Contracting Logics

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Abstract. In this paper, inspired in the field of belief revision, it is presented a novel operation for defining a new logic given a known logic. The operation consists in removing some (maybe undesirable) derived rule from a logic. Besides removing the 'undesirable' rule, this operation (called contraction) should change the logic in a minimal way. This paper presents formal definitions for contraction operations over logics, both as sets of rationality postulates and by means of concrete constructions. This allowed us to generalize several notions of maximality of logics presented in the literature. Furthermore, the proposed constructions are applied to the study of some paraconsistent and intermediate logics.

1 Introduction

Belief revision is a subfield of knowledge representation that studies the dynamics of propositional theories [AGM85]. The dynamics of the theories is given by a set of operations (contraction, revision, expansion etc.) which are defined via sets of rationality postulates.

The first papers in the field restricted themselves to the study of the dynamics of theories within supra-classical logics, but recently it was showed how this can be generalized to several other non-classical logics [Rib12]. In this paper it is presented a way to generalize belief revision techniques even more.

Instead of considering operations for changing a theory within a given logic, it is presented operations that change the *logic* itself. This paper focuses on contraction operations i.e. operations that given a logic returns another logic where certain rule doesn't hold.

Firstly a set of rationality postulates that the operation should satisfy is presented. This includes some kind of minimality criterion concerning the change needed to perform the operation. The techniques involved lead us naturally to the notion of maximality of logics. The constructions proposed here generalize several notions of maximality considered in the literature. After this, representation theorems relating the constructions defined here with sets of postulates are obtained. Finally, several examples involving paraconsistent, paracomplete and super-intuitionistic logics are shown.

2 Preliminaries

In this section we recall the basic notions to be used along the paper.

Definition 1 *A (Tarskian)* consequence system *is a pair* $\langle For, Cn\rangle$ *such that* For *is a set (whose elements are called* formulas*) and* $Cn : \wp For \rightarrow \wp For$ *is a map satisfying, for every* $\Gamma, \Delta \subseteq For$*:*

- $\Gamma \subseteq Cn(\Gamma)$ *(extensiveness)*
- *if* $\Gamma \subseteq \Delta$ *then* $Cn(\Gamma) \subseteq Cn(\Delta)$ *(monotonicity)*
- $Cn(Cn(\Gamma)) \subseteq Cn(\Gamma)$ *(idempotence)*

The map $Cn : \wp For \rightarrow \wp For$ is called a *consequence operator*. We say that $\langle For, Cn\rangle$ is *compact* if $Cn(\Gamma) = \bigcup\{Cn(\Gamma_0) \ : \ \Gamma_0 \text{ is a finite subset of } \Gamma\}$.

The usual consequence systems are defined over formal languages generated by signatures.

Definition 2 *A (propositional)* signature *is a family of sets* $C = \{C_n\}_{n\in\mathbb{N}}$ *such that* $C_i \cap C_j = \emptyset$ *if* $i \neq j$*, and* $C_0 \neq \emptyset$*.*

The elements of C_n are called *n-ary connectives*. The elements of C_0, in particular, are called *propositional variables*. Let Ξ be a given set of symbols called *schema variables*.

Definition 3 *Let* $C = \{C_n\}_{n\in\mathbb{N}}$ *be a signature. The propositional* schema language generated by C *is the algebra* $L(C, \Xi)$ *of type* C *freely generated by* $\Xi \cup C_0$*. The propositional* language generated by C *is the algebra* $L(C)$ *of type* C *freely generated by* C_0*.*

The elements of $L(C, \Xi)$ (of $L(C)$, resp.) are called *schema formulas* (*formulas*, resp.) over C. Note that $C_0 \subseteq L(C) \subseteq L(C, \Xi)$.

Definition 4 *A Hilbert calculus is a pair* $H = \langle C, R\rangle$ *such that* C *is a signature, and* R *is a set of* inference rules*, that is, a set of pairs* $\langle\Delta, \varphi\rangle$ *where* $\Delta \cup \{\varphi\}$ *is a finite subset of* $L(C, \Xi)$*.*

When $\Delta = \emptyset$ we say that the inference rule is an *axiom*. The notion of derivation in a Hilbert calculus is the usual one. Before that it is necessary to introduce the notion of substitution.

A *substitution over* C is a map $\sigma : \Xi \rightarrow L(C, \Xi)$. A substitution can be extended to a unique C-homomorphism $\hat{\sigma} : L(C, \Xi) \rightarrow L(C, \Xi)$ as usual. We denote by $\hat{\sigma}(\Delta)$ the set of schema formulas $\{\hat{\sigma}(\psi) \ : \ \psi \in \Delta\}$, for $\Delta \subseteq L(C, \Xi)$. Finally, let $Subs(C) = \{\sigma \ : \ \sigma \text{ is a substitution over } C\}$.

Definition 5 *A* derivation *of* $\varphi \in L(C, \Xi)$ *in* H *from a set* $\Gamma \subseteq L(C, \Xi)$ *is a sequence* $\varphi_1 \ldots \varphi_n$ *such that* $\varphi_n = \varphi$ *and, for* $i = 1, \ldots, n$*, each* φ_i *is either an element of* Γ *or there is a substitution* σ *and an inference rule* $\langle\Delta, \varphi\rangle$ *in* H *such that* $\hat{\sigma}(\varphi)$ *is* φ_i *and, for every* $\psi \in \Delta$*,* $\hat{\sigma}(\psi)$ *is* φ_j *for some* $j < i$*.*

We say that φ *is* derived from Γ *in* H*, denoted by* $\Gamma \vdash_H \varphi$*, if there is a derivation of it in* H *from* Γ*.*

If H is a Hilbert calculus then its set of rules will be frequently denoted by R_H. Given two Hilbert calculi $H_i = \langle C, R_i \rangle$ over C (for $i = 1, 2$) we say that H_1 is a subcalculus of H_2, denoted by $H_1 \subseteq H_2$, if $R_1 \subseteq R_2$. Given $H = \langle C, R \rangle$ over C then the *closure* of the set of rules R is the set of rules $Cl(R) = \{\langle \Delta, \varphi \rangle : \Delta \vdash_H \varphi\}$. The *closure of* H is the Hilbert calculus $Cl(H) = \langle C, Cl(R) \rangle$. If $r \in Cl(R)$ then r is a *derived rule* of H. Let $Cn_H : \wp(L(C, \Xi)) \to \wp(L(C, \Xi))$ be the consequence operator generated by H as expected: $Cn_H(\Gamma) = \{\varphi : \Gamma \vdash_H \varphi\}$. This consequence operator is Tarskian and *structural*, that is: for every substitution σ and every set of schema formulas Γ, $\hat{\sigma}(Cn_H(\Gamma)) \subseteq Cn_H(\hat{\sigma}(\Gamma))$. We say that $\langle L(C, \Xi), Cn_H \rangle$ is the consequence system, associated to the Hilbert calculus H. If $R(C) =_{def} \wp_f(L(C, \Xi)) \times L(C, \Xi)$ denotes the set of all rules over C, it is easy to see that $\langle R(C), Cl \rangle$ is a compact and structural consequence system, where $\wp_f(X)$ denotes the set of finite subsets of a set X, and where $\hat{\sigma}(r)$ is defined in the obvious way for every $r \in R(C)$ and every $\sigma \in Subs(C)$.

3 Rationality Postulates

Given a Hilbert calculus H over C we want to define an operation $-$ that for each rule $r = \langle \Delta, \varphi \rangle$ over C returns a Hilbert calculus $H - r$ over C that satisfies certain properties:

(inclusion) $H - r \subseteq H$

Inclusion states that the result of a contraction in H is a subcalculus of H.

(success) If $\varphi \notin \Delta$ (i.e. $r \notin Cl(\emptyset)$) then $\Delta \nvdash_{H-r} \varphi$.

Success states that the resulting calculus should not derive the rule r, unless this is impossible. The only case when it is impossible to remove r from H is when φ is a substitution of some element of Δ i.e. if r is a derived rule in *any* Hilbert calculus, by extensiveness.

(failure) If $\varphi \in \Delta$ (i.e. $r \in Cl(\emptyset)$) then $H \subseteq H - r$.

Failure states that if r is a derived rule in *any* Hilbert calculus, then the contraction $-$ over H should not remove any element from H. Together with inclusion, failure postulate states that $H - r = H$ if $r \in Cl(\emptyset)$. In other words, in this case the contraction should fail.

Besides these, we need some postulate that guaranties the minimality of change i.e. that guarantees that only derivations relevant to prove r should be removed. Next section investigate possible minimality postulates.

So far postulates for contraction over an arbitrary Hilbert calculus were presented. A criticism one can make to this approach is that it compromises itself with one specific axiomatization for a logic.

One way to abstract away the specificity of certain choice of a set of rules R is to consider the closure of $H = \langle C, R \rangle$, that is, the calculus $Cl(H) = \langle C, Cl(R) \rangle$. Observe that $\langle L(C, \Xi), Cl(R) \rangle$ is a Tarskian, compact and structural consequence system which represents the logic generated by H. For this reason it is more robust to consider the contraction over the closure of H instead of H itself.

Let us then define a contraction $-$ over a closed Hilbert calculus H (i.e. where $H = Cl(H)$). In this case we want the result of a contraction $-$ over H to be also a closed Hilbert calculus:

(closure) $H - r = Cl(H - r)$

The other postulates for contraction over closed Hilbert calculi are the same we already stated.

3.1 Minimality Criteria

In belief revision literature we can find several minimality postulates: recovery, relevance, core-retainment, fullness etc. (see [Han99]). We will focus here in two of these postulates, namely, relevance and fullness.

(fullness) If $r' \in R_H$ and $r' \notin R_{H-r}$ then $r \in Cl(R_{H-r} \cup \{r'\})$.

Fullness states that if some rule r' was removed from H after contraction then re-inserting r' should recover the rule r.

It will be sometimes useful to weaken this postulate to guarantee that only axioms $r' = \langle \emptyset, \varphi' \rangle$ may be removed from H.

(weak fullness) If $r' \in R_H$ is an axiom and $r' \notin R_{H-r}$ then $r \in Cl(R_{H-r} \cup \{r'\})$.

However, it is pointed out in the literature that these postulates may be too strong for certain purposes. Relevance is a weaker version of this postulate (see [Han91]):

(relevance) If $r' \in R_H$ and $r' \notin R_{H-r}$ then there is H' such that $H - r \subseteq H' \subseteq H$, $r \notin Cl(H')$ and $r \in Cl(R_{H'} \cup \{r'\})$.

Relevance states that if r' is removed then there is some intermediary calculus H' such that r is not a derived rule in H' but r is a derived rule if r' is added. Again we can define a weak version of the postulate which is concerned only with derived axioms, not derived rules in general:

(weak relevance) If $r' \in R_H$ is an axiom and $r' \notin R_{H-r}$ then there is H' such that $H - r \subseteq H' \subseteq H$, $r \notin Cl(H')$ and $r \in Cl(R_{H'} \cup \{r'\})$.

Notice that relevance implies failure:

Lemma 6 *Let H be a Hilbert calculus over C. If $-$ over H satisfies relevance then H satisfies failure.*

As will be shown in section 4, these minimality postulates are related with a construction called *remainder set*.

4 Maximality

In previous sections we presented a set of postulates for a contraction operation over a Hilbert calculus which may or may not be closed. We argued that minimality is a desirable property of the operation, i.e. contraction should change the logic as little as possible. This desiderata is closely related to the notion of maximal logics. In this section we will investigate different definitions for maximal logic presented in the literature and we will present a definition which generalizes them.

Usually, a logic $\mathbf{L}_1$ is said to be *maximal* w.r.t. another logic $\mathbf{L}_2$ when both are defined over the same language, the consequence relation $\vdash_{\mathbf{L_1}}$ of $\mathbf{L}_1$ is contained in $\vdash_{\mathbf{L_2}}$ and, if φ is a schema formula such that $\vdash_{\mathbf{L_2}} \varphi$ but $\nvdash_{\mathbf{L_1}} \varphi$ then the extension of $\mathbf{L}_1$ obtained by adding φ as a valid schema coincides with $\mathbf{L}_2$. In our framework if $\mathbf{L_1}$ is given by the closure of a Hilbert calculus H_1 then $Cl(R_{\mathbf{L_1}} \cup \{\langle \emptyset, \varphi \rangle\}) = R_{\mathbf{L_2}}$ whenever $\langle \emptyset, \varphi \rangle \in R_{\mathbf{L_2}} \setminus R_{\mathbf{L_1}}$.

For example, a logic over a signature C is *Post complete* if it is maximal w.r.t. the trivial logic $\mathbf{Triv_C} = \langle C, R(C) \rangle$ over C.

Another definition of maximality found in the literature comes from [AAZ10]. In this article the authors defined a logic $\mathbf{L}$ as being maximal w.r.t. a rule (in their case the principle of explosion $\langle \{\neg\xi_1, \xi_1\}, \xi_2 \rangle$). They define two types of maximality: strong and weak. The logic $\mathbf{L_1}$ seeing as a closed Hilbert calculus is strongly maximal w.r.t. a rule $r \notin R_{\mathbf{L_1}}$ if for every logic $\mathbf{L_2}$ such that $R_{\mathbf{L_1}} \subset R_{\mathbf{L_2}}$ we have that $r \in Cl(\mathbf{L_2})$. Using this definition the authors proved that several three-valued logics, such as $\mathbf{P^1}$ and $\mathbf{J_3}$, are maximal w.r.t. the principle of explosion.

We will borrow a concept from belief revision literature to generalize the notions of maximality.

Definition 7 (Remainder set) *Let H be a Hilbert calculus over C and R be a set of rules over C. A* remainder set *$H \perp R$ is a set such that $X \in H \perp R$ iff:*

1 $X \subseteq H$ (X is a subcalculus of H).
2 $R \nsubseteq Cl(R_X)$ (there is some $r \in R$ that is not a derived rule in X).
3 If $X \subset X' \subseteq H$ then $R \subseteq Cl(R_{X'})$ (X is maximal).

The remainder set $H \perp R$ is the set of all maximal subcalculus X of H such that some rule in R is not a derived rule in X. We can also define a weak version of remainder set, denoted by $H \perp_w R$, by changing item 3 in the above definition to:

3' For any axiom $r \in R_H \setminus R_X$ we have that $R \subseteq Cl(R_X \cup \{r\})$.

Example 8 *Consider the following Hilbert calculus $H_{CPL} = \langle C, R \rangle$ for Classical Propositional Logic (*__CPL__*). Let $C = \{\mathbb{P}, \{\neg\}, \{\wedge, \vee, \to\}\}$ where $\mathbb{P}$ denote the set of propositional variables, and let R be the following set of rules*[1]*:*

[1] In this example we will identify an axiom $\langle \emptyset, \varphi \rangle$ with φ.

(Ax$_1$) $\xi_1 \to (\xi_2 \to \xi_1)$ *(Ax$_2$)* $\xi_1 \to (\xi_2 \to (\xi_1 \wedge \xi_2))$
(Ax$_3$) $(\xi_1 \wedge \xi_2) \to \xi_1$ *(Ax$_4$)* $(\xi_1 \wedge \xi_2) \to \xi_2$
(Ax$_5$) $\xi_1 \to (\xi_1 \vee \xi_2)$ *(Ax$_6$)* $\xi_2 \to (\xi_1 \vee \xi_2)$
(Ax$_7$) $\xi_1 \to (\neg\xi_1 \to \xi_2)$ *(Ax$_8$)* $\xi_1 \vee \neg\xi_1$
(Ax$_9$) $(\xi_1 \to (\xi_2 \to \xi_3)) \to ((\xi_1 \to \xi_2) \to (\xi_1 \to \xi_3))$
(Ax$_{10}$) $(\xi_1 \to \xi_2) \to ((\xi_3 \to \xi_2) \to (\xi_1 \vee \xi_3 \to \xi_2))$
(Ax$_{11}$) $(\xi_1 \to \xi_2) \to ((\xi_1 \to \neg\xi_2) \to \neg\xi_1)$
(MP) $\langle\{\xi_1, \xi_1 \to \xi_2\}, \xi_2\rangle$

Let $r = \langle\{\neg\neg\xi\}, \xi\rangle$*. It is well known that* $r \in Cl(H_{\mathbf{CPL}})$*. Consider now the Hilbert calculus* $H_{\mathbf{Int}} = \langle C, R \setminus \{Ax_8\}\rangle$ *for Intuitionistic logic. It is also well known that* $r \notin Cl(H_{\mathbf{Int}})$*. Hence, it is trivial to show that* $H_{\mathbf{Int}} \in H_{\mathbf{CPL}}\perp\{r\}$*.*

Notice that this is strongly dependent on the choice of the rules. This is why it is important to consider the contraction operation not over arbitrary Hilbert calculi, but over closed Hilbert calculi.[2]

The above example shows the limitation of using an arbitrary Hilbert calculus H in the definition of remainder set. We will be more interested in applications where H is closed. In this case, we can prove that the elements of $H\perp R$ are also closed:

Lemma 9 *If* $H = Cl(H)$ *and* $X \in H\perp R$ *then* $X = Cl(X)$*.*

As a first application of our framework, let us show that the notions of maximality presented above can be represented using remainder sets:

- A logic $\mathbf{L_1}$ seeing as a closed Hilbert calculus is maximal w.r.t. a logic $\mathbf{L_2}$ iff $\mathbf{L_1} \in \mathbf{L_2}\perp_w R_{\mathbf{L_2}}$.
- A logic $\mathbf{L}$ over a signature C is strongly maximal w.r.t. a rule r iff $\mathbf{L} \in \mathbf{Triv_C}\perp\{r\}$.
- A logic $\mathbf{L}$ over C is weakly maximal w.r.t. a rule r iff $\mathbf{L} \in \mathbf{Triv_C}\perp_w\{r\}$.
- A logic $\mathbf{L}$ over C is Post complete iff $\mathbf{L} \in \mathbf{Triv_C}\perp_w R_{\mathbf{Triv_C}}$.

Now let us show some useful propositions about remainder sets.

Proposition 10 *Let* H *be an arbitrary Hilbert calculus and let* R_1 *and* R_2 *be sets of rules. If* $X \in H\perp R_1$*,* $R_2 \subseteq R_1 \subseteq R_H$ *and* $R_2 \nsubseteq Cl(R_X)$ *then* $X \in H\perp R_2$*.*

This proposition states that if we know that $X \in H\perp R_1$ and we know that a subset R_2 of R_1 also contains some rule that is not derived from X then $X \in H\perp R_2$.

Recall that $\mathbf{P^1}$ was introduced in [Set73] as a three valued paraconsistent logic axiomatized by a Hilbert calculus over a signature just containing $\neg$ and $\to$. We know from [Set73] that $\mathbf{P^1}$ is maximal w.r.t. $\mathbf{CPL}$, that is $\mathbf{P^1} \in \mathbf{CPL}\perp_w R_{\mathbf{CPL}}$, when both logics are considered as closed Hilbert calculi. Furthermore, we know that $r = \langle\{\xi_1, \neg\xi_1\}, \xi_2\rangle \notin R_{\mathbf{P^1}}$ and that $r \in R_{\mathbf{CPL}}$. Hence, we have the following corollary of Proposition 10:

[2] Notice that similar problems arise when using belief base contraction instead of belief set contraction.

Corollary 11 $\mathbf{P^1} \in \mathbf{CPL}\bot_w\{\langle\{\xi_1, \neg\xi_1\}, \xi_2\rangle\}$.

The logic $\mathbf{I^1}$ was introduced in [SC95] as a three valued paracomplete logic (dual to $\mathbf{P^1}$) axiomatized by a Hilbert calculus over a signature just containing $\neg$ and $\rightarrow$. From [SC95] we know that $\mathbf{I^1}$ is maximal w.r.t. **CPL**, i.e. $\mathbf{I^1} \in \mathbf{CPL}\bot_w R_{\mathbf{CPL}}$. Furthermore, the rule of double negation $r = \langle\{\neg\neg\xi_1\}, \xi_1\rangle$ doesn't hold in $\mathbf{I^1}$.

The Smetanich's logic **Sm** is another example of logic where the rule of double negation fails. **Sm** is the greatest super-intuitionistic logic (cf. [CZ97]), i.e. $\mathbf{Int} \subset \mathbf{Sm} \in \mathbf{CPL}\bot_w R_{\mathbf{CPL}}$. From this, clearly $\mathbf{I^1}$ is not super-intuitionistic.

Since $\langle\{\neg\neg\xi_1\}, \xi_1\rangle \in R_{\mathbf{CPL}}$, we have the following corollaries:

Corollary 12 $\mathbf{I^1} \in \mathbf{CPL}\bot_w\{\langle\{\neg\neg\xi_1\}, \xi_1\rangle\}$.

Corollary 13 $\mathbf{Sm} \in \mathbf{CPL}\bot_w\{\langle\{\neg\neg\xi_1\}, \xi_1\rangle\}$.

Proposition 10 showed a relation that holds when we fix the Hilbert calculus H and change the set of rules. The following proposition states a relation that holds when we fix the set of rules R and change the Hilbert calculus.

Proposition 14 *Let H_1 and H_2 be arbitrary Hilbert calculi and let R be a set of rules. If $X \in H_1\bot R$ and $X \subseteq H_2 \subseteq H_1$ then $X \in H_2\bot R$.*

The logic $\mathbf{J_3}$, introduced in [DdC70], is a paraconsistent three valued logic that can be defined over the signature $C = \langle\mathbb{P} \cup \{\bot\}, \{\neg\}, \{\wedge, \vee, \rightarrow\}\rangle$. From [AAZ10] we know that $\mathbf{J_3}$ is strongly maximal w.r.t. the principle of explosion, that is $\mathbf{J_3} \in \mathbf{Triv_C}\bot\{\langle\{\neg\xi_1, \xi_1\}, \xi_2\rangle\}$. The following corollary is a consequence of the fact that $\mathbf{J_3} \subseteq \mathbf{CPL} \subseteq \mathbf{Triv_C}$:

Corollary 15 $\mathbf{J_3} \in \mathbf{CPL}\bot\{\langle\{\neg\xi_1, \xi_1\}, \xi_2\rangle\}$.

The following results analyze the connection between the notions of remainder sets and weak remainder sets just introduced. The main purpose is to obtain sufficient conditions in order to guarantee the equivalence between both notions.

Lemma 16 *Let $H = \langle C, R\rangle$ be a Hilbert calculus over C, $r = \langle\{\gamma_1, \ldots, \gamma_n\}, \varphi\rangle$ a rule over C and $H^r = \langle C, R \cup \{r\}\rangle$. Assume that H has (possibly derived) connectives $\rightarrow$ (binary) and $\sim$ (unary) such that the following holds:*

1. $\xi_1, (\xi_1 \rightarrow \xi_2) \vdash_H \xi_2$;
2. $\vdash_H \xi_1 \rightarrow (\xi_2 \rightarrow \xi_1)$;
3. $\sim\xi_1 \vdash_H (\xi_1 \rightarrow \xi_2)$;
4. *If $\Gamma, \xi_1 \vdash_{H^r} \xi_2$ and $\Gamma, \sim\xi_1 \vdash_{H^r} \xi_2$ then $\Gamma \vdash_{H^r} \xi_2$, for every Γ.*

Let ξ be a schema variable not occurring in r. Then

$$(\xi \rightarrow \gamma_1), \ldots, (\xi \rightarrow \gamma_n) \vdash_{H^r} (\xi \rightarrow \varphi).$$

Corollary 17 *Let H and H^r as in Lemma 16. Assume that H satisfies the* Deduction Meta-Theorem *(MTD) with respect to $\to$: $\Gamma, \alpha \vdash_H \beta$ implies $\Gamma \vdash_H (\alpha \to \beta)$, for every α, β. Then H^r also satisfies MTD with respect to $\to$.*

Corollary 18 *Let H and H^r as in Corollary 17 for $r = \langle\{\gamma_1, \ldots, \gamma_n\}, \varphi\rangle$. Let $H_r = \langle C, R \cup \{\langle\emptyset, \gamma_1 \to (\gamma_2 \to (\ldots \to (\gamma_n \to \varphi)\ldots))\rangle\}\rangle$. Then the consequence systems generated by H^r and H_r coincide, that is: $Cl(H^r) = Cl(H_r)$.*

Theorem 19 *Let H and H' be closed Hilbert calculi over C such that $H \subseteq H'$. Assume that H satisfies the conditions of Corollary 17 for every rule r over C, and that H' satisfies MTD with respect to $\to$. Then, $H \in H' \bot_w R$ iff $H \in H' \bot R$ for every $R \subseteq R(C)$.*

With Theorem 19 we can now prove that both $\mathbf{I^1}$ and $\mathbf{P^1}$ are not only weakly maximal, but also strongly maximal w.r.t. **CPL**:

Proposition 20 *Let* **CPL** *be the Classical Propositional Logic in the signature containing just $\to$ and $\neg$, seeing as closed Hilbert set. Then we have the following:*

1. $\mathbf{I^1} \in \mathbf{CPL} \bot R_{\mathbf{CPL}}$ *and*
2. $\mathbf{I^1} \in \mathbf{CPL} \bot \{\langle\{\neg\neg\xi_1\}, \xi_1\rangle\}$.

Proposition 21 *Let* **CPL** *be the Classical Propositional Logic as in Proposition 20. Then the following holds:*

1. $\mathbf{P^1} \in \mathbf{CPL} \bot R_{\mathbf{CPL}}$ *and*
2. $\mathbf{P^1} \in \mathbf{CPL} \bot \{\langle\{\neg\xi_1, \xi_1\}, \xi_2\rangle\}$.

5 Representation Theorems

In Section 3 we enumerated a set of rationality postulates for a contraction operation. In Section 4 we claimed that this operation is related with a notion of maximality which we explored. This section presents constructions for contraction. Each of them is proved to be characterized by a specific set of rationality postulates.

The following is an important lemma related to remainder sets called *upper-bound lemma*. This result was adapted from a similar one in belief revision field found in [AM81].

Lemma 22 (upper-bound) *Let H and X be Hilbert Calculi such that $X \subseteq H$ and let R be a finite set of rules, all over a signature C. If $R \nsubseteq Cl(X)$ then*

1. *there is some H' such that $X \subseteq H' \in H \bot R$ and*
2. *there is some H'' such that $X \subseteq H'' \in H \bot_w R$.*

Consider now a Hilbert calculus H and a rule r both over the same signature. A (strong) *subset selection function* is any function Υ that satisfies the following properties:

1. $\Upsilon(H, r) \neq \emptyset$;
2. $\Upsilon(H, r) \subseteq H\bot\{r\}$ if $H\bot\{r\} \neq \emptyset$;
3. $\Upsilon(H, r) = \{H\}$ if $H\bot\{r\} = \emptyset$.

A *weak subset selection function* is defined analogously using weak remainder set instead of remainder set. Now consider the following construction for some selection function Υ:

$$H -_{\Upsilon} r = \bigcap \Upsilon(H, r)$$

In the belief revision field this construction is called *partial meet contraction* (cf. [AGM85]). Any partial meet contraction satisfies *success*, *inclusion* and *failure*. It also satisfies *relevance* or *weak relevance* depending if Υ is a weak or a strong selection function. Furthermore, if H is closed, by Lemma 9 and the fact that the intersection of closed Hilbert calculi is closed, partial meet contraction satisfies *closure*.

Theorem 23 *For any weak subset selection function Υ, the contraction $-_{\Upsilon}$ over H defined as $H -_{\Upsilon} r = \bigcap \Upsilon(H, r)$ satisfies* success, inclusion, failure *and* weak relevance*. Furthermore:*

- *If Υ is a (strong) subset selection function then $-_{\Upsilon}$ also satisfies* relevance*.*
- *If H is closed then $-_{\Upsilon}$ satisfies* closure

Besides satisfying the postulates, the following theorem shows that, in fact, the postulates fully characterize the construction.

Theorem 24 *Let H be a Hilbert calculus. If a contraction $-$ over H satisfies* success, inclusion, failure *and* weak relevance *then there is some weak subset selection function Υ such that:*

$$H - r = H -_{\Upsilon} r = \bigcap \Upsilon(H, r)$$

If $-$ satisfies relevance *then the above equation holds for some (not weak) subset selection function. In this case,* failure *property become redundant by Lemma 6.*

Now let us define an *element selection function* as any function Ψ such that:

1. $\Psi(H, r) \in H\bot\{r\}$ if $H\bot\{r\} \neq \emptyset$.
2. $\Psi(H, r) = H$ if otherwise.

Weak element selection function is defined analogously using a weak remainder set. A *maxi-choice contraction* is defined as follows:

$$H -_{\Psi} r = \Psi(H, r)$$

Weak maxi-choice contraction is defined analogously.

As for partial meet contraction, maxi-choice contraction is fully characterized by a set of postulates, namely: *success*, *inclusion*, *failure* and *fullness*. Furthermore, weak maxi-choice contraction is characterized by the same postulates with fullness exchanged by *weak fullness*.

Theorem 25 *Let* H *be a Hilbert calculus. An operation* $-$ *over* H *is a maxi-choice contraction iff it satisfies* success, inclusion, failure *and* fullness. $-$ *is a weak maxi-choice contraction iff instead of fullness it satisfies* weak fullness.

Example 26 *Consider* $\mathbf{P^1}$ *and* **CPL** *defined in the same signature of* $\mathbf{J_3}$ *and let* $r = \langle\{\neg\xi_1, \xi_1\}, \xi_2\rangle$. *For certain choice of element selection functions we have that* $\mathbf{P^1} = \mathbf{CPL} -_{\Psi_1} r$ *and* $\mathbf{J_3} = \mathbf{CPL} -_{\Psi_2} r$. *Furthermore, for certain choice of* subset *selection function* Υ, *we have* $\mathbf{P^1} \cap \mathbf{J_3} = \mathbf{CPL} -_{\Upsilon} r$. *An analogous result can be obtained by considering weak selection functions,* $\mathbf{I^1}$, **Sm** *and* **CPL** *(over the same signature) and* $r = \langle\{\neg\neg\xi_1\}, \xi_1\rangle$. *Note that* $\mathbf{I^1}$ *can be choose even for non weak selection functions.*

6 Conclusion and Future Work

In this paper it was presented a formal framework for defining new logics by contracting a derived rule from a known logic. The framework consists in defining operations $-$ over possibly closed Hilbert Calculi. These operation are related with contraction operation in belief revision field. Constructions for the operations as well as postulates that characterize them were presented.

This framework is related with the study of maximal logics w.r.t. another logic (cf. [Set73,SC95]) and the study of maximal logics w.r.t. a principle (cf. [AAZ10]). Furthermore, the relation between the notion of strong remainder set and weak remainder set was analyzed. As an application it was proved that the logics $\mathbf{P^1}$ and $\mathbf{I^1}$ are strongly maximal w.r.t. Classical Propositional Logic.

This paper focused in the contraction of *one* rule from a logic. It would be interesting to analyze the result of contracting a set of rules. Once again the field of belief revision may help us in this task.

As future work we intend to extend this framework to sequent calculi and also to study the analogous of a revision operation over logics.

Acknowledgements: This research was financed by FAPESP (Brazil), Thematic Project LogCons 2010/51038-0. The first author was also supported by a Post-doctoral grant by FAPESP, process 2011/01384-1. The second author was also financed by an individual research grant from The National Council for Scientific and Technological Development (CNPq), Brazil, process 305237/2011-0.

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Appendix A: Proofs of the main results

Lemma 6: **Proof.** Let r be a rule over C such that $r \in Cl(\emptyset)$. If $H \not\subseteq H - r$ then there is $r' \in R_H \setminus R_{H-r}$. By relevance, there is H' such that $r \notin Cl(H')$. But this is an absurd, by monotonicity of Cl. ■

Lemma 9: **Proof.** Let $H = Cl(H)$, and $X \in H \bot R$. By extensiveness it follows that $X \subseteq Cl(X)$. Now, suppose that $X \subset Cl(X)$. By monotonicity $Cl(X) \subseteq Cl(H) = H$, and so $X \subset Cl(X) \subseteq H$. Since $X \in H \bot R$ we have, by idempotence, that $R \subseteq Cl(R_X)$, which is a contradiction. Hence, there is no $r \in Cl(X) \setminus X$ i.e. $X = Cl(X)$. ■

Proposition 10: **Proof.** To prove that $X \in H \bot R_2$ we need to show 1) that $X \subseteq H$, 2) $R_2 \not\subseteq Cl(R_X)$ and 3) if $X \subset X' \subseteq H$ then $R_2 \subseteq Cl(R_{X'})$. 1) follows directly from the fact that $X \in H \bot R_1$ and 2) follows by hypothesis. To prove 3) notice that if $X \subset X' \subseteq H$ then $R_1 \subseteq Cl(R_{X'})$ and, since $R_2 \subseteq R_1$, $R_2 \subseteq Cl(X')$. ■

Proposition 14: **Proof.** We need to prove 1) that $X \subseteq H_2$, 2) $R \not\subseteq Cl(R_X)$ and 3) if $X \subset X' \subseteq H_2$ then $R \subseteq Cl(R_{X'})$. 1) and 2) follow directly from hypothesis. Now let consider X' such that $X \subset X' \subseteq H_2$. We have by hypothesis that $H_2 \subseteq H_1$ and, hence, $X' \subseteq H_1$. It follows that $R \subseteq Cl(R'_{X'})$, because $X \in H_1 \bot R$. ■

[2] Lemmas 6 and 9 were adapted from [Han99].

Lemma 16: **Proof.** For $i = 1, \ldots, n$ it holds that $(\xi \to \gamma_i), \xi \vdash_{H^r} \gamma_i$, by (1), and so $(\xi \to \gamma_1), \ldots, (\xi \to \gamma_n), \xi \vdash_{H^r} \gamma_i$, for every i. Since $\gamma_1, \ldots, \gamma_n \vdash_{H^r} \varphi$ then $(\xi \to \gamma_1), \ldots, (\xi \to \gamma_n), \xi \vdash_{H^r} \varphi$. But then $(\xi \to \gamma_1), \ldots, (\xi \to \gamma_n), \xi \vdash_{H^r} (\xi \to \varphi)$, since $\vdash_{H^r} \varphi \to (\xi \to \varphi)$ and by (1). On the other hand $\sim\xi \vdash_{H^r} (\xi \to \varphi)$, by (3), and so $(\xi \to \gamma_1), \ldots, (\xi \to \gamma_n), \sim\xi \vdash_{H^r} (\xi \to \varphi)$. By (4) it follows that $(\xi \to \gamma_1), \ldots, (\xi \to \gamma_n) \vdash_{H^r} (\xi \to \varphi)$ as required. ■

Corollary 17: **Proof.** Assume that $\Gamma, \alpha \vdash_{H^r} \beta$. By induction on the length of a derivation $\varphi_1 \ldots \varphi_k$ of β from $\Gamma \cup \{\alpha\}$ in H^r it will be shown that $\Gamma \vdash_{H^r} (\alpha \to \beta)$. Since H satisfies MTD with respect to $\to$, the only case to be analyzed is when β is obtained by the use of the rule $r = \langle\{\gamma_1, \ldots, \gamma_n\}, \varphi\rangle$. Thus, there is some substitution σ such that $\beta = \hat{\sigma}(\varphi)$ and $\{\hat{\sigma}(\gamma_1), \ldots, \hat{\sigma}(\gamma_n)\} \subseteq \{\varphi_1, \ldots, \varphi_{k-1}\}$. By induction hypothesis, $\Gamma \vdash_{H^r} (\alpha \to \hat{\sigma}(\gamma_i))$ for every i. By Lemma 16, $(\alpha \to \hat{\sigma}(\gamma_1)), \ldots, (\alpha \to \hat{\sigma}(\gamma_n)) \vdash_{H^r} (\alpha \to \hat{\sigma}(\varphi))$ (by taking $\sigma(\xi) = \alpha$). Therefore $\Gamma \vdash_{H^r} (\alpha \to \hat{\sigma}(\varphi))$, that is, $\Gamma \vdash_{H^r} (\alpha \to \beta)$. ■

Corollary 18: **Proof.** Clearly $\gamma_1, \ldots, \gamma_n \vdash_{H^r} \varphi$, and then $\vdash_{H^r} \gamma_1 \to (\gamma_2 \to (\ldots \to (\gamma_n \to \varphi) \ldots)$, by MTD. Thus $Cl(H_r) \subseteq Cl(H^r)$.

On the other hand, by (i) of Lemma 16 it holds that $\gamma_1, \ldots, \gamma_n \vdash_{H_r} \varphi$ and so $Cl(H^r) \subseteq Cl(H_r)$. This completes the proof. ■

Theorem 19: **Proof.** Assume that H and H' satisfy the hypothesis of the theorem. The 'if' part is obviously true. For the 'only if' part, assume that $H \in H' \bot_w R$ and let $r = \langle\{\gamma_1, \ldots, \gamma_n\}, \varphi\rangle$ be a rule such that $r \in R_{H'} \setminus R_H$. Since H' satisfies MTD with respect to $\to$ if follows that $\vdash_{H'} \gamma_1 \to (\gamma_2 \to (\ldots \to (\gamma_n \to \varphi) \ldots))$. Since H satisfies (1) of Lemma 16 it follows that $\nvdash_H \gamma_1 \to (\gamma_2 \to (\ldots \to (\gamma_n \to \varphi) \ldots))$. Given that $H \in H' \bot_w R$ it follows that $R \nsubseteq Cl(R_H)$ but $R \subseteq Cl(H_r)$, where $H_r = \langle C, R_H \cup \{\langle\emptyset, \gamma_1 \to (\gamma_2 \to (\ldots \to (\gamma_n \to \varphi) \ldots))\rangle\}\rangle$. But $Cl(H_r) = Cl(H^r)$, by Corollary 18, where $H^r = \langle C, R_H \cup \{r\}\rangle$. Then $R \subseteq Cl(H^r)$ and so $H \in H' \bot R$. ■

Proposition 20: **Proof.** 1. It is known that the logic $\mathbf{I^1}$ is maximal with respect to **CPL** in the signature just containing $\to$ and $\neg$ (cf.[SC95]). That is, $\mathbf{I^1} \in \mathbf{CPL} \bot_w R_{\mathbf{CPL}}$. Let $\sim\alpha =_{def} (\alpha \to \neg\alpha)$ and $\alpha \vee \beta =_{def} (\neg(\beta \to \beta) \to \beta) \to ((\alpha \to \alpha) \to \alpha)$. Then $\vdash_{\mathbf{I^1}} (\sim\alpha \vee \alpha)$ for every α. On the other hand, it is easy to show the following: $\Gamma, \alpha \vdash_{(\mathbf{I^1})^r} \gamma$ and $\Gamma, \beta \vdash_{(\mathbf{I^1})^r} \gamma$ implies that $\Gamma, (\alpha \vee \beta) \vdash_{(\mathbf{I^1})^r} \gamma$, for every $\Gamma, \alpha, \beta, \gamma$ and for every rule r, where $(\mathbf{I^1})^r$ is as in Lemma 16. Thus, $\mathbf{I^1}$ satisfies the conditions (1)-(4) of Lemma 16, for every rule r. On the other hand, both **CPL** and $\mathbf{I^1}$ satisfy MTD with respect to $\to$. Then, by Theorem 19 it follows that $\mathbf{I^1}$ is strongly maximal with respect to **CPL**, that is, $\mathbf{I^1} \in \mathbf{CPL} \bot R_{\mathbf{CPL}}$.

2. We know from Corollary 12 that $\mathbf{I^1} \in \mathbf{CPL} \bot_w \{\langle\{\neg\neg\xi_1\}, \xi_1\rangle\}$. The proof that $\mathbf{I^1} \in \mathbf{CPL} \bot \{\langle\{\neg\neg\xi_1\}, \xi_1\rangle\}$ is analogous to that of item 1. ■

Proposition 21: **Proof.** The proof is similar to that of Proposition 20.
1. We begin by observing that the logic $\mathbf{P^1}$ is maximal with respect to **CPL** in the signature just containing $\to$ and $\neg$ (cf. [Set73]). That is, $\mathbf{P^1} \in \mathbf{CPL}\bot_w R_{\mathbf{CPL}}$. It is enough to define in $\mathbf{P^1}$ a classical negation $\sim$ and a disjunction $\vee$ which guarantee, as in the case of $\mathbf{I^1}$, the satisfaction of conditions (1)-(4) of Lemma 16, for every rule r. The derived connectives $\sim\alpha =_{def} \neg(\neg\alpha \to \alpha)$ and $\alpha \vee \beta =_{def} (\sim\alpha \to \beta)$ satisfy the required properties. Since both $\mathbf{P^1}$ and **CPL** satisfy MTD with respect to $\to$ then, by Theorem 19, it follows that $\mathbf{P^1}$ is strongly maximal with respect to **CPL**. That is, $\mathbf{P^1} \in \mathbf{CPL}\bot R_{\mathbf{CPL}}$.

2. We know from Corollary 11 that $\mathbf{P^1} \in \mathbf{CPL}\bot_w\{\langle\{\xi_1, \neg\xi_1\}, \xi_2\rangle\}$. The proof that $\mathbf{P^1} \in \mathbf{CPL}\bot\{\langle\{\xi_1, \neg\xi_1\}, \xi_2\rangle\}$ is analogous to that of item 1. ■

Lemma 22: **Proof.** Let C be the signature of H and enumerate the rules in R_H: $\{r_1, r_2, \ldots\}$. Let $R_0 = R_X$ and for each $i \geq 1$ let:

$$R_i = \begin{cases} R_{i-1} \cup \{r_i\} & \text{if } R \not\subseteq Cl(R_{i-1} \cup \{r_i\}) \\ R_{i-1} & \text{otherwise.} \end{cases}$$

Now consider $H' = \langle C, \bigcup_i R_i\rangle$. Clearly, $X \subseteq H'$. Suppose that $R = \{r'_1, \ldots, r'_n\}$ is contained in $Cl(R_{H'})$. Since Cl is compact, there exists a finite set $R'_j \subseteq R_{H'}$ such that $r'_j \in Cl(R'_j)$ for $j = 1, \ldots, n$. But $R_i \subseteq R_{i+1}$ and so $R \subseteq Cl(R_j)$ for some j, which is a contradiction. We conclude that $R \not\subseteq Cl(R_{H'})$.

The other conditions for $H' \in H\bot R$ are easy to verify.[3] ■

Theorem 23: **Proof.** *Inclusion* follows trivially and *success* follows directly from the upper-bound lemma with $X = \emptyset$. To prove *relevance* note that if $r' \in R_H \setminus \bigcap\Upsilon(H, r)$ then there is $X \in \Upsilon(H, r)$ such that $r' \notin X$. Of course $\bigcap\Upsilon(H, r) \subseteq X \subseteq H$, $r \notin Cl(X)$ and, since X is maximal, then $r \in Cl(R_X \cup \{r'\})$. This same argument holds if r' is an axiom and Υ is a weak subset selection funcion, hence *weak relevance* also holds. Finally, if H is closed then *closure* follows directly from Lemma 9. ■

Theorem 24: **Proof.** We will show only a sketch of a proof for strong subset selection function. The proof for weak subset selection function is completely analogous. Let $\Upsilon(H, r) = \{X \in H\bot\{r\} \ : \ H - r \subseteq X\}$ if $H\bot\{r\} \neq \emptyset$ and $\Upsilon(H, r) = \{H\}$ otherwise. We need to show that Υ is well defined, that it is a selection function and that $H - r = \bigcap\gamma(H, r)$.

Proving that Υ is well defined is trivial, since we defined Υ over the generators (that is, the pairs $\langle H, r\rangle$).

It is also trivial to verify that $\Upsilon(H, r) \subseteq H\bot\{r\}$. From *success*, *inclusion* and the upper-bound lemma, we show that $\Upsilon(H, r) \neq \emptyset$.

Now if $H\bot\{r\} = \emptyset$ then $r \in Cl(\emptyset)$. In this case $\bigcap\Upsilon(H, r) = H$ and, by *failure* and *inclusion* we have that $H - r = H$. If $H\bot\{r\} \neq \emptyset$ then trivially

[3] This proof was adapted from a very similar to the proof of Lindenbaum's lemma found in [Wój88].

$H - r \subseteq \bigcap \Upsilon(H, r)$. To prove the converse, suppose by absurdum that $r' \notin H - r$. If $r' \notin H$ then $r' \notin \bigcap \Upsilon(H, r)$ and we are done. Consider then that $r' \in H$. Then by *relevance* there is H' such that $H - r \subseteq H' \subseteq H$, $r \notin Cl(H')$ and $r \in Cl(R_{H'} \cup \{r'\})$. By the upper bound lemma there is X such that $H' \subseteq X \in H \bot \{r\}$. It follows that $r' \notin X \in \Upsilon(H, r)$ and, hence, $r' \notin \bigcap \Upsilon(H, r)$. ■

Theorem 25: **Proof.** We will sketch the proof of maxi-choice characterization. The proof for weak maxi-choice characterization is completely analogous.

Let $\Psi(H, r) = H - r$. We must prove that $\Psi(H, r) \in H \bot \{r\}$ if $H \bot \{r\} \neq \emptyset$ and $\Psi(H, r) = H$ otherwise.

The second situation follows directly from *failure*. Now let us assume that $r \notin Cl(\emptyset)$, then $\Psi(H, r) \in H \bot \{r\}$ by *success*, *inclusion* and *fullness*. ■