Maximality in finite-valued Łukasiewicz logics defined by order filters

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1 Preliminaries and first results

In this talk we consider the logics $L^i_n$, obtained from the $(n + 1)$-valued Łukasiewicz logics $L_{n+1}$ by taking the order filter generated by $i/n$ as the set of designated elements. The $(n + 1)$-valued Łukasiewicz logic can be semantically defined as the matrix logic $L_{n+1} = \langle LV_{n+1}, \{1\} \rangle$,

where $LV_{n+1} = (LV_{n+1}, \neg, \to)$ with $LV_{n+1} = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$, and the operations are defined as follows: for every $x, y \in LV_{n+1}$, $\neg x = 1 - x$ and $x \to y = \min\{1, 1 - x + y\}$.

Observe that $L_2$ is the usual presentation of classical propositional logic CPL as a matrix logic over the two-element Boolean algebra $B_2$ with domain $\{0, 1\}$ and signature $\{\neg, \to\}$. The logics $L_2$ can also be presented as Hilbert calculi that are axiomatic extensions of the infinite-valued Łukasiewicz logic $L_\infty$.

The following operations can be defined in every algebra $LV_{n+1}$: $x \odot y = \neg(x \to \neg y) = \max\{0, x + y - 1\}$ and $x \oplus y = \neg x \to y = \min\{1, x + y\}$. For every $n > 1$, $x^n = x \odot \cdots \odot x$ ($n$-times) and $nx = x \oplus \cdots \oplus x$ ($n$-times).

For $1 \leq i \leq n$ let $F_{i/n} = \{x \in LV_{n+1} : x \geq i/n\} = \left\{\frac{i}{n}, \ldots, \frac{n-1}{n}, 1\right\}$ be the order filter generated by $i/n$, and let $L^i_n = \langle LV_{n+1}, F_{i/n} \rangle$ be the corresponding matrix logic. From now on, the consequence relation of $L^i_n$ is denoted by $\models L^i_n$. Observe that $L_{n+1} = L^0_n$ for every $n$. In particular, CPL is $L^1_1$ (that is, $L_2$). If $1 \leq i, m \leq n$, we can also consider the following matrix logic: $L^{i/n}_{m/n} = \langle LV_{m+1}, F_{i/n} \cap LV_{m+1} \rangle$.

The logic $L^2_3 = \langle LV_3, \{1, 1/2\} \rangle$ was already known as the 3-valued paraconsistent logic $J_3$, introduced by da Costa and D’Ottaviano see [4] in order to obtain an example of a paraconsistent logic maximal w.r.t. CPL.

Definition 1. Let $L_1$ and $L_2$ be two standard propositional logics defined over the same signature $\Theta$ such that $L_1$ is a proper sublogic of $L_2$. Then, $L_1$ is maximal w.r.t. $L_2$ if, for every formula $\varphi$ over $\Theta$, if $\vdash_{L_2} \varphi$ but $\not\vdash_{L_1} \varphi$, then the logic $L^+_1$ obtained from $L_1$ by adding $\varphi$ as a theorem, coincides with $L_2$.

In order to study maximality among finite-valued Łukasiewicz logics defined by order filters we obtain the following sufficient condition:
Theorem 1. Let $L_1 = \langle A_1, F_1 \rangle$ and $L_2 = \langle A_2, F_2 \rangle$ be two distinct finite matrix logics over a same signature $\Theta$ such that $A_2$ is a subalgebra of $A_1$ and $F_2 = F_1 \cap A_2$. Assume the following:

1. $A_1 = \{0, 1, a_1, \ldots, a_k, a_{k+1}, \ldots, a_n\}$ and $A_2 = \{0, 1, a_1, \ldots, a_k\}$ are finite such that $0 \notin F_1$, $1 \in F_2$ and $\{0, 1\}$ is a subalgebra of $A_2$.

2. There are formulas $\top(p)$ and $\bot(p)$ in $L(\Theta)$ depending at most on one variable $p$ such that $e(\top(p)) = 1$ and $e(\bot(p)) = 0$, for every evaluation $e$ for $L_1$.

3. For every $k + 1 \leq i \leq n$ and $1 \leq j \leq n$ (with $i \neq j$) there exists a formula $\alpha_j^i(p)$ in $L(\Theta)$ depending at most on one variable $p$ such that, for every evaluation $e$, $e(\alpha_j^i(p)) = a_j$ if $e(p) = a_i$.

Then, $L_1$ is maximal w.r.t. $L_2$.

We use this result to prove that

Theorem 2. Let $1 \leq i, m \leq n$. Then $L_i^1$ is maximal w.r.t. $L_m^i/n$ if the following condition holds: there is some prime number $p$ and $k \geq 1$ such that $n = p^k$, and $m = p^{k-1}$.

Corollary 1. Let $1 \leq i \leq p$. For every prime number $p$, $L_p^i$ is maximal w.r.t. CPL.

Notice that the above corollary generalizes the well known result: $L_{p+1}$ is maximal w.r.t. CPL for every prime number $p$.

Definition 2. Let $L_1$ and $L_2$ be two standard propositional logics defined over the same signature $\Theta$ such that $L_1$ is a proper sublogic of $L_2$. Then, $L_1$ is strongly maximal w.r.t. $L_2$ if, for every finitary rule $\varphi_1, \ldots, \varphi_n/\psi$ over $\Theta$, if $\varphi_1, \ldots, \varphi_n \vdash_{L_2} \psi$ but $\varphi_1, \ldots, \varphi_n \not\vdash_{L_1} \psi$, then the logic $L_1^*$ obtained from $L_1$ by adding $\varphi_1, \ldots, \varphi_n/\psi$ as structural rule, coincides with $L_2$.

Let $i$ be a strictly positive integer, the $i$-explosion rule is the rule $(exp_i) \frac{i(\varphi \land \neg \varphi) \bot}{\bot}$.

Lemma 1. For every $1 \leq i \leq n$, the rule $(exp_i)$ is not valid in $L_n^i$.

Corollary 2. Let $1 \leq i \leq p$. For every prime number $p$, $L_p^i$ is not strongly maximal w.r.t. CPL.

2 Equivalent systems

Blok and Pigozzi introduce in [3] the notion of equivalent deductive systems in the following sense: Two propositional deductive systems $S_1$ and $S_2$ in the same language $L$ are equivalent iff there are two translations $\tau_1, \tau_2$ (finite subsets of $L$-propositional formulas in one variable) such that:

- $\Gamma \vdash_{S_1} \varphi$ iff $\tau_1(\Gamma) \vdash_{S_2} \tau_1(\varphi)$,

- $\Delta \vdash_{S_2} \psi$ iff $\tau_2(\Delta) \vdash_{S_1} \tau_2(\psi)$,

- $\varphi \not\vdash_{S_1} \tau_2(\tau_1(\varphi))$,

- $\psi \not\vdash_{S_2} \tau_1(\tau_2(\psi))$.

Theorem 3. For every $n \geq 2$ and every $1 \leq i \leq n$, $L_n^i$ and $L_n^{n+1}$ are equivalent deductive systems.
From the equivalence among $\mathbb{L}_n^i$ and $\mathbb{L}_{n+1}$, we can obtain, by translating the axiomatization of the finite valued Lukasiewicz logic $\mathbb{L}_{n+1}$, a calculus sound and complete with respect $\mathbb{L}_n^i$ that we denote by $H_n^i$.

Since $\mathbb{L}_\infty$ is algebraizable and the class $MV$ of all MV-algebras is its equivalent quasivariety semantics, finitary extensions of $\mathbb{L}_\infty$ are in 1 to 1 correspondence with quasivarieties of MV-algebras. Actually, there is a dual isomorphism from the lattice of all finitary extensions of $\mathbb{L}_\infty$ and the lattice of all quasivarieties of $MV$. Moreover, if we restrict this correspondence to varieties of $MV$ we get the dual isomorphism from the lattice of all varieties of $MV$ and the lattice of all axiomatic extensions of $\mathbb{L}_\infty$. Since $\mathbb{L}_{n+1} = \mathbb{L}_n^n$ is an axiomatic extension of $\mathbb{L}_\infty$, $\mathbb{L}_{n+1}$ is an algebraizable logic with the class $MV_n = \mathcal{Q}(\mathbb{L}V_{n+1})$, the quasivariety generated by $\mathbb{L}V_{n+1}$, as its equivalent variety semantics. It follows from the previous theorem that $\mathbb{L}_n^n$, for every $1 \leq i \leq n$, is also algebraizable with the same class of $MV_n$-algebras as its equivalent variety semantics. Thus, the lattices of all finitary extensions of $\mathbb{L}_n^n$ are isomorphic, and in fact, dually isomorphic to the lattice of all subquasivarieties of $MV_n$, for all $0 < i < n$.

Therefore maximality conditions in the lattice of finitary (axiomatic) extensions correspond to minimality conditions in the lattice of subquasivarieties (subvarieties). Thus, given two finitary extensions $L_1$ and $L_2$ of a given logic $L_n^n$, where $K_{L_1}$ and $K_{L_2}$ are its associated $MV_n$-quasivarieties, $L_1$ is strongly maximal with respect $L_2$ iff $K_{L_1}$ is a minimal subquasivariety of $MV_n$ among those $MV_n$-quasivarieties properly containing $K_{L_2}$. Moreover, if $L_1$ and $L_2$ are axiomatic extensions of $\mathbb{L}_n^n$, then $K_{L_1}$ and $K_{L_2}$ are indeed $MV_n$-varieties. In that case, $L_1$ is maximal with respect $L_2$ iff $K_{L_1}$ is a minimal subvariety of $MV_n$ among those $MV_n$-varieties properly containing $K_{L_2}$.

The lattice of all axiomatic extensions $\mathbb{L}_\infty$ is fully described also by Komori in [7], thus from the equivalence of Theorem 3, we can obtain the following maximality conditions for all axiomatic extensions of $\mathbb{L}_n^n$.

**Theorem 4.** Let $0 < i, m \leq n$ be natural numbers such that $m|n$. If $L$ is an axiomatic extension of $\mathbb{L}_n^n$, then $L$ is maximal with respect to $\mathbb{L}_m^i$ if $L = \mathbb{L}_m^n \cap \mathbb{L}_m^{i/n}$ for some prime number $p$ with $p|n$ and a natural $k \geq 0$ such that $p^k|m$ and $p^{k+1} \nmid m$.

As a corollary we obtain that the sufficient condition of Theorem 2 is also necessary.

**Corollary 3.** Let $1 \leq i, m \leq n$. Then $\mathbb{L}_n^i$ is maximal w.r.t. $\mathbb{L}_m^n$ if and only if there is some prime number $p$ and $k \geq 1$ such that $n = p^k$, and $m = p^{k-1}$.

To obtain results on strong maximality we need to study finitary extensions of $\mathbb{L}_\infty$. The task of fully describing the lattice of all finitary extensions of $\mathbb{L}_\infty$, isomorphic to the lattice of all subquasivarieties of $MV$, turns to be an heroic task since the class of all MV-algebras is $Q$-universal [1]. For the finite valued case it is much simpler, since $MV_n$ is a locally finite discriminator variety. Any locally finite quasivariety is generated by its critical algebras [5]. Critical MV-algebras were fully described in [6] and using this description we can obtain some results on strong maximality.

First we need to introduce the following matrix logics: For every $1 \leq i, m \leq n$,

\[ \mathbb{L}_n^i = \langle \mathbb{L}V_{n+1} \times \mathbb{L}V_2, F_i/n \times \{1\} \rangle \quad \mathbb{L}_m^{i/n} = \langle \mathbb{L}V_{m+1} \times \mathbb{L}V_2, (F_i/n \cap \mathbb{L}V_{m+1}) \times \{1\} \rangle \]

**Theorem 5.** Let $0 < i \leq n$ be natural numbers, let $p$ be a prime number and let $r = \max\{ j \in \mathbb{N} : p^j|n \}$. Then we have: For every $j$ such that $(i - 1)p < j < ip$, $\mathbb{L}_n^i \cap \mathbb{L}_r^{i/p}$ is strongly maximal with respect to $\mathbb{L}_n^i$. Moreover, every finitary extension of some $\mathbb{L}_k^i$ is strongly maximal with respect $\mathbb{L}_n^i$ iff it is one of the preceding types.
As a particular case we can also prove the following result.

**Theorem 6.** Let $p$ be a prime number. Then, for every $j$ such that $0 < j \leq p$:

- $\mathcal{L}_p^j$ is strongly maximal with respect to CPL and it is axiomatized by $H_p^j$ plus the $j$-explosion rule $(\exp_j) j(\varphi \land \neg \varphi)/\bot$.
- $\mathcal{L}_p^j$ is strongly maximal w.r.t. $\mathcal{L}_p^j$.

### 3   Ideal paraconsistent logics

Arieli, Avron and Zamansky introduced in [2] the concept of *ideal paraconsistent logics*.

**Definition 3.** Let $L$ be a propositional logic defined over a signature $\Theta$ (with consequence relation $\vdash_L$) containing at least a unary connective $\neg$ and a binary connective $\rightarrow$ such that:

(i) $L$ is paraconsistent w.r.t. $\neg$, i.e. there are formulas $\varphi, \psi \in L(\Theta)$ such that $\varphi, \neg \varphi \vdash_L \psi$; and $\rightarrow$ is a deductive implication, i.e. $\Gamma \cup \{\varphi\} \vdash_L \psi$ iff $\Gamma \vdash_L \varphi \rightarrow \psi$.

(ii) There is a presentation of CPL as a matrix logic $L = \mathcal{A}$ over the signature $\Theta$ such that the domain of $\mathcal{A}$ is $\{0, 1\}$, and $\neg$ and $\rightarrow$ are interpreted as the usual 2-valued negation and implication of CPL, respectively, such that $L$ is a sublogic of CPL.

Then, $L$ is said to be an *ideal paraconsistent logic* if it is maximal w.r.t. CPL, and every proper extension of $L$ over $\Theta$ is not $\neg$-paraconsistent.

**Lemma 2.** Let $0 < i \leq n$. $\mathcal{L}_n^i$ is paraconsistent w.r.t. $\neg$ iff $\frac{i}{n} \leq \frac{1}{2}$

Since for every $0 < i \leq n$, there is a term definable implication $\Rightarrow_n^i$ which is deductive implication next result follows from Theorem 6

**Theorem 7.** Let $p$ be a prime number, and let $1 \leq i < p$ such that $i/p \leq 1/2$. Then, $\mathcal{L}_p^i$ is a $(p + 1)$-valued ideal paraconsistent logic.\(^1\)

### References


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\(^1\)Strictly speaking, in this claim we implicitly assume that the signature of $\mathcal{L}_p^j$ has been changed by adding the definable implication $\Rightarrow_n^i$ as a primitive connective.