* $\mathsf{Val}_k^{\mathbf{T}}$ *are called* kth-level valuations *or* valuations of level k for $\mathbf{T}$.
(b) The set of $\mathbf{T}$*-valuations is given by* $\mathsf{Val}_{\mathbf{T}} = \bigcap_{k\geq 0} \mathsf{Val}_k^{\mathbf{T}}$. □

A formula α is a tautology in $\mathbf{T}$ when it receives the truth-value t^n for every $\mathbf{T}$-valuation. In formal terms:

Definition 2.2 *Let* α *be a formula of* $\mathbf{T}$*. Then:*

$$\vDash_{\mathbf{T}} \alpha \quad \textit{iff} \quad v(\alpha) = t^n \textit{ for every } v \in \mathsf{Val}_{\mathbf{T}}. \qquad \square$$

Intuitively, a tautology in $\mathbf{T}$ is a formula such that, at certain level k of valuations, it takes the truth-value t^n in every row. In formal terms:

Proposition 2.3 *Let* α *be a formula of* $\mathbf{T}$*. Then:*

$$\vDash_{\mathbf{T}} \alpha \quad \textit{iff there exists } k \geq 0 \textit{ such that } \mathsf{Val}_k^{\mathbf{T}}(\alpha) = \{t^n\}.$$

Proof. Suppose that $\vDash_{\mathbf{T}} \alpha$. By Definition 2.2, $\mathsf{Val}_{\mathbf{T}}(\alpha) = \{t^n\}$. Thus, by Definition 2.1 (b), there exists some $k \geq 0$ such that $\mathsf{Val}_k^{\mathbf{T}}(\alpha) = \{t^n\}$. Conversely, if $\mathsf{Val}_k^{\mathbf{T}}(\alpha) = \{t^n\}$ for some k then, by Definition 2.1 (a), $\mathsf{Val}_{k'}(\alpha) = \{t^n\}$ for every $k' \geq k$. Thus, $v(\alpha) = t^n$ for every $v \in \mathsf{Val}_{\mathbf{T}}$. That is, $\vDash_{\mathbf{T}} \alpha$, by Definition 2.2. ■

The soundness and completeness theorem for $\mathbf{T}$ with respect to the semantics of Nmatrices with level valuations will be proved now. In order to simplify the proof, the following axiomatics for $\mathbf{T}$ will be adopted:

$$\mathbf{T} = \mathbf{Tm} \cup \{(Nec)\}.$$

Theorem 2.4 (Soundness of T) *Let* α *be a formula of* $\mathbf{T}$*. Then:*

$$\vdash_{\mathbf{T}} \alpha \textit{ implies } \vDash_{\mathbf{T}} \alpha$$

Proof. Suppose that $\vdash_{\mathbf{T}} \alpha$. The proof will be done by induction on the length k of a derivation of α in $\mathbf{T}$. If $k = 0$ then α is an axiom of $\mathbf{T}$. Then, by Theorem 1.2 and by Definition 2.1, $\mathsf{Val}_0^{\mathbf{T}}(\alpha) \subseteq t$. From this it follows that $\mathsf{Val}_1^{\mathbf{T}}(\alpha) = \{t^n\}$ and so $\vDash_{\mathbf{T}} \alpha$, by Proposition 2.3.

Assume the result holds for any theorem γ of $\mathbf{T}$ which admits a derivation of length k (induction hypothesis), and let α be a theorem of $\mathbf{T}$ with a derivation of length $k+1$. If α is an axiom of $\mathbf{T}$, the proof is as above. If α is obtained from formulas β and $\beta \to \alpha$ by (MP) then, by induction hypothesis, $\vDash_{\mathbf{T}} \beta$ and $\vDash_{\mathbf{T}} \beta \to \alpha$ and so $\mathsf{Val}_{\mathbf{T}}(\beta) = \mathsf{Val}_{\mathbf{T}}(\beta \to \alpha) = \{t^n\}$. By definition of the Nmatrix $\mathcal{N}_{\to}^{\mathbf{T}}$ for the implication operator $\to$, it follows that $\mathsf{Val}_{\mathbf{T}}(\alpha) = \{t^n\}$, that is, $\vDash_{\mathbf{T}} \alpha$. Finally, if α is the formula $\Box\beta$, obtained from β by the Necessitation rule (Nec) then, by induction hypothesis, $\vDash_{\mathbf{T}} \beta$ and so $\mathsf{Val}_k^{\mathbf{T}}(\beta) = \{t^n\}$ for some $k \geq 0$, by Proposition 2.3. Thus, $\mathsf{Val}_k(\alpha) \subseteq t$, by definition of the Nmatrix $\mathcal{N}_{\Box}^{\mathbf{T}}$, hence $\mathsf{Val}_{k+1}(\alpha) = \{t^n\}$. This shows that $\vDash_{\mathbf{T}} \alpha$, by Proposition 2.3. ■

In order to prove the completeness of $\mathbf{T}$, some notions and results stated for $\mathbf{Tm}$ will be used.

Definition 2.5 *The set of* $\mathbf{T}$-canonical valuations *is the set*

$$\mathit{Val}_{can}^{\mathbf{T}} = \{\mathbb{V}_{\mathbf{Tm}}^{\Delta} \ : \ \Delta \ \textit{ is a maximal consistent set in } \mathbf{T}\},$$

where $\mathbb{V}_{\mathbf{Tm}}^{\Delta}$ *is given as in Definition 1.3.*

Lemma 2.6 $\mathit{Val}_{can}^{\mathbf{T}} \subseteq \mathit{Val}_0^{\mathbf{T}}$.

Proof. It is an immediate consequence of the definition above by adapting the proof of Lemma 1.4, given that (DN) and the axioms of $\mathbf{Tm}$ hold in $\mathbf{T}$. ■

Lemma 2.7 *If* $\mathit{Val}_{can}^{\mathbf{T}} \subseteq \mathit{Val}_k^{\mathbf{T}}$ *then* $\mathit{Val}_{can}^{\mathbf{T}} \subseteq \mathit{Val}_{k+1}^{\mathbf{T}}$.

Proof. Assuming that $\mathsf{Val}_{can}^{\mathbf{T}} \subseteq \mathsf{Val}_k^{\mathbf{T}}$, let $\mathbb{V}_{\Delta} \in \mathsf{Val}_{can}^{\mathbf{T}}$. It will be shown that $\mathbb{V}_{\Delta} \in \mathsf{Val}_{k+1}^{\mathbf{T}}$. Let α be a formula such that $\mathsf{Val}_k^{\mathbf{T}}(\alpha) \subseteq t$. Since $\mathsf{Val}_{can}^{\mathbf{T}} \subseteq \mathsf{Val}_k^{\mathbf{T}}$, it follows that $\mathbb{V}_{\Delta} \in \mathsf{Val}_k^{\mathbf{T}}$ and then $\mathbb{V}_{\Delta}(\alpha) \in t$ by Definition 2.1.

Suppose that $\neg\alpha \in \Delta$; then $\mathbb{V}_{\Delta}(\alpha) \in f$, an absurd. Therefore $\neg\alpha \notin \Delta$ and so $\neg\alpha \to \beta \in \Delta$ for every β, since Δ is a maximal consistent set in $\mathbf{T}$. In particular, $\neg\alpha \to \alpha \in \Delta$. But $\vdash_{\mathbf{T}} (\neg\alpha \to \alpha) \to \alpha$, since it is a theorem of $\mathbf{PC}$, and so $\vdash_{\mathbf{T}} \alpha$, by (MP). Then $\vdash_{\mathbf{T}} \Box\alpha$, by (Nec), and so $\Box\alpha \in \Delta$. From this it follows that $\mathbb{V}_{\Delta}(\alpha) = t^n$, by Definition 1.3. Therefore $\mathbb{V}_{\Delta} \in \mathsf{Val}_{k+1}^{\mathbf{T}}$, by Definition 2.1. ■

Corollary 2.8 $\mathsf{Val}^{\mathbf{T}}_{can} \subseteq \mathsf{Val}_{\mathbf{T}}$.

Proof. It is an immediate consequence of Lemma 2.6, Lemma 2.7 and Definition 2.1. ■

Theorem 2.9 (Completeness of T) *Let α be a formula of* $\mathbf{T}$. *Then:*

$$\vDash_{\mathbf{T}} \alpha \textit{ implies } \vdash_{\mathbf{T}} \alpha.$$

Proof. Suppose that $\nvdash_{\mathbf{T}} \alpha$. Then, by the Theorem of Lindenbaum and Łos, there exists a maximal consistent set Δ in $\mathbf{T}$ such that $\alpha \notin \Delta$ and so $\neg\alpha \in \Delta$. If $\vDash_{\mathbf{T}} \alpha$ then there exists $k \geq 0$ such that $\mathsf{Val}^{\mathbf{T}}_k(\alpha) = \{t^n\}$, by Proposition 2.3. Since $\mathsf{Val}^{\mathbf{T}}_{can} \subseteq \mathsf{Val}_{\mathbf{T}}$ by Corollary 2.8, it follows by Definition 2.1 (iii) that $\mathsf{Val}^{\mathbf{T}}_{can} \subseteq \mathsf{Val}^{\mathbf{T}}_k$. Thus, given $\mathbb{V}_\Delta \in \mathsf{Val}^{\mathbf{T}}_{can}$ it follows that $\mathbb{V}_\Delta$ is a $\mathbf{T}$-valuation such that $\mathbb{V}_\Delta(\alpha) = t^n$ (given that $\mathbb{V}_\Delta \in \mathsf{Val}^{\mathbf{T}}_k$) and so $\alpha \in \Delta$, an absurd. Therefore $\nvDash_{\mathbf{T}} \alpha$. ■

This shows that the Nmatrices for **Tm** enriched with level valuations constitute an adequate semantics for **T**. In what follows, a similar (and original) result will be obtained for **B** with respect to the semantics of Nmatrices for **TBm** extended with level valuations. The system **B** can be defined as follows:

$$\mathbf{B} = \mathbf{TBm} \cup \{(Nec)\}.$$

The proof of soundness and completeness of **B** w.r.t. Nmatrices with level valuations is analogous to that presented for **T**, but now using the original Nmatrix for **TBm** that we propose in the previous section. The definition of level valuations is similar to the given for **T**.

Definition 2.10 *Let $\mathsf{Val}^{\mathbf{B}}$ be the set of* $\mathbf{TBm}$*-valuations (see Definition 1.1).*
(a) Let $(\mathsf{Val}^{\mathbf{B}}_k)_{k\in\mathbb{N}}$ be the sequence of subsets of $\mathsf{Val}^{\mathbf{B}}$ defined as follows: $\mathsf{Val}^{\mathbf{B}}_0 = \mathsf{Val}^{\mathbf{B}}$ and, for every $k \geq 0$,

$$\mathsf{Val}^{\mathbf{B}}_{k+1} = \{v \in \mathsf{Val}^{\mathbf{B}}_k \ : \ \textit{for every formula } \alpha,\ \mathsf{Val}^{\mathbf{B}}_k(\alpha) \subseteq t \textit{ implies } v(\alpha) = t^n\}$$

where $\mathsf{Val}^{\mathbf{B}}_k(\alpha) = \{v(\alpha) \ : \ v \in \mathsf{Val}^{\mathbf{B}}_k\}$, for every k and α. Elements in $\mathsf{Val}^{\mathbf{B}}_k$ are called kth-level valuations *or* valuations of level k for **B**.
(b) The set of $\mathbf{B}$*-valuations is given by $\mathsf{Val}_{\mathbf{B}} = \bigcap_{k\geq 0} \mathsf{Val}^{\mathbf{B}}_k$.* □

Definition 2.11 *Let α be a formula of* $\mathbf{B}$. *Then:*

$$\models_{\mathbf{B}} \alpha \textit{ iff } v(\alpha) = t^n \textit{ for every } v \in \mathsf{Val}_{\mathbf{B}}. \qquad \square$$

Proposition 2.12 *Let α be a formula of* $\mathbf{B}$. *Then:*

$$\models_{\mathbf{B}} \alpha \textit{ iff there exists } k \geq 0 \textit{ such that } \mathsf{Val}_k^{\mathbf{B}}(\alpha) = \{t^n\}.$$

Proof. Analogous to that of Proposition 2.3. ■

Theorem 2.13 (Soundness of B) *Let α be a formula of* $\mathbf{B}$. *Then:*

$$\vdash_{\mathbf{B}} \alpha \textit{ implies } \models_{\mathbf{B}} \alpha.$$

Proof. Analogous to that of Theorem 2.4, by substituting Theorem 1.2 and Definition 2.1 by Theorem 1.8 and Definition 2.10. ■

Definition 2.14 *The set of* $\mathbf{B}$-canonical valuations *is the set*

$$\mathsf{Val}_{can}^{\mathbf{B}} = \{\mathbb{V}_{\mathbf{TBm}}^{\Delta} \ : \ \Delta \ \textit{ is a maximal consistent set in } \mathbf{B}\},$$

where $\mathbb{V}_{\mathbf{TBm}}^{\Delta}$ is given as in Definition 1.3. □

Lemma 2.15 $\mathsf{Val}_{can}^{\mathbf{B}} \subseteq \mathsf{Val}_0^{\mathbf{B}}$.

Proof. Analogous to that of Lemma 2.6, by substituting Lemma 1.4 by Lemma 1.12. ■

Lemma 2.16 *If* $\mathsf{Val}_{can}^{\mathbf{B}} \subseteq \mathsf{Val}_k^{\mathbf{B}}$ *then* $\mathsf{Val}_{can}^{\mathbf{B}} \subseteq \mathsf{Val}_{k+1}^{\mathbf{B}}$.

Proof. Analogous to that of Lemma 2.7, by substituting Definition 2.1 by Definition 2.10. ■

Corollary 2.17 $\mathsf{Val}_{can}^{\mathbf{B}} \subseteq \mathsf{Val}_{\mathbf{B}}$.

Proof. It is an immediate consequence of Lemma 2.15, Lemma 2.16 and Definition 2.10. ■

Theorem 2.18 (Completeness of B) *Let α be a formula of* $\mathbf{B}$. *Then:*

$$\models_{\mathbf{B}} \alpha \textit{ implies } \vdash_{\mathbf{B}} \alpha.$$

Proof. Analogous to that of Theorem 2.9, by substituting Proposition 2.3 by Proposition 2.12, Corollary 2.8 by Corollary 2.17 and Definition 2.1 by Definition 2.10. ■

The next modal system to be analyzed is **S4**. As it will be shown, the technique used for **T** and **B** can also be applied to **S4**, obtaining so an adequate semantics of Nmatrices with level valuations. In order to do this, an adequate Hilbert-style axiomatization of **S4** will be considered, instead of the natural-deduction system considered by Kearns:

$$\mathbf{S4} = \mathbf{T4m} \cup \{(Nec)\}.$$

Definition 2.19 *Let $\mathsf{Val}^{\mathbf{S4}}$ be the set of* **T4m***-valuations (see Definition 1.1).*
(a) Let $(\mathsf{Val}_k^{\mathbf{S4}})_{k\in\mathbb{N}}$ be the sequence of subsets of $\mathsf{Val}^{\mathbf{S4}}$ defined as follows: $\mathsf{Val}_0^{\mathbf{S4}} = \mathsf{Val}^{\mathbf{S4}}$ and, for every$k \geq 0$,

$$\mathsf{Val}_{k+1}^{\mathbf{S4}} = \{v \in \mathsf{Val}_k^{\mathbf{S4}} \ : \ \textit{for every formula } \alpha, \ \mathsf{Val}_k^{\mathbf{S4}}(\alpha) \subseteq t \textit{ implies } v(\alpha) = t^n\}$$

*where $\mathsf{Val}_k^{\mathbf{S4}}(\alpha) = \{v(\alpha) \ : \ v \in \mathsf{Val}_k^{\mathbf{S4}}\}$ for every k and α. Elements in $\mathsf{Val}_k\frac{1}{2}$ are called k*th-level valuations *or* valuations of level k for **S4**.
(b) The set of **S4***-valuations is given by $\mathsf{Val}_{\mathbf{S4}} = \bigcap_{k\geq 0} \mathsf{Val}_k^{\mathbf{S4}}$.* □

Definition 2.20 *Let α be a formula of* **S4***. Then:*

$$\vDash_{\mathbf{S4}} \alpha \textit{ iff } v(\alpha) = t^n \textit{ for every } v \in \mathsf{Val}_{\mathbf{S4}}.$$ □

Proposition 2.21 *Let α be a formula of* **S4***. Then:*

$$\vDash_{\mathbf{S4}} \alpha \textit{ iff there exists } k \geq 0 \textit{ such that } \mathsf{Val}_k^{\mathbf{S4}}(\alpha) = \{t^n\}.$$

Proof. Analogous to that of Proposition 2.3. ■

Theorem 2.22 (Soundness of S4) *Let α be a formula of* **S4***. Then:*

$$\vdash_{\mathbf{S4}} \alpha \textit{ implies } \vDash_{\mathbf{S4}} \alpha.$$

Proof. Analogous to that of Theorem 2.4, by substituting Theorem 1.2, Proposition 2.3 and Definition 2.1 by Theorem 1.9, Proposition 2.21 and Definition 2.19, respectively. ■

Definition 2.23 *The set of* **S4**-canonical valuations *is the set*

$$\mathsf{Val}_{can}^{\mathbf{S4}} = \{\mathbb{V}_{\mathbf{T4m}}^{\Delta} \ : \ \Delta \ \textit{ is a maximal consistent set of } \mathbf{S4}\},$$

where $\mathbb{V}_{\mathbf{4m}}^{\Delta}$ is given as in Definition 1.3. □

Lemma 2.24 $\mathsf{Val}^{\mathbf{S4}}_{can} \subseteq \mathsf{Val}^{\mathbf{S4}}_0$.

Proof. Analogous to that of Lemma 2.6, by substituting Lemma 1.4 by Lemma 1.15. ■

Lemma 2.25 *If* $\mathsf{Val}^{\mathbf{S4}}_{can} \subseteq \mathsf{Val}^{\mathbf{S4}}_k$ *then* $\mathsf{Val}^{\mathbf{S4}}_{can} \subseteq \mathsf{Val}^{\mathbf{S4}}_{k+1}$.

Proof. Analogous to that of Lemma 2.7, by substituting Definition 2.1 by Definition 2.19. ■

Corollary 2.26 $\mathsf{Val}^{\mathbf{S4}}_{can} \subseteq \mathsf{Val}_{\mathbf{S4}}$.

Proof. It is an immediate consequence of Lemma 2.24, Lemma 2.25 and Definition 2.19. ■

Theorem 2.27 (Completeness of S4) *Let α be a formula of* **S4**. *Then:*

$$\models_{\mathbf{S4}} \alpha \text{ implies } \vdash_{\mathbf{S4}} \alpha.$$

Proof. Analogous to that of Theorem 2.9, by substituting Proposition 2.3 by Proposition 2.20, Corollary 2.8 by Corollary 2.26 and Definition 2.1 by Definition 2.19, respectively. ■

Remark 2.28 *The adequacy of* **S4** *with respect to Nmatrices with level valuations has important consequences for the study of the complexity of (fragments of) Intuitionistic propositional logic (***IPC***). Indeed, as it is well-known, the so-called Gödel-McKinsey-Tarski translation allows to interpret* **IPC** *inside* **S4**. *Given that the semantics of Nmatrices with level valuations for* **S4** *constitutes a decision procedure for* **S4**, *the study of the computational complexity of this semantics will constitute an upper bound for the computational complexity of* **IPC** *and its fragments. That is, the study of the complexity of the semantics for* **S4** *proposed here can throw some light on the question of complexity of* **IPC** *and related systems.* □

Finally, the modal system **S5** can be defined as the union of **S4** and **B** or, alternatively:

$$\mathbf{S5} = \mathbf{5m} \cup \{(Nec)\}.$$

The proof of soundness and completeness of **S5** with respect to the semantics of Nmatrices with level valuations is as follows:

Definition 2.29 *Let $\mathsf{Val}^{\mathbf{S5}}$ be the set of* $\mathbf{T45m}$*-valuations (see Definition 1.1).*
(a) Let $(\mathsf{Val}_k^{\mathbf{S5}})_{k\in\mathbb{N}}$ be the sequence of subsets of $\mathsf{Val}^{\mathbf{S5}}$ defined as follows: $\mathsf{Val}_0^{\mathbf{S5}} = \mathsf{Val}^{\mathbf{S5}}$ and, for $k \geq 0$,

$$\mathsf{Val}_{k+1}^{\mathbf{S5}} = \{v \in \mathsf{Val}_k^{\mathbf{S5}} \ : \ \textit{for every formula } \alpha,\ \mathsf{Val}_k^{\mathbf{S5}}(\alpha) \subseteq t \textit{ implies } v(\alpha) = t^n\}$$

where $\mathsf{Val}_k^{\mathbf{S5}}(\alpha) = \{v(\alpha) \ : \ v \in \mathsf{Val}_k^{\mathbf{S5}}\}$ for every k and α. Elements of $\mathsf{Val}_k^{\mathbf{S5}}$ are called kth-level valuations *or* valuations of level k for $\mathbf{S5}$.
(b) The set of $\mathbf{S5}$*-valuations is given by $\mathsf{Val}_{\mathbf{S5}} = \bigcap_{k\geq 0} \mathsf{Val}_k^{\mathbf{S5}}$.* □

Definition 2.30 *Let α be a formula of* $\mathbf{S5}$*. Then:*

$$\models_{\mathbf{S5}} \alpha \textit{ iff } v(\alpha) = t^n \textit{ for every } v \in \mathsf{Val}_{\mathbf{S5}}.$$ □

Proposition 2.31 *Let α be a formula of* $\mathbf{S5}$*. Then:*

$$\models_{\mathbf{S5}} \alpha \textit{ iff there exists } k \geq 0 \textit{ such that } \mathsf{Val}_k^{\mathbf{S5}}(\alpha) = \{t^n\}.$$

Proof. Analogous to that of Proposition 2.3. ■

Theorem 2.32 (Soundness of S5) *Let α be a formula of* $\mathbf{S5}$*. Then:*

$$\vdash_{\mathbf{S5}} \alpha \textit{ implies } \models_{\mathbf{S5}} \alpha.$$

Proof. Analogous to that of Theorem 2.4, by substituting Theorem 1.2 and Definition 2.1 by Theorem 1.10 and Definition 2.29, respectively. ■

Definition 2.33 *The set of* $\mathbf{S5}$-canonical valuations *is the set*

$$\mathsf{Val}_{can}^{\mathbf{S5}} = \{\mathbb{V}_{\mathbf{T45m}}^{\Delta} \ : \ \Delta \textit{ is a maximal consistent set in } \mathbf{S5}\},$$

where $\mathbb{V}_{\mathbf{5m}}^{\Delta}$ is given as in Definition 1.3. □

Lemma 2.34 $\mathsf{Val}_{can}^{\mathbf{S5}} \subseteq \mathsf{Val}_0^{\mathbf{S5}}$.

Proof. Analogous to that of Lemma 2.6, by substituting Lemma 1.4 by Lemma 1.18. ■

Lemma 2.35 *If $\mathsf{Val}_{can}^{\mathbf{S5}} \subseteq \mathsf{Val}_k^{\mathbf{S5}}$ then $\mathsf{Val}_{can}^{\mathbf{S5}} \subseteq \mathsf{Val}_{k+1}^{\mathbf{S5}}$.*

Proof. Analogous to that of Lemma 2.7, by substituting Definition 2.1 by Definition 2.29. ■

Corollary 2.36 $\mathsf{Val}^{\mathbf{S5}}_{can} \subseteq \mathsf{Val}_{\mathbf{S5}}$.

Proof. It is an immediate consequence of Lemma 2.34, Lemma 2.35 and Definition 2.29. ■

Theorem 2.37 (Completeness of S5) *Let α be a formula of* **S5**. *Then:*

$$\vDash_{\mathbf{S5}} \alpha \textit{ implies } \vdash_{\mathbf{S5}} \alpha.$$

Proof. Analogous to that of Theorem 2.9, by substituting Proposition 2.3 by Proposition 2.31, Corollary 2.8 by Corollary 2.36 and Definition 2.1 by Definition 2.29, respectively. ■

To summarize, it was shown that the systems **Tm**, **TBm**, **T4m** and **T45m** are adequate w.r.t. a semantics of 4-valued Nmatrices. By adding level valuations to this semantics, it was obtained adequacy for **T**, **B S4** and **S5**, which are obtained from the previous ones by adding the necessitation rule. The relationship between the 8 systems already analyzed can be displayed in Figure 1.

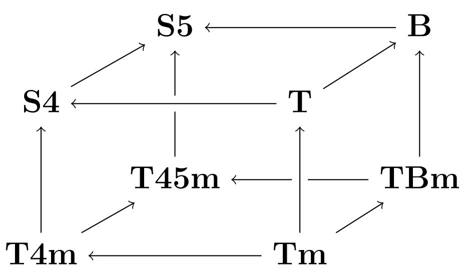

Figure 1: Modal systems adequate for 4-valued Nmatrices

In the next section it will be shown that, by interpreting the modality $\Box$ as a deontic operator, four truth-values are not enough in order to obtain adequacy w.r.t. Nmatrices (with or without level valuations). However, by using suitable six-valued Nmatrices, new semantical characterizations will be obtained.

3 More Ivlev-like systems and Nmatrices

In the previous sections, the modal operators $\Box$ and $\Diamond$ were intuitively interpreted as "it is necessary that" and "it is possible that", by using four truth-values. This is the so-called *alethic* interpretation. However, this is not the unique interpretation for such modalities: as it is well known, that operators admit several other interpretations, depending on the axioms and rules governing them. Of course, by changing the interpretation of the meaning of the modal operators, then the interpretation of the truth-values of the Nmatrices interpreting them should also change. For instance, in an *epistemic* interpretation for the $\Box$ operator in the systems **S4** and **S5** studied in the previous section, $\Box p$ reads as "it is known that p" (see, for instance, [14]), and the four truth-valued should be interpreted accordingly.

Deontic Logics propose a different interpretation for such operators (see, for instance, [24]): $\Box$ is interpreted as *it is obligatory that*, while $\Diamond$ is intuitively interpreted as *it is permitted that*. Under this interpretation, it would not be expected that principles like $\Box\alpha \to \alpha$ and $\alpha \to \Diamond\alpha$ would be valid. However, a weaker principle would be more convenient: $\Box p \to \Diamond p$ could be accepted in this context (the so-called axiom (D)), which would require additional truth-values corresponding to new scenarios with respect to the former interpretation of modalities. Again, another interpretation could be given for modality satisfying axiom (D) instead of (T) (in combination with another ones, such as axioms (4) and (5): a *doxastic* interpretation could be given for $\Box$ and so $\Box p$ could be interpreted as "it is believed that p" (see [14]).

In the deontic interpretation, not only the previous four cases should be considered, but two new situations should be admitted: "p is obligatory and it is false", and its dual "p is forbidden and it is true". This lead us naturally to a six truth-values semantics. This is the task of this section and the following one, in which the results of the two previous sections (by using four-valued Nmatrices) will be obtained for systems in which axiom (T) is replaced by (D), constituting a novel semantical characterization (as well as a novel decision procedure) for such systems by means of six-valued Nmatrices.

Let us begin with variant of system **Tm** in which axiom (T) is replaced by (D):

(D) $\Box p \to \Diamond p$

The (weakly)deontic system characterized by six-valued Nmatrices obtained by a suitable modification of Ivlev's **Tm** will be called **Dm**. Concerning its axiomatization, the following Hilbert calculus will be considered:

Definition 3.1

$$\mathbf{Dm} = \mathbf{PC} \cup \{(K), (K1), (K2), (M1), (M2), (M3), (M4), (D), (DN)\}$$

Consider now the following terminology:[9] a proposition is said to be *fulfilled* whenever either it is obligatory and it is the case, or it is forbidden and it is not the case (that is, it is the case of its negation). A proposition is said to be *infringed* whenever either it is obligatory and it is not the case, or it is forbidden and it is the case. Finally, a proposition is said to be *optional* if it is neither obligatory nor forbidden. It generates six truth-values as follows: T^+ (p is fulfilled), C^+ (p is optional), F^+ (p is infringed), T^- ($\neg p$ is infringed), C^- ($\neg p$ is optional) and F^- ($\neg p$ is fulfilled). Consider the following abbreviations:

$$T = \{T^+, T^-\} \qquad C = \{C^+, C^-\} \qquad F = \{F^+, F^-\}$$
$$+ = \{T^+, C^+, F^+\} \qquad - = \{T^-, C^-, F^-\}$$

The sets T, C and F are interpreted as *obligatory*, *optional* and *forbidden*, respectively, while $+$ and $-$ are, by definition, the set of designated and of non-designated truth-values.

The semantics proposed for the operators $\to, \neg$ and $\Box$ is as follows:

$\to$	T	C	F
T	T	C	F
C	T	$T \cup C$	C
F	T	T	T

α	$\Box\alpha$
T	$+$
C	$-$
F	$-$

α	$\neg\alpha$
T	F
C	C
F	T

$\to$	$+$	$-$
$+$	$+$	$-$
$-$	$+$	$+$

α	$\neg\alpha$
$+$	$-$
$-$	$+$

Any valuation over the multivalued functions above (recall Definition 1.1) is called a **Dm**-valuation. Observe that there are two multivalued functions for $\to$ and $\neg$. Putting it all together, it is equivalent to consider the following standard (that is, deterministic) truth-table for $\neg$, and the following multivalued function for $\to$:

[9] It should be observed once again that the deontic interpretation for the modalities and for the truth-values given in this section and the following one is merely suggested.

$\mathcal{M}^6_\neg$

α	$\neg\alpha$
T^+	F^-
T^-	F^+
C^+	C^-
C^-	C^+
F^+	T^-
F^-	T^+

$\mathcal{N}^6_\rightarrow$

$\rightarrow$	T^+	T^-	C^+	C^-	F^+	F^-
T^+	T^+	T^-	C^+	C^-	F^+	F^-
T^-	T^+	T^+	C^+	C^+	F^+	F^+
C^+	T^+	T^-	$\{T^+, C^+\}$	$\{T^-, C^-\}$	C^+	C^-
C^-	T^+	T^+	$\{T^+, C^+\}$	$\{T^+, C^+\}$	C^+	C^+
F^+	T^+	T^-	T^+	T^-	T^+	T^-
F^-	T^+	T^+	T^+	T^+	T^+	T^+

By defining $\Diamond$ as usual, the following multivalued functions are obtained for the modal operators:

$\mathcal{N}^{\mathbf{D}}_\Box$

α	$\Box\alpha$
T^+	$\{T^+, C^+, F^+\}$
T^-	$\{T^+, C^+, F^+\}$
C^+	$\{T^-, C^-, F^-\}$
C^-	$\{T^-, C^-, F^-\}$
F^+	$\{T^-, C^-, F^-\}$
F^-	$\{T^-, C^-, F^-\}$

$\mathcal{N}^{\mathbf{D}}_\Diamond$

α	$\Diamond\alpha$
T^+	$\{T^+, C^+, F^+\}$
T^-	$\{T^+, C^+, F^+\}$
C^+	$\{T^+, C^+, F^+\}$
C^-	$\{T^+, C^+, F^+\}$
F^+	$\{T^-, C^-, F^-\}$
F^-	$\{T^-, C^-, F^-\}$

Observe that the multivalued functions for $\Diamond$ and $\vee$ can be written in a more compact way:

α	$\Diamond\alpha$
T	$+$
C	$+$
F	$-$

$\vee$	T	C	F
T	T	T	T
C	T	$T \cup C$	C
F	T	C	F

$\vee$	$+$	$-$
$+$	$+$	$+$
$-$	$+$	$-$

Theorem 3.2 (Soundness of Dm) *Let α be a formula of* **Dm**. *Then:*

$$\vdash_{\mathbf{Dm}} \alpha \textit{ implies } \vDash_{\mathbf{Dm}} \alpha$$

Proof. The proof is analogous to the classical case concerning the axioms and rules of **PC**. The only case to be analyzed is axiom (D). Thus, let v be a valuation and p a propositional variable. If $v(\Box p) \in -$ then $v(D) \in +$ and if $v(\Box p) \in +$ then $v(p) \in T$. Therefore $v(D) \in +$. ■

The completeness of **Dm** will be shown now by defining a canonical valuation by means of a maximal constant set of formulas.

Observe that a maximal consistent set Δ in **Dm** has the following properties: $\neg\alpha \in \Delta$ iff $\alpha \notin \Delta$, and $\alpha \rightarrow \beta$ iff either $\alpha \notin \Delta$ or $\beta \in \Delta$.

Definition 3.3 *Let* Δ *be a maximal consistent set in* **Dm**. *The canonical valuation associated to* Δ *is the function*

$$\mathbb{V}^{\Delta}_{\mathbf{Dm}} : For_{\mathbf{Dm}} \longrightarrow \{T^+, C^+, F^+, T^-, C^-, F^-\}$$

defined as follows:

1. $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\alpha) \in +$ *iff* $\alpha \in \Delta$
2. $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\alpha) \in -$ *iff* $\neg\alpha \in \Delta$
3. $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\alpha) \in T$ *iff* $\Box\alpha \in \Delta$
4. $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\alpha) \in C$ *iff* $\Diamond\alpha \in \Delta$ *and* $\Diamond\neg\alpha \in \Delta$
5. $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\alpha) \in F$ *iff* $\Box\neg\alpha \in \Delta$.

$\Box$

Lemma 3.4 $\mathbb{V}^{\Delta}_{\mathbf{Dm}}$ *is a* **Dm**-*valuation.*

Proof.
CASE 1: α is $\neg\beta$.
(i) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta) \in T$ then $\Box\beta \in \Delta$. By (DN) it follows that $\Box\neg\neg\beta \in \Delta$ and so $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\neg\beta) \in F$.
(ii) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta) \in C$ then $\Diamond\beta, \Diamond\neg\beta \in \Delta$. By (DN), $\Diamond\neg\neg\beta \in \Delta$ and so $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\neg\beta) \in C$.
(iii) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta) \in F$ then $\Box\neg\beta \in \Delta$ whence $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\neg\beta) \in T$.
(iv) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta) \in +$ then $\beta \in \Delta$. By (DN) it follows that $\neg\neg\beta \in \Delta$ and so $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\neg\beta) \in -$.
(v) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta) \in -$ then $\neg\beta \in \Delta$, therefore $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\neg\beta) \in +$.
CASE 2: α is $\Box\beta$.
(i) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta) \in T$ then $\Box\beta \in \Delta$ and $\mathbb{V}_{\Delta}(\Box\beta) \in +$.
(ii) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta) \in C$ then $\Diamond\neg\beta \in \Delta$. From this, by definition of $\Diamond$ and (DN) it follows that $\neg\Box\beta \in \Delta$ and $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\Box\beta) \in -$.
(iii) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta) \in F$ then $\mathbb{V}_{\Delta}(\beta) \notin T$ e $\Box\beta \notin \Delta$. Thus $\neg\Box\beta \in \Delta$ and $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\Box\beta) \in -$.
CASE 3: α is $\beta \to \gamma$.
(i) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\gamma) \in T$ then $\Box\gamma \in \Delta$. By (M2) it follows that $\Box\gamma \to \Box(\beta \to \gamma) \in \Delta$ and, by (MP), $\Box(\beta \to \gamma) \in \Delta$. Therefore $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta \to \gamma) \in T$.
(ii) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta) \in F$ then $\neg\Diamond\beta \in \Delta$. By (M1) and (MP), $\Box(\beta \to \gamma) \in \Delta$.

Hence $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta \to \gamma) \in T$.
(iii) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta) \in T$ and $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\gamma) \in C$ then $\Box\beta \in \Delta$ and $\Diamond\gamma, \Diamond\neg\gamma \in \Delta$. Since $\Diamond\gamma \in \Delta$ then $\Diamond(\beta \to \gamma) \in \Delta$, by (M3). On the other hand, since $\neg\Diamond\gamma \in \Delta$ by (D) it follows that $\neg\Box\gamma \in \Delta$. From this $\Box\beta \wedge \neg\Box\gamma \in \Delta$ and so $\neg(\Box\beta \to \Box\gamma) \in \Delta$. Thus $\neg\Box(\beta \to \gamma) \in \Delta$, by (K). Hence $\Diamond\neg(\beta \to \gamma) \in \Delta$ and so $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta \to \gamma) \in C$.
(iv) $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta) \in T$ and $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\gamma) \in F$ then $\Box\beta, \Box\neg\gamma \in \Delta$ and so $\Box\beta \wedge \Box\neg\gamma \in \Delta$. By (K2), $\neg(\Diamond\neg\beta \vee \Diamond\neg\neg\gamma) \to \neg\Diamond(\neg\beta \vee \neg\neg\gamma) \in \Delta$ and so $(\Box\beta \wedge \Box\neg\gamma) \to \Box(\beta \wedge \neg\gamma) \in \Delta$. Thus, $\Box(\beta \wedge \neg\gamma) \in \Delta$ by (MP) and so $\Box\neg(\beta \to \gamma) \in \Delta$. That is, $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta \to \gamma) \in F$.
(v) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta) \in C$ and $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\gamma) \in C$ and so $\Diamond\beta, \Diamond\neg\beta, \Diamond\gamma, \Diamond\neg\gamma \in \Delta$. If $\Box(\beta \to \gamma) \in \Delta$ then $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta \to \gamma) \in T$. Otherwise, $\Diamond\neg(\beta \to \gamma) \in \Delta$ since Δ is a maximal consistent set. Additionally, $\Diamond\gamma \in \Delta$ and so $\Diamond(\beta \to \gamma) \in \Delta$, by $(M3)$. This shows that $\Diamond(\beta \to \gamma), \Diamond\neg(\beta \to \gamma) \in \Delta$ and then $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta \to \gamma) \in C$.
(vi) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta) \in C$ and $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\gamma) \in F$ then $\Diamond\beta, \Diamond\neg\beta \in \Delta$ and $\Box\neg\gamma \in \Delta$. Since $\Diamond\neg\beta \in \Delta$ then $\Diamond(\beta \to \gamma) \in \Delta$, by $(M4)$. On the other hand, since $\Diamond\beta \in \Delta$ and $\neg\Diamond\gamma \in \Delta$ then $\Diamond\beta \wedge \neg\Diamond\gamma \in \Delta$ and $\neg(\Diamond\beta \to \Diamond\gamma) \in \Delta$. From this $\neg\Box(\beta \to \gamma) \in \Delta$, by (K1). Therefore $\Diamond\neg(\beta \to \gamma) \in \Delta$. This shows that $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta \to \gamma) \in C$.
(vii) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\gamma) \in +$ then $\gamma \in \Delta$ and $(\beta \to \gamma) \in \Delta$. Therefore $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta \to \gamma) \in +$.
(viii) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta) \in -$ then $\neg\beta \in \Delta$ and $(\beta \to \gamma) \in \Delta$. Therefore $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta \to \gamma) \in +$.
(ix) If $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta) \in +$ and $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\gamma) \in -$ then $\beta \in \Delta$ e $\neg\gamma \in \Delta$. Thus, $\beta \wedge \neg\gamma \in \Delta$ whence $\neg(\beta \to \gamma) \in \Delta$. Therefore $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\beta \to \gamma) \in -$. ■

Lemma 3.5 *Let Δ be a maximal consistent set in* **Dm**. *Then, there exists a* **Dm**-*valuation $\mathbb{V}$ such that, for every formula α:*

$$\mathbb{V}(\alpha) \in + \textit{ iff } \alpha \in \Delta.$$

Proof. By Definition 3.3, $\mathbb{V}^{\Delta}_{\mathbf{Dm}}$ is a function such that $\mathbb{V}^{\Delta}_{\mathbf{Dm}}(\alpha) \in +$ iff $\alpha \in \Delta$. By Lemma 3.4, $\mathbb{V}^{\Delta}_{\mathbf{Dm}}$ is a **Dm**-valuation. ■

Theorem 3.6 (Completeness of Dm) *Let α be a formula of* **Dm**. *Then:*

$$\vDash_{\mathbf{Dm}} \alpha \quad \textit{implies} \quad \vdash_{\mathbf{Dm}} \alpha.$$

Proof. Suppose that $\nvDash_{\mathbf{Dm}} \alpha$. Then there exists a maximal consistent set Δ in **Dm** such that $\alpha \notin \Delta$. By Lemma 3.5, there exists a **Dm**-valuation $\mathbb{V}$ such that $\mathbb{V}(\alpha) \notin +$. Therefore $\nvDash_{\mathbf{Dm}} \alpha$ ■

The following systems can be naturally defined, by using axioms (B), (4) and (5):

Definition 3.7

$$\mathbf{DBm} = \mathbf{Dm} \cup \{(B)\}$$

$$\mathbf{D4m} = \mathbf{Dm} \cup \{(4)\}$$

$$\mathbf{D45m} = \mathbf{D4m} \cup \{(5)\}$$

It is easy to prove that **D45m** = **DBm**∪**D4m**. The (multivalued)functions proposed for the modality $\Box$ corresponding to each axiom of Definition 3.7 are displayed below.

$\mathcal{N}_{\Box}^{\mathbf{DB}}$

α	$\Box\alpha$
T^+	$+$
T^-	F^+
C^+	$-$
C^-	F^-
F^+	$-$
F^-	F^-

$\mathcal{N}_{\Box}^{\mathbf{D4}}$

α	$\Box\alpha$
T^+	T^+
T^-	T^+
C^+	$-$
C^-	$-$
F^+	$-$
F^-	$-$

$\mathcal{N}_{\Box}^{\mathbf{D5}}$

α	$\Box\alpha$
T^+	T^+
T^-	T^+
C^+	F^-
C^-	F^-
F^+	F^-
F^-	F^-

DBm-valuations are defined according to Definition 1.1 by using the corresponding multivalued functions. **D4m**-valuations and **D45m**-valuations are defined analogously.

In what follows, soundness and completes of these systems w.r.t. the corresponding Nmatrices will be obtained.

Theorem 3.8 (Soundness of DBm) *Let α be a formula of* **DBm**. *Then:*

$$\vdash_{\mathbf{DBm}} \alpha \textit{ implies } \vDash_{\mathbf{DBm}} \alpha$$

Proof. With respect to the axioms and rules of **Dm** the proof is analogous to the one of Theorem 3.2. Concerning axiom (B), let p be a propositional variable, and let v be a valuation. If $v(p) \in +$ then $v(\neg p) \in -$. Hence $v(\Box\neg p) = F$ and $v(\Diamond p) = T$. Therefore $v(\Box\Diamond p) \in +$ and $v(B) \in +$. ■

Theorem 3.9 (Soundness of D4m) *Let α be a formula of* **D4m**. *Then:*

$$\vdash_{\mathbf{D4m}} \alpha \textit{ implies } \vDash_{\mathbf{D4m}} \alpha$$

Proof. Concerning the axioms and rules of **Dm**, the proof is analogous to the one of Theorem 3.2. With respect to axiom (4), let p be a propositional variable, and let v be a valuation. If $v(p) \in +$ then $v(\Box p) = T^+$ and $v(\Box\Box p) = T^+$. Therefore $v(4) \in +$. ■

Theorem 3.10 (Soundness of D45m) *Let α be a formula of* **D45m**. *Then:*

$$\vdash_{\mathbf{D45m}} \alpha \textit{ implies } \vDash_{\mathbf{D45m}} \alpha$$

Proof. With respect to the axioms and rules of **Dm** the proof is analogous to the one of Theorem 3.2, by substituting $v(\alpha) \in +$ by $v(\alpha) = T^+$ and $v(\alpha) \in -$ by $v(\alpha) = F^-$ if α has the form $\Box\beta$ or $\Diamond\beta$. Concerning axiom (4), the proof is analogous to that of Theorem 3.9. With respect to axiom (5), if $v(\Diamond p) \in +$ then $v(\Box\neg p) \in -$ and $v(\neg p) \notin T$. Then $v(p) \in T$, which implies that $v(\Diamond p) = T^+$ and $v(\Box\Diamond p) = T^+$. Therefore $v(5) \in +$. ■

Definition 3.11 *Let Δ be a maximal consistent set in* **DLm**, *for* $\mathbf{DLm} \in \{\mathbf{DBm}, \mathbf{D4m}, \mathbf{D45m}\}$. *The canonical valuation associated to Δ is a function $\mathbb{V}^{\Delta}_{\mathbf{DLm}} : For_{\mathbf{DLm}} \longrightarrow \{T^+, C^+, F^+, T^-, C^-, F^-\}$ defined as follows:*

1. $\mathbb{V}^{\Delta}_{\mathbf{DLm}}(\alpha) \in +$ *iff* $\alpha \in \Delta$
2. $\mathbb{V}^{\Delta}_{\mathbf{DLm}}(\alpha) \in -$ *iff* $\neg\alpha \in \Delta$
3. $\mathbb{V}^{\Delta}_{\mathbf{DLm}}(\alpha) \in T$ *iff* $\Box\alpha \in \Delta$
4. $\mathbb{V}^{\Delta}_{\mathbf{DLm}}(\alpha) \in C$ *iff* $\Diamond\alpha \in \Delta$ *and* $\Diamond\neg\alpha \in \Delta$
5. $\mathbb{V}^{\Delta}_{\mathbf{DLm}}(\alpha) \in F$ *iff* $\Box\neg\alpha \in \Delta$

□

Lemma 3.12 $\mathbb{V}^{\Delta}_{\mathbf{DLm}}$ *is a* **DLm***-valuation.*

Proof. Analogous to that of Lemma 3.4. ■

Lemma 3.13 *Let Δ be as in Definition 3.11. Then, there exists a* **DLm***-valuation $\mathbb{V}$ such that, for every formula α:*

$$\mathbb{V}(\alpha) \in + \textit{ iff } \alpha \in \Delta.$$

Proof. By Definition 3.11, $\mathbb{V}^{\Delta}_{\mathbf{DLm}}$ is a function such that $\mathbb{V}^{\Delta}_{\mathbf{DLm}}(\alpha) \in +$ iff $\alpha \in \Delta$. By Lemma 3.12, $\mathbb{V}^{\Delta}_{\mathbf{DLm}}$ is a **DLm**-valuation. ∎

Lemma 3.14 $\mathbb{V}^{\Delta}_{\mathbf{DBm}}$ *is a* **DBm**-*valuation.*

Proof. The proof for **CASE 1** and **CASE 3** is identical to that of Lemma 3.4. With respect to **CASE 2**, the same observation holds when $\mathbb{V}_{\Delta}(\beta) \in +$. It remains to analyze three items.

CASE 2: α is $\Box\beta$.

(i) If $\mathcal{V}^{\Delta}_{\mathbf{DBm}}(\beta) = T^-$ then $\Box\beta, \neg\beta \in \Delta$. By (B) it holds that $\Box\Diamond\neg\beta \in \Delta$ and so $\Box\neg\Box\beta \in \Delta$. Therefore $\mathcal{V}^{\Delta}_{\mathbf{DBm}}(\Box\beta) = F^+$.

(ii) If $\mathcal{V}^{\Delta}_{\mathbf{DBm}}(\beta) = C^-$ then $\neg\beta, \Diamond\beta, \Diamond\neg\beta \in \Delta$. On the one hand, $\neg\Box\beta \in \Delta$. On the other hand, $\Box\Diamond\neg\beta \in \Delta$, by (B). From this, $\Box\neg\Box\beta \in \Delta$ and so $\mathcal{V}^{\Delta}_{\mathbf{DBm}}(\Box\beta) = F^+$.

(iii) If $\mathcal{V}^{\Delta}_{\mathbf{DBm}}(\beta) = F^-$ then $\Box\neg\beta, \neg\beta \in \Delta$. Hence $\Diamond\neg\beta \in \Delta$, by (D), and so $\neg\Box\beta \in \Delta$. Besides this, by (B) it can be concluded that $\Box\Diamond\neg\beta \in \Delta$ and $\Box\neg\Box\beta \in \Delta$. Therefore $\mathcal{V}^{\Delta}_{\mathbf{DBm}}(\Box\beta) = F^-$. ∎

Lemma 3.15 *Let* Δ *be a maximal consistent set in* **DBm**. *Then, there exists a* **DBm**-*valuation* $\mathbb{V}$ *such that, for every formula* α*:*

$$\mathbb{V}(\alpha) \in + \textit{ iff } \alpha \in \Delta.$$

Proof. Analogous to that for Lemma 3.5, by substituting Lemma 3.4 by Lemma 3.14. ∎

Theorem 3.16 (Completeness of DBm) *Let* α *be a formula of* **DBm**. *Then:*

$$\vDash_{\mathbf{DBm}} \alpha \quad \textit{implies} \quad \vdash_{\mathbf{DBm}} \alpha.$$

Proof. Suppose that $\nvdash_{\mathbf{DBm}} \alpha$. Then, there exists a maximal consistent set Δ in **DBm** such that $\alpha \notin \Delta$. By Lemma 3.15, there exists a **DBm**-valuation $\mathbb{V}$ such that $\mathbb{V}(\alpha) \notin +$. Therefore $\nvDash_{\mathbf{DBm}} \alpha$ ∎

Now, the completeness of **D4m** and **D45m** w.r.t. Nmatrices will be proved.

Lemma 3.17 $\mathbb{V}^{\Delta}_{\mathbf{D4m}}$ *is a* **D4m**-*valuation.*

Proof. The proof for **CASE 1** and **CASE 3** is identical to that for Lemma 3.4. Concerning **CASE 2**, the same holds when $\mathbb{V}^{\Delta}_{\mathbf{D4m}}(\beta) \in C$ or $\mathbb{V}^{\Delta}_{\mathbf{D4m}}\Delta(\beta) \in F$. It remains to analyze the case when $\mathbb{V}^{\Delta}_{\mathbf{D4m}}(\beta) \in T$.

CASE 2: α is $\Box\beta$.

If $\mathbb{V}^{\Delta}_{\mathbf{D4m}}(\beta) \in T$ then $\Box\beta \in \Delta$. By (4), $\Box\Box\beta \in \Delta$ whence $\mathbb{V}^{\Delta}_{\mathbf{D4m}}(\Box\beta) \in T$. ■

Lemma 3.18 *Let Δ be a maximal consistent set in* **D4m**. *Then, there exists a* **D4m***-valuation $\mathbb{V}$ such that, for every formula α:*

$$\mathbb{V}(\alpha) \in + \quad \textit{iff} \quad \alpha \in \Delta.$$

Proof. Analogous to that for Lemma 3.5, by substituting Lemma 3.4 by Lemma 3.17. ■

Theorem 3.19 (Compleness of D4m) *Let α be a formula of* **D4m**. *Then:*

$$\vDash_{\mathbf{D4m}} \alpha \quad \textit{implies} \quad \vdash_{\mathbf{D4m}} \alpha.$$

Proof. Suppose that $\nvdash_{\mathbf{D4m}} \alpha$. Then, there exists a maximal consistent set Δ in **D4m** such that $\alpha \notin \Delta$. By Lemma 3.18, there exists a **D4m**-valuation $\mathbb{V}$ such that $\mathbb{V}(\alpha) \notin +$. Therefore $\nvDash_{\mathbf{D4m}} \alpha$ ■

Lemma 3.20 $\mathbb{V}^{\Delta}_{\mathbf{D45m}}$ *is a* **D45m***-valuation.*

Proof. The proof for **CASE 1** and **CASE 3** is identical to that for Lemma 3.4. With respect to **CASE 2**, the proof is identical to that of Lemma 3.17 when $V_{\Delta}(\beta) \in T$. It remains to analyze two items.

CASE 2: α is $\Box\beta$.

(i) If $\mathbb{V}^{\Delta}_{\mathbf{D45m}}(\beta) \in C$ then $\Diamond\beta, \Diamond\neg\beta \in \Delta$. Since $\Diamond\neg\beta \in \Delta$ then $\neg\Box\beta \in \Delta$, and $\Box\Diamond\neg\beta \in \Delta$, by (5). Hence $\Box\neg\Box\beta \in \Delta$ and so $\mathbb{V}^{\Delta}_{\mathbf{D45m}}(\Box\beta) = F^{-}$.

(ii) If $\mathbb{V}_{\Delta}(\beta) \in F$ then $\neg\Diamond\beta \in \Delta$ and, by (D), $\neg\Box\beta \in \Delta$. Since $\neg\Diamond\beta \in \Delta$ it follows that $\Box\neg\beta \in \Delta$ and, again by (D), $\Diamond\neg\beta \in \Delta$. Thus $\Box\Diamond\neg\beta \in \Delta$, by (5). Therefore $\Box\neg\Box\beta \in \Delta$ and $\mathbb{V}^{\Delta}_{\mathbf{D45m}}(\Box\beta) = F^{-}$. ■

Lemma 3.21 *Let Δ be a maximal consistent set in* **D45m**. *Then, there exists a* **D45m***-valuation $\mathbb{V}$ such that, for every formula α:*

$$\mathbb{V}(\alpha) \in + \quad \textit{iff} \quad \alpha \in \Delta.$$

Proof. Analogous to that for Lemma 3.5, by substituting Lemma 3.4 by Lemma 3.20. ■

Theorem 3.22 (Completeness of D45m) *Let α be a formula of* **D45m**. *Then:*

$$\models_{\mathbf{D45m}} \alpha \quad \textit{implies} \quad \vdash_{\mathbf{D45m}} \alpha.$$

Proof. Suppose that $\nvdash_{\mathbf{D45m}} \alpha$. Then, there exists a maximal consistent set Δ in **D45m** such that $\alpha \notin \Delta$. By Lemma 3.21, there exists a **D45m**-valuation $\mathbb{V}$ such that $\mathbb{V}(\alpha) \notin +$. Therefore $\nvDash_{\mathbf{D45m}} \alpha$. ∎

4 Kearns' level valuations for other normal modal logics

As it was done in Section 2, if the necessitation rule (Nec) is added to the Ivlev-like modal systems studied in the previous section, well-known normal modal systems extending the so-called *Standard Deontic Logic* (**SDL**) are obtained (see [14]). All these systems will be semantically characterized by the corresponding Nmatrices restricted by level valuations. As observed in the previous section, a modal operator $\Box$ satisfying these properties could also be interpreted as an epistemic or doxastic operator logic, in the framework of the modal logic of knowledge and belief (see for instance [14]).

Definition 4.1

$\mathbf{KD} = \mathbf{Dm} \cup \{(Nec)\}$ *(also known as* **SDL***)*

$\mathbf{KDB} = \mathbf{KD} \cup \{(B)\}$

$\mathbf{KD4} = \mathbf{KD} \cup \{(4)\}$

$\mathbf{KD45} = \mathbf{KD4} \cup \{(5)\}$

It should be observed that the equivalence of the Hilbert calculi introduced in Definition 4.1 and the usual calculi for these logics can be easily stated, as it was done with the definition of the calculus for **T** from **Tm**.

Definition 4.2 *Let* $\mathbf{DLm} \in \{\mathbf{Dm}, \mathbf{DBm}, \mathbf{D4m}, \mathbf{D45m}\}$*, and consider* $\mathbf{DL} \in \{\mathbf{KD}, \mathbf{KDB}, \mathbf{KD4}, \mathbf{KD45}\}$. *Let* $\mathsf{Val}^{\mathbf{DL}}$ *be the set of* **DLm***-valuations (recall Definition 1.1).*

(a) Let $(\mathsf{Val}_k^{\mathbf{DL}})_{k\in\mathbb{N}}$ *be the sequence of subsets of* $\mathsf{Val}^{\mathbf{DL}}$ *defined as follows:* $\mathsf{Val}_0^{\mathbf{DL}} = \mathsf{Val}^{\mathbf{DL}}$ *and, for* $k \geq 0$,

$$\mathsf{Val}_{k+1}^{\mathbf{DL}} = \{v \in \mathsf{Val}_k^{\mathbf{DL}} \ : \ \textit{for every } \alpha,\ \mathsf{Val}_k^{\mathbf{DL}}(\alpha) \subseteq + \textit{ implies } v(\alpha) = T^+\}$$

where $\mathit{Val}_k^{\mathbf{DL}}(\alpha) = \{v(\alpha) \ : \ v \in \mathit{Val}_k^{\mathbf{DL}}\}$, *for every* k *and* α. *Elements in* $\mathit{Val}_k^{\mathbf{DL}}$ *are called* kth-level valuations *or* valuations of level k for $\mathbf{DL}$.

(b) The set of $\mathbf{DL}$*-valuations is given by* $\mathit{Val}_{\mathbf{DL}} = \bigcap_{k\geq 0} \mathit{Val}_k^{\mathbf{DL}}$. $\square$

Definition 4.3 *Let* $\mathbf{DL} \in \{\mathbf{KD}, \mathbf{KDB}, \mathbf{KD4}, \mathbf{KD45}\}$ *and let* α *be a formula in* $\mathbf{DL}$. *Then:*

$$\vDash_{\mathbf{DL}} \alpha \ \text{ iff } \ v(\alpha) = T^+ \text{ for every } v \in \mathit{Val}_{\mathbf{DL}}. \qquad \square$$

Proposition 4.4 *Let* $\mathbf{DL} \in \{\mathbf{KD}, \mathbf{KDB}, \mathbf{KD4}, \mathbf{KD45}\}$ *and let* α *be a formula in* $\mathbf{DL}$. *Then:*

$$\vDash_{\mathbf{DL}} \alpha \ \text{ iff there exists } k \geq 0 \text{ such that } \mathit{Val}_k^{\mathbf{DL}}(\alpha) = \{T^+\}.$$

Proof. Analogous to that of Proposition 2.3 ■

Theorem 4.5 (Soundness of DL) *Let* $\mathbf{DL} \in \{\mathbf{KD}, \mathbf{KDB}, \mathbf{KD4}, \mathbf{KD45}\}$ *and let* α *be a formula in* $\mathbf{DL}$. *Then:*

$$\vdash_{\mathbf{DL}} \alpha \text{ implies } \vDash_{\mathbf{DL}} \alpha.$$

Proof. The proof concerning the axioms and rules of **DLm** is analogous to that for theorems 3.2, 3.8, 3.9 and 3.10. With respect to the (Nec) rule, the argument is similar to that for Theorem 2.4. The only detail to be observed is that, if $\mathsf{Val}_k^{\mathbf{DL}}(\beta) = \{T^+\}$ then $\mathsf{Val}_k^{\mathbf{DL}}(\Box\beta) \subseteq +$ and so $\mathsf{Val}_{k+1}^{\mathbf{DL}}(\Box\beta) = \{T^+\}$. ■

Definition 4.6 *Let* $\mathbf{DL}$ *as above. The set of* $\mathbf{DL}$-canonical valuations *is the set*

$$\mathit{Val}_{can}^{\mathbf{DL}} = \{\mathbb{V}_{\mathbf{DL}}^{\Delta} \ : \ \Delta \ \text{ is a maximal consistent set in } \mathbf{DL}\},$$

where each $\mathbb{V}_{\mathbf{DL}}^{\Delta}$ *is given according to Definition 3.11.* $\square$

Lemma 4.7 $\mathit{Val}_{can}^{\mathbf{DL}} \subseteq \mathit{Val}_0^{\mathbf{DL}}$.

Proof. It is an immediate consequence of Definition 4.6 and Definition 3.11, by adapting the proof of lemmas 3.4, 3.14, 3.17 and 3.20, taking into account that each axiom of **DLm** holds in **DL**. ■

Lemma 4.8 *If* $\mathit{Val}_{can}^{\mathbf{DL}} \subseteq \mathit{Val}_k^{\mathbf{DL}}$ *then* $\mathit{Val}_{can}^{\mathbf{DL}} \subseteq \mathit{Val}_{k+1}^{\mathbf{DL}}$.

Proof. Analogous to that for Lemma 2.7. The only modification is the following: given $\mathbb{V}_{\mathbf{DL}}^{\Delta} \in \mathsf{Val}_{can}^{\mathbf{DL}}$, if α is a formula such that $\mathsf{Val}_{k}^{\mathbf{DL}}(\alpha) \subseteq +$ then $\mathbb{V}_{\Delta}(\alpha) \in +$. Since $\Box\alpha \in \Delta$ (the proof is analogous to that of Lemma 2.7) then $\mathbb{V}_{\Delta}(\alpha) \in T$, by Definition 3.11. From this, $\mathbb{V}_{\Delta}(\alpha) = T^{+}$ and so $\mathbb{V}_{\Delta} \in \mathsf{Val}_{k+1}^{\mathbf{DL}}$, by Definition 4.2. ■

Corollary 4.9 $\mathit{Val}_{can}^{\mathbf{DL}} \subseteq \mathit{Val}_{\mathbf{DL}}$.

Proof. It is an immediate consequence of Lemma 4.7 and Lemma 4.8. ■

Theorem 4.10 (Completeness of DL) *Let* $\mathbf{DL} \in \{\mathbf{D}, \mathbf{KDB}, \mathbf{KD4}, \mathbf{KD45}\}$ *and let* α *be a formula of* $\mathbf{DL}$. *Then:*

$$\vDash_{\mathbf{DL}} \alpha \textit{ implies } \vdash_{\mathbf{DL}} \alpha.$$

Proof. Suppose that $\nvdash_{\mathbf{DL}} \alpha$. Then, there exists a maximal consistent set Δ in $\mathbf{DL}$ such that $\alpha \notin \Delta$, and so $\neg\alpha \in \Delta$. Suppose that $\vDash_{\mathbf{DL}} \alpha$. Then, there exists $k \geq 0$ such that $\mathsf{Val}_{k}^{\mathbf{DL}}(\alpha) = \{T^{+}\}$, by Proposition 4.4. Given that $\mathsf{Val}_{can}^{\mathbf{DL}} \subseteq \mathsf{Val}_{k}^{\mathbf{DL}}$, by Corollary 4.9, it follows that $\mathbb{V}_{\Delta}(\alpha) = T^{+}$. But $\neg\alpha \in \Delta$ and so $\mathbb{V}_{\Delta}(\alpha) \in -$, a contradiction. Therefore $\nvDash_{\mathbf{DL}} \alpha$. ■

To summarize, it was shown in the previous section that the systems **Dm**, **DBm**, **D4m** and **D45m** are characterized by 6-valued Nmatrices. By taking into account the level valuations, a semantical characterization (as well as a novel decision procedure) for **D**, **KDB KD4** and **KD45** is obtained. The relationship between all these systems can be visualized in figure 2.

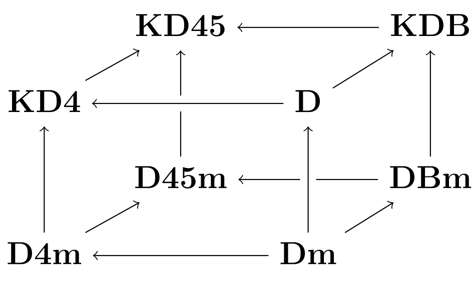

Figure 2: Modal systems adequate for 6-valued Nmatrices

5 Concluding remarks and future work

This paper intends to contribute to the development of alternatives to Kripke semantics for modal logics, with interesting properties such as decidability. With this aim, the pioneering work of J. Kearns and J. Ivlev on a the semantical characterization of some modal systems by means of what came to be known as nondeterministic matrices was revisited and expanded to several systems. In particular, an elegant decision procedure for several well-known modal systems is presented: **T**, **B**, **S4**, **S5**, and the systems obtained from these by replacing axiom (T) by axiom (D), namely **D**, **KDB**, **KD4** and **KD45**. The first group of systems are characterized by four-valued Nmatrices with level valuations (that is, by restricting the valuations over the Nmatrices in a suitable way). The second group of systems, being weaker, need more truth-values. As it was proved in the second part of this paper, six truth-values are enough.

The semantics of Nmatrices with level valuations can be considered as a genuine contribution to the development of alternatives to Kripke semantics for modal logics. Moreover, this approach can seen as a kind of answer to Dugundji's "objection" against finite matrices for modal logics. Indeed, the results here mentioned show that modal logics are perfectly treatable by means of finite non-deterministic matrices, a conception of modal semantics completely different from Kripke-style semantics, and closer to finite valued logics.

The extension of this technique to normal, non-serial modal systems such as **K** and some of its extensions require eight-valued Nmatrices. This method can also be extended to some non-normal modal systems, by modifying the level valuations rule. All of this will be treated in a future paper.

From the results here obtained, it is clear that the semantical counterpart of the necessitation rule is the refinement of the valuations considered over the given Nmatrices, expressed by level valuations.

The fact that Nmatrices with level valuations constitute a decision procedure for **S4** opens interesting possibilities for the study of the complexity of Intuitionistic propositional logic (**IPC**) and its fragments, taking into account the Gödel-McKinsey-Tarski translation between **IPC** and **S4**. Moreover, it is possible to define four-valued Nmatrices with level valuations characterizing **IPC** in a direct way, by suitable modification of the corresponding one for **S4**. This will also shown in a future paper.

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References

[1] A. Avron. Non-deterministic matrices and modular semantics of rules. *Logica Universalis*, pages 155–173, 2007.

[2] A. Avron and I. Lev. Canonical propositional Gentzen-type systems. In *Proceedings of the First International Joint Conference on Automated Reasoning (IJCAR '01)*, pages 529–544, London, 2001. Springer-Verlag.

[3] A. Avron and A. Zamansky. Non-deterministic semantics for logical systems. *Handbook of Philosophical Logic*, 16:227–304, 2011.

[4] W. Carnielli. Many–valued logics and plausible reasoning. In *Proceedings of the XX International Congress on Many–Valued Logics, University of Charlotte, USA*, pages 328–335. IEEE Computer Society, 1990.

[5] W. Carnielli. Possible-Translations Semantics for Paraconsistent Logics. In D. Batens, C. Mortensen, G. Priest, and J. P. Van Bendegem, editors, *Frontiers of Paraconsistent Logic: Proceedings of the I World Congress on Paraconsistency*, Logic and Computation Series, pages 149–163. Baldock: Research Studies Press, King's College Publications, 2000.

[6] W. Carnielli and M.E. Coniglio. Splitting logics. In S. Artemov, H. Barringer, A. Garcez, L. Lamb, and J. Woods, editors, *We Will Show Them! Essays in Honour of Dov Gabbay*, volume 1, pages 389–414. College Publications, 2005.

[7] W. Carnielli, M.E. Coniglio, and J. Marcos. Logics of Formal Inconsistency. In D.M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic (2nd. edition)*, volume 14, pages 1–93. Springer, 2007.

[8] A. Chagrov and M. Zakharyaskev. *Modal Logic*. Clarendon Press, 1997.

[9] M. E. Coniglio and N. M. Peron. Dugundji's theorem revisited. *Logica Universalis*, 8(3-4):407–422, 2014.

[10] J. Dugundji. Note on a property of matrices for Lewis and Langford's calculi of propositions. *The Journal of Symbolic Logic*, 5(4):150–151, 1940.

[11] L. Fariñas del Cerro. Un principe de résolution en logique modale. *R.A.I.R.O. Informatique Théorique*, 18:161–170, 1984.

[12] L. Fariñas del Cerro, M. E. Coniglio, and N.M. Peron. Non-deterministic matrices for modal logic. In *17th Brazilian Logic Conference (XVII EBL) - Book of Abstracts*, pages 47–47. Brazilian Logic Society (SBL), 2014. `http://www.uff.br/ebl/`.

[13] L. Fariñas del Cerro and N.M. Peron. Nmatrices for modal logic. In J.-Y. Béziau, A. Buchsbaum, and A. Costa-Leite, editors, *Handbook of the 4th World Congress and School on Universal Logic*, pages 115–115, 2013.
`http://www.uni-log.org/start4.html`.

[14] V. Hendricks and J. Symons. Epistemic logic. In E.N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Spring 2014 edition, 2014. `http://plato.stanford.edu/archives/spr2014/entries/logic-epistemic/`.

[15] J. V. Ivlev. A semantics for modal calculi. *Bulletin of the Section of Logic*, 17(3/4):114–121, 1988.

[16] J. V. Ivlev. *Modal Logic* (in Russian). Moscow University Publ., 1991.

[17] J. V. Ivlev. Semantics of restricted state-descriptions for quasi-matrix logic. *Bulletin of the Section of Logic*, 27(1/2):74–76, 1998.

[18] J. V. Ivlev. Quasi-matrix logic as a paraconsistent logic for dubitable information. *Logic and Logical Philosophy*, 8:91–97, 2000.

[19] J. Kearns. Modal semantics without possible worlds. *The Journal of Symbolic Logic*, 46(1):77–86, 1981.

[20] A. Kouznetsov. Quasi-matrix Deontic Logic. In A. Lomuscio and D. Nute, editors, *Deontic Logic in Computer Science. Proceedings of DEON 2004*, pages 191–208, Berlin, 2004. Springer-Verlag.

[21] S. A. Kripke. Semantic analysis of modal logic I. Normal propositional calculi. *Zeitschrift für mathemathische Logik und Grundlagen der Mathematik*, 9:67–96, 1963.

[22] S. A. Kripke. Semantical analysis of modal logic II. Non-normal modal propositional calculi. In J. W. Addison, A. Tarski, and L. Henkin, editors, *The Theory of Models*, pages 206–220, Amsterdam, 1966. North-Holland.

[23] J. Marcos. Possible-translations semantics for some weak classically-based paraconsistent logics. *Journal of Applied Non-Classical Logics*, 18(1):7–28, 2008.

[24] P. McNamara. Deontic logic. In E.N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Winter 2014 edition, 2014. `http://plato.stanford.edu/archives/win2014/entries/logic-deontic/`.

[25] N.M. Peron. *(In)Completude Modal por (N)Matrizes Finitas (Modal (In)Completeness by Finite (N)Matrices, in Portuguese)*. PhD thesis, IFCH, State University of Campinas, Brazil, 2014. `http://www.bibliotecadigital.unicamp.br/document/?code=000932590`.