Modal extensions of sub-classical logics for recovering classical logic

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Abstract. In this paper we introduce non-normal modal extensions of the sub-classical logics **CLoN**, **CluN** and **CLaN**, in the same way that $\mathbf{S0.5}^0$ extends classical logic. The first modal system is both paraconsistent and paracomplete, while the second one is paraconsistent and the third is paracomplete. Despite being non-normal, these systems are sound and complete for a suitable Kripke semantics. We also show that these systems are appropriate for interpreting $\Box$ as "is provable in classical logic". This allows us to recover the theorems of propositional classical logic within three sub-classical modal systems.

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Introduction

In Modal Logic, the operator $\Box$ has been interpreted in many ways: "necessary", "obligatory","always the case", and so on. Nevertheless Lewis, who has been acclaimed the inventor of formal modal logic, focuses on the modal concept of "deducible".

When Lewis formulated the systems **S1-S5**, he used $\Diamond$ and $\prec$ as primitive operators. This being so, it seems reasonable to suppose that $\Diamond\alpha$ should be interpreted as "α is consistent", and $\alpha \prec \beta$ as "β is deducible from α".[1]

It is worth noting that the first modal axiomatization in terms of $\Box$ was proposed by Gödel in 1933 (see [10]). Gödel in fact used the symbol B (from German "Beweisbar" - provable), showing a new axiomatization for **S4** in terms of this operator, in which $\Diamond\alpha \equiv_{def} \sim B{\sim}\alpha$ while $\alpha \prec \beta \equiv_{def} B(\alpha \to \beta)$.

In 1957 (see [14]), Lemmon showed that not only **S4** but the whole Lewis hierarchy can be formalized in terms of $\Box$. By changing the primitive

[1] That is what Boolos defends in [3] p. xvii-xviii, citing [16].

operators, Lemmon proposed two more hierarchies: **D1-D5** and **E1-E5**. He also presented two systems weaker than **S1**: **S0.9** and **S0.5**. The latter, according to the author, deserves particular attention, since $\Box\alpha$ can be interpreted as "α is tautology in propositional logic", in other words, α is provable in propositional classical logic.

Even weaker systems can be defined, as proposed in [8] by Hughes and Cresswell[2], by excluding the axiom $\Box\alpha \supset \alpha$ of **S0.5** and thus obtaining $\mathbf{S0.5}^0$. This system does not seem to have received much interest in the literature, and until now a reasonable interpretation of $\Box$ has not been offered. Since $\mathbf{S0.5}^0$ extends the propositional classical calculus, if we interpret $\Box\alpha$ as "α is provable in propositional logic", we would be forced to accept $\Box\alpha \supset \alpha$ as a theorem.

However, that is not the case when the propositional fragment is sub-classical. In this paper three sub-classical logics (one paraconsistent, another paracomplete and the third paraconsistent and paracomplete) are modally extended, based on $\mathbf{S0.5}^0$, thus obtaining three systems appropriate for interpreting $\Box\alpha$ as "α is provable in classical logic". This allows the theorems of propositional classical logic to be recovered within three sub-classical modal systems.

1. Modal and non-modal languages

In this article we will deal with modal extensions of non-modal propositional logics. For this reason, this section defines the different languages that we use throughout this paper.

Let $\mathcal{V} = \{p_n \ : \ n \in \mathbb{N}\}$ be a given set of propositional variables, which are the basis of the propositional languages to be defined here. Consider now the following signatures:

- $S = \{\bot, \wedge, \vee, \supset, \equiv\}$;
- $S_\neg = S \cup \{\neg\}$;
- $S_{\neg\Box} = S_\neg \cup \{\Box\}$.

We may then define the following sets of formulas:[3]

- For, generated by S from $\mathcal{V}$;
- $For_\neg$, generated by $S_\neg$ from $\mathcal{V}$;
- $For_{\neg\Box}$, generated by $S_{\neg\Box}$ from $\mathcal{V}$;
- $\overline{For}$, generated by S from $\mathcal{V} \cup \{\star\alpha \ : \ \star \in \{\neg, \Box\} \ \text{ and } \ \alpha \in For_{\neg\Box}\}$;
- $\overline{For}_\neg$, generated by $S_\neg$ from $\mathcal{V} \cup \{\Box\alpha \ : \ \alpha \in For_{\neg\Box}\}$.

Note that, as sets, $\overline{For}_\neg$ and $For_{\neg\Box}$ coincide. As free algebras, however, they are different. The former does not have $\Box$ as an operator, and so formulas like $\Box\alpha$ are treated as propositional variables. Similarly, the sets $\overline{For}$ and $For_{\neg\Box}$ coincide, but they are different algebras: expressions like $\Box\alpha$ or $\neg\alpha$ are propositional variables in $\overline{For}$.

[2]The completeness of **S0.5** was originally demonstrated in [7]
[3]Technically speaking, they are free algebras.

2. Normal and non-normal systems

Following Segerberg, we call *normal*[4] a modal system in which the following axiom holds:

(K) $\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$

as well as the inference rule:

(N) $\dfrac{\vdash \alpha}{\vdash \Box\alpha}$

where α and β are arbitrary formulas of the language $For_{\neg\Box}$, and $\vdash \alpha$ means that α is a theorem of the system. By enriching the classical propositional calculus **PC** (defined over the language $For_{\neg}$) by the above conditions, we obtain the minimal normal system **K**. By adding to **K** the axiom

(T) $\Box\alpha \supset \alpha$

we obtain the system **KT**, commonly known as **T**.

Usually *non-normal* systems are constructed by weakening the rule N. In order to do this, Lemmon offers two alternatives:

(N') $\dfrac{\vdash \alpha \supset \beta}{\vdash \Box\alpha \supset \Box\beta}$

(N") $\dfrac{\vdash_{\overline{\mathbf{PC}}} \alpha}{\vdash \Box\alpha}$

where $\overline{\mathbf{PC}}$ is the classical propositional calculus defined over $\overline{For}_{\neg}$ and α is a formula of $\overline{For}_{\neg}$ (or, equivalently, of $For_{\neg\Box}$). By replacing the rule N in **KT** by N' or N", we obtain respectively the systems **E2** and **S0.5**. If T is excluded, the systems **C2** and $\mathbf{S0.5}^0$ will be obtained[5].

From the semantic point of view, *normal* models are triples $\mathcal{M} = \langle W, R, v\rangle$ in which W is a set of *possible worlds*, R is a relation in W and v a valuation. Kripke proved that **KT** is complete with respect to the structures $\mathcal{M} = \langle W, R\rangle$ where R is reflexive, and Segerberg proved the completeness of **K** by abandoning the reflexivity condition.

Kripke's results are not restricted to normal models. In those systems where N is replaced by N', we simply have to consider the models $\mathcal{M} = \langle W, D, R, v\rangle$ in which D is a subset of W. The elements of D are called *distinguished* worlds, while R is a relation over these elements. It can be proved that **C2** is complete with respect to the class of structures $\mathcal{M} = \langle W, D, R\rangle$, while **E2** is complete with respect to the class of such structures where R is reflexive.

Concerning **S0.5**, the condition is more specific. Take a model $\mathcal{M} = \langle W, v\rangle$ in which w^* is the only distinguished world. In this world:

$$v(\Box\alpha, w^*) = 1 \text{ iff } v(\alpha, w) = 1 \text{ for all } w \in W$$

[4]The first distinction between normal and non-normal system was proposed by Kripke in [12] and [13], but we follow the canonical separation of [17] p. 12.

[5]The system **C2** by [15] appears in [13] as $\mathbf{E2}^0$.

For the other worlds, $v(\Box\alpha, w)$ takes an arbitrary value. But in $\mathbf{S0.5}^0$ the clause above should be slightly changed to:

$$v(\Box\alpha, w^*) = 1 \text{ iff } v(\alpha, w) = 1 \text{ for all } w \in W \text{ in which } w \neq w*$$

therefore invalidating the axiom T.

As previously mentioned, according to Lemmon it is possible to interpret $\Box\alpha$ in **S0.5** as "α is provable in **CP**". From the completeness of **CP**, it is clear that if α is provable then α is the case, which shows that the presence of the axiom T is appropriate. In the following sections, suitable generalizations of $\mathbf{S0.5}^0$ based on sub-classical propositional systems will be introduced. Moreover, it will be shown that adding T to such modal systems will produce the collapse with **CP** at the propositional level.

3. Sub-classical logics extended à la $\mathbf{S0.5}^0$

There are many systems that restrict the classical propositional calculus, so any choice seems, in a sense, arbitrary. However, we chose three examples that seem to be more appropriate for several reasons. First, because these systems are examples of two large families of propositional logics: the paraconsistent and the paracomplete ones. Second, two of these systems are in some sense dual: while in one of them the axiom $\alpha \vee \neg\alpha$ does not hold, in the other one, $\alpha \supset (\neg\alpha \supset \beta)$ does not hold either. Finally, these systems have the particle $\bot$, and so a classical negation can be additionally defined. We will see that this latter phenomenon (the existence of two negations) implies two results: the ability to define multiple dual modal operators and the collapse of them in our interpretation.

Remember that For is the algebra of formulas generated by the set of connectives $\{\bot, \wedge, \vee, \supset, \equiv\}$ and by the set $\mathcal{V}$ of propositional variables (cf. Section 1).

Definition 3.1. The calculus **CLoN**[6] is defined over For by the following rules and axioms:

Axioms schemes:

(A⊃1). $\alpha \supset (\beta \supset \alpha)$
(A⊃2). $((\alpha \supset \beta) \supset \alpha) \supset \alpha$
(A⊃3). $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$
(A⊥). $\bot \supset \alpha$
(A∧1). $(\alpha \wedge \beta) \supset \alpha$
(A∧2). $(\alpha \wedge \beta) \supset \beta$
(A∧3). $\alpha \supset (\beta \supset (\alpha \wedge \beta))$
(A∨1). $\alpha \supset (\alpha \vee \beta)$
(A∨2). $\beta \supset (\alpha \vee \beta)$
(A∨3). $(\alpha \supset \gamma) \supset ((\beta \supset \gamma) \supset ((\alpha \vee \beta) \supset \gamma))$
(A≡1). $(\alpha \equiv \beta) \supset (\alpha \supset \beta)$
(A≡2). $(\alpha \equiv \beta) \supset (\beta \supset \alpha)$

[6]The systems **CLoN**, **CLuN** and **CLaN** were proposed in [2].

(A≡3). $(\alpha \supset \beta) \supset ((\beta \supset \alpha) \supset (\alpha \equiv \beta))$

Inference rule:

(MP). $\dfrac{\alpha, \alpha \to \beta}{\beta}$ □

The paraconsistent calculus **CLuN** is obtained by adding to **CLoN** the operator $\neg$ and the axiom:

$$\alpha \vee \neg\alpha$$

The paracomplete calculus **CLaN** is obtained by adding to **CLoN** the operator $\neg$ and the axiom:

$$\alpha \supset (\neg\alpha \supset \beta)$$

Finally, **PC** is obtained by adding to **CLoN** the operator $\neg$ and the two axioms above.

Remark 3.2. In [2] it is claimed that **CLoN** is the positive fragment of **PC**. This is justified by the fact that the negation $\neg$ plays no role at all in **CLoN**, and so this system could be considered as both a paraconsistent and a paracomplete logic with respect to $\neg$. However, it should be observed that propositional classical logic is in fact equivalent to **CLoN** (and so it is contained in **CLuN** and **CLaN**), since classical negation can be defined in these systems as $\sim\alpha =_{def} \alpha \supset \bot$. From this perspective, **CLoN** is just classical logic, while **CLuN** and **CLaN** are a paraconsistent and a paracomplete extension of it, respectively (with respect to $\neg$). Note that $\sim$ and $\neg$ collapse in **PC**. Since we will focus almost exclusively on the negation $\neg$, we will still consider **CLoN**, **CLuN** and **CLaN** as being sub-classical logics. On the other hand, the soundness and completeness theorems we present here can be also obtained if we start from the positive classical logic $\mathbf{PC}^+$ instead of **CLoN**. It is worth noting that in [1] the logic **PI** was considered, which is $\mathbf{PC}^+$ plus the axiom $\alpha \vee \neg\alpha$ (or, equivalently, **CLuN** without $\bot$). □

From now on, by **L** we will denote any of the systems **CLoN**, **CLuN**, **CLaN**, **PC**. Observe that while **CLoN** is defined over For, the other systems are defined over $For_\neg$. Recalling the notation from Section 1, since all these systems are structural (that is, the consequence relation is stable under substitutions), it is possible to consider the same systems, but now defined over $\overline{For}$ in the case of **CLoN** or over $\overline{For}_\neg$ (in the case of the other systems). The corresponding system defined over the extended set of formulas $For_{\neg\Box}$ will be denoted from now on by $\overline{\mathbf{L}}$. Note that the case $\mathbf{L} = \mathbf{PC}$ was already considered in Section 2. With this notation, consider now the following generalization of N”:

$$(\text{N''}_{\mathbf{L}}) \quad \frac{\vdash_{\overline{\mathbf{L}}} \alpha}{\vdash \Box\alpha}$$

It is clear that $\text{N''}_{\mathbf{PC}}$ coincides with N”.

Now, for $\mathbf{L} \in \{\mathbf{CLoN}, \mathbf{CLuN}, \mathbf{CLaN}, \mathbf{PC}\}$, consider the following modal extension of **L**, defined as follows:

- $\mathbf{CLoN0.5} = \mathbf{CLoN} + (\mathrm{K}) + (\mathrm{N}''_{\mathbf{CLoN}}) + \Box(\alpha \vee \neg\alpha) + \Box(\alpha \supset (\neg\alpha \supset \beta))$
- $\mathbf{CLuN0.5} = \mathbf{CLuN} + (\mathrm{K}) + (\mathrm{N}''_{\mathbf{CLuN}}) + \Box(\alpha \supset (\neg\alpha \supset \beta))$
- $\mathbf{CLaN0.5} = \mathbf{CLaN} + (\mathrm{K}) + (\mathrm{N}''_{\mathbf{CLaN}}) + \Box(\alpha \vee \neg\alpha)$
- $\mathbf{S0.5}^0 = \mathbf{PC} + (\mathrm{K}) + (\mathrm{N}''_{\mathbf{PC}})$

Note that $\mathbf{S0.5}^0$ was already considered in Section 2. From now on, **LM** will denote the modal extension of **L** as defined above. Observe that the language of every **LM** is generated by $S_{\neg\Box}$ over $\mathcal{V}$.

The structural consequence relation generated by the Hilbert calculi **L**, $\overline{\mathbf{L}}$ and **LM** will be denoted by $\vdash_{\mathbf{L}}$, $\vdash_{\overline{\mathbf{L}}}$ and $\vdash_{\mathbf{LM}}$, respectively (this notation was already used in order to express the inference rule $\mathrm{N}''_{\mathbf{L}}$). Note that in the case of **LM**, the rule $\mathrm{N}''_{\mathbf{L}}$ only applies to theorems of $\overline{\mathbf{L}}$, while MP can always be applied. Two important theorems are shared by all those logics:

Theorem 3.3 (Deduction Theorem).

$$\Gamma, \alpha \vdash \beta \textit{ iff } \Gamma \vdash \alpha \supset \beta$$

Proof. Simply observe that the axioms (A⊃1) and (A⊃3) are sufficient conditions to demonstrate the classical case, as proved for instance in [18], and that $\mathrm{N}''_{\mathbf{L}}$ preserves theorems, following the same argument as in [5], p. 36. □

Theorem 3.4 (Proof by Cases).

$$\textit{if } \Gamma, \alpha \vdash \beta \textit{ and } \Delta, \alpha \supset \bot \vdash \beta \textit{ then } \Gamma, \Delta \vdash \beta$$

Proof. Simply check that in **CLaN** we have $\alpha \vee (\alpha \supset \bot)$ as theorem. □

In the next section the completeness of every system **LM** with respect to (modified) Kripke models will be obtained. After this, we will show that $\Box\alpha$ can be seen as "α is proved in **PC**".

4. Completeness

The semantics of each of the systems **LM** can be obtained from the following notion of Kripke model:

Definition 4.1. A Kripke model for **CLoN0.5** is a quadruple $\langle W, D, R, v\rangle$ in which W is a set, D is a proper subset of W, $R \subseteq W \times W$ is a relation on W, and $v : For_{\neg\Box} \times W \to \{0, 1\}$ is a function that satisfies the following conditions, for all elements w of W (here, CL denotes the image of R):

1. $v(\bot, w) = 0$
2. $v(\alpha \wedge \beta\, w) = 1$ iff $v(\alpha, w) = v(\beta, w) = 1$
3. $v(\alpha \vee \beta\, w) = 0$ iff $v(\alpha, w) = v(\beta, w) = 0$
4. $v(\alpha \supset \beta, w) = 0$ iff $v(\alpha, w) = 1$ and $v(\beta, w) = 0$
5. $v(\alpha \equiv \beta, w) = 1$ iff $v(\alpha, w) = v(\beta, w)$

6. if $w \in D$ then: $v(\Box\alpha, w) = 1$ iff $v(\alpha, w') = 1$ for all $w' \in W$ such that wRw'
7. $v(\neg\alpha, w') = 1$ iff $v(\alpha, w') = 0$ for all $w' \in CL$.

Note that as a consequence of the clauses *6* and *7* we have the following property:

if $w \in D$ then: $v(\Box\neg\alpha, w) = 1$ iff $v(\alpha, w') = 0$ for all w' such that wRw'.

The elements of D are called *normal* worlds, while those of $W \setminus D$ are the *non-normal* ones. On the other hand, the worlds in CL are *classical* worlds.

Definition 4.2. A model for **CLuN0.5** is a model for **CLoN0.5** by adding:

8. $v(\alpha, w) = 0$ implies $v(\neg\alpha, w) = 1$. □

Definition 4.3. A model for **CLaN0.5** is a model for **CLoN0.5** by adding:

9. $v(\alpha, w) = 1$ implies $v(\neg\alpha, w) = 0$. □

Definition 4.4. A model for $\mathbf{S0.5}^0$ is a model for **CLuN0.5** and for **CLaN0.5**.

The (local) consequence relation $\vDash_{\mathbf{LM}}$ generated by the corresponding class of Kripke models is defined as usual, but taking into account only the normal worlds:

Definition 4.5. For $\mathbf{L} \in \{\mathbf{CLoN}, \mathbf{CLuN}, \mathbf{CLaN}, \mathbf{PC}\}$ and $\Gamma \cup \{\alpha\} \subseteq For_{\neg\Box}$, we say that α is a semantic consequence of Γ in **LM**, denoted by $\Gamma \vDash_{\mathbf{LM}} \alpha$ if, for every Kripke model for **LM** and for every $w \in D$: if $v(w, \gamma) = 1$ for every $\gamma \in \Gamma$ then $v(w, \alpha) = 1$. In particular, a formula $\alpha \in For_{\neg\Box}$ is valid in **CLoN0.5**, **CLuN0.5**, **CLaN0.5** and $\mathbf{S0.5}^0$, respectively, if and only if $v(w, \alpha) = 1$ for all $w \in D$, in all the corresponding models.

Before we show that every system **LM** is sound, it should be emphasized that all these Kripke models have in common the fact that the worlds in CL (that is, in the image of the relation R) have classical behavior.Thus, in **CLoN0.5** and **CLuN0.5** it is possible to have $v(\alpha, w) = v(\neg\alpha, w) = 1$ if $w \notin CL$, while it is impossible if $w \in CL$. On the other hand, in **CLoN0.5** and **CLaN0.5** it is possible to have $v(\alpha, w) = v(\neg\alpha, w) = 0$ if $w \notin CL$, while it is impossible if $w \in CL$. This behavior is allowed by the definition of Kripke semantics given above and by the axioms $\Box(\alpha \supset (\neg\alpha \supset \beta))$ and $\Box(\alpha \vee \neg\alpha)$, respectively. The $\mathbf{S0.5}^0$ case forces that all the elements of W have classical behavior, and so clause *7* becomes redundant.

It will be seen below that these four modal systems are sound with respect to the semantics above:

Theorem 4.6 (CLoN0.5-soundness). *Let* $\Gamma \cup \{\alpha\}$ *be a set of formulas of* $For_{\neg\Box}$. *Then:*

$$\Gamma \vdash_{\mathbf{CLoN0.5}} \alpha \textit{ implies } \Gamma \vDash_{\mathbf{CLoN0.5}} \alpha$$

Proof. The proof will be restricted to the modal axioms and the rule N"$_{\mathbf{CLoN}}$, since the other cases are analogous to classical logic.

Suppose that $v(\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta), w) = 0$ in a certain model $\mathcal{M}$, and $w \in D$. By clause *4* of Definition 4.1, this implies that $v(\Box(\alpha \supset \beta), w) = 1$ and $v(\Box\alpha \supset \Box\beta, w) = 0$. Then, by clause *4* again, $v(\Box\alpha, w) = 1$ and $v(\Box\beta, w) = 0$. Therefore, by clause *6* of Definition 4.1, $v(\alpha, w') = 1$ and $v(\beta, w') = 0$ for some $w' \in W$ such that wRw'. In the same way, from $v(\Box(\alpha \supset \beta), w) = 1$ we conclude by clauses *4* and by *6* that $v(\alpha, w') = 0$ or $v(\beta, w') = 1$, which is a contradiction. Therefore axiom K is semantically valid in **CLoN0.5**.

Assume that $v(\Box(\alpha \vee \neg\alpha), w) = 0$ for some model $\mathcal{M}$ and $w \in D$. Then, by clause *6* we know that $v(\alpha \vee \neg\alpha, w') = 0$ for some $w' \in W$ such that wRw'. Therefore, by clause *3* we have $v(\alpha, w') = v(\neg\alpha, w') = 0$, contradicting clause *7* (note that $w' \in CL$).

Now, suppose that $v(\Box(\alpha \supset (\neg\alpha \supset \beta)), w) = 0$ in a certain model $\mathcal{M}$, and $w \in D$. Thus, it exists w' such that wRw' and $v(\alpha \supset (\neg\alpha \supset \beta)), w') = 0$. Therefore, by clause *4* applied two times we have that $v(\alpha, w') = v(\neg\alpha, w') = 1$ for some $w' \in CL$, again contradicting clause *7* (since $w' \in CL$).

Finally, if α is a theorem of $\overline{\mathbf{CLoN}}$ then, by clauses *1-5* of Definition 4.1, $v(\alpha, w') = 1$ for all $w' \in W$. In particular, $v(\alpha, w') = 1$ for all $w' \in CL$ and so, by clause *6*, $v(\Box\alpha, w) = 1$ for all $w \in D$. □

Theorem 4.7 (CLuN0.5-soundness). *Let $\Gamma \cup \{\alpha\}$ be a set of formulas of $For_{\neg\Box}$. Then:*

$$\Gamma \vdash_{\mathbf{CLuN0.5}} \alpha \text{ implies } \Gamma \vDash_{\mathbf{CLuN0.5}} \alpha$$

Proof. In light of Theorem 4.6, it suffices to analyze the validity of axiom $\alpha \vee \neg\alpha$ and the rule N"$_{\mathbf{CLoN}}$.

Consider a model $\mathcal{M}$ and a $w \in D$ such that $v(\alpha \vee \neg\alpha, w) = 0$. Then, by clause *3* of Definition 4.1, it holds that $v(\alpha, w) = v(\neg\alpha, w) = 0$, which contradicts clause *8* of Definition 4.2.

Finally, if α is a theorem of $\overline{\mathbf{CLuN}}$ then, by clauses *1-5* of Definition 4.1 and by clause *8* of Definition 4.2, $v(\alpha, w') = 1$ for all $w' \in W$. In particular, $v(\alpha, w') = 1$ for all $w' \in CL$, and so, by clause *6*, $v(\Box\alpha, w) = 1$ for all $w \in D$. □

Theorem 4.8 (CLaN0.5-soundness). *Let $\Gamma \cup \{\alpha\}$ be a set of formulas of $For_{\neg\Box}$. Then:*

$$\Gamma \vdash_{\mathbf{CLaN0.5}} \alpha \text{ implies } \Gamma \vDash_{\mathbf{CLaN0.5}} \alpha$$

Proof. By using Theorem 4.6 again, it is enough to analyze the validity of axiom $\alpha \supset (\neg\alpha \supset \beta)$ and the rule N"$_{\mathbf{CLaN}}$.

Consider a model $\mathcal{M}$ and a $w \in D$ in which $v(\alpha \supset (\neg\alpha \supset \beta), w) = 0$. Then, by clause *4* of Definition 4.1 used two times, we have that $v(\alpha, w) = v(\neg\alpha, w) = 1$, which contradicts clause *9* of Definition 4.3.

The proof of the validity of the rule N"$_{\mathbf{CLaN}}$ is similar to that of Theorem 4.7. □

Theorem 4.9 ($\mathbf{S0.5^0}$-soundness). *Let $\Gamma \cup \{\alpha\}$ be a set of formulas of $For_{\neg\Box}$. Then:*

$$\Gamma \vdash_{\mathbf{S0.5^0}} \alpha \text{ implies } \Gamma \vDash_{\mathbf{S0.5^0}} \alpha$$

Proof. It is immediate, by combining the proofs of Theorem 4.7 and Theorem 4.8. In particular, the proof of the validity of the rule N"$_{\mathbf{PC}}$ is similar to that of the Theorem 4.7. □

In order to prove completeness, we must consider some definitions. From now on, "wrt" will be used to abbreviate "with respect to".

Definition 4.10. Let $\mathbf{L} \in \{\mathbf{CLoN}, \mathbf{CLuN}, \mathbf{CLaN}, \mathbf{PC}\}$, and let $\Gamma \cup \{\alpha\}$ be a set of formulas of $For_{\neg\Box}$. We say that Γ is α-saturated wrt **LM** when:

(i) $\Gamma \nvdash_{\mathbf{LM}} \alpha$;
(ii) $\Gamma \cup \{\beta\} \vdash_{\mathbf{LM}} \alpha$ for any formula β in $For_{\neg\Box}$ such that $\beta \notin \Gamma$.

Analogously, we define the notion of α-saturated sets wrt $\overline{\mathbf{L}}$. □

The important theorem below guarantees that is possible to construct an α-saturated set in all those logics.

Theorem 4.11 (Lindenbaum-Łos). *Let $\Gamma \cup \{\alpha\}$ a set of formulas of $For_{\neg\Box}$ such that $\Gamma \nvdash_{\mathbf{LM}} \alpha$. Then it is possible to extend Γ to an α-saturated set Δ wrt* **LM**. *The same result holds for $\overline{\mathbf{L}}$.*

Proof. A general proof can be found in Theorem 22.2 of [19], covering a wide class of logics, including the systems above. □

We will now look at some fundamental lemmas in order to obtain completeness.

Lemma 4.12. *Let $\mathcal{L} \in \{\overline{\mathbf{CLoN}}, \mathbf{CLoN0.5}\}$. Let Γ be a set of formulas α-saturated wrt $\mathcal{L}$, and let β, γ be formulas of $For_{\neg\Box}$. Then:*

(i) *Δ is a closed theory in $\mathcal{L}$, that is: $\Delta \vdash_{\mathcal{L}} \beta$ iff $\beta \in \Delta$;*
(ii) *$\bot \notin \Delta$;*
(iii) *$\beta \wedge \gamma \in \Delta$ iff $\beta \in \Delta$ and $\gamma \in \Delta$;*
(iv) *$\beta \vee \gamma \in \Delta$ iff $\beta \in \Delta$ or $\gamma \in \Delta$;*
(v) *$\beta \supset \gamma \in \Delta$ iff $\beta \notin \Delta$ or $\gamma \in \Delta$;*
(vi) *$\beta \equiv \gamma \in \Delta$ iff $\beta \in \Delta$ and $\gamma \in \Delta$ or $\beta \notin \Delta$ and $\gamma \notin \Delta$.*

Proof. For the clauses (i),(iii),(iv) and (v) simply check [4] p. 39.
Clause (ii): If $\bot \in \Delta$ then, by MP and $A_\bot$, it follows that $\alpha \in \Delta$, which is an absurd.
Clause (vi): If $\beta \equiv \gamma \in \Delta$ then, by (i), $A{\equiv}1$, $A{\equiv}2$ and MP, it follows that $\beta \supset \gamma \in \Delta$ and $\gamma \supset \beta \in \Delta$. So, by clause (v), we have that $\beta \in \Delta$ and $\gamma \in \Delta$ or $\beta \notin \Delta$ and $\gamma \notin \Delta$. Conversely, if $\beta \in \Delta$ and $\gamma \in \Delta$ or $\beta \notin \Delta$ and $\gamma \notin \Delta$, then, by (i), $A{\equiv}3$, (v) and MP, we have that $\beta \equiv \gamma \in \Delta$. □

Lemma 4.13. *Let $\mathcal{L} \in \{\overline{\mathbf{CLuN}}, \mathbf{CLuN0.5}\}$. Let Γ be a set of formulas α-saturated wrt $\mathcal{L}$, and let β, γ be formulas of $For_{\neg\Box}$. Then clauses (i) to (vi) of Lemma 4.12 are valid and, additionally:*

(vii) *$\beta \notin \Delta$ implies $\neg\beta \in \Delta$.*

Proof. If $\neg\beta \notin \Delta$ and $\beta \notin \Delta$, then, by (iv), $\beta \vee \neg\beta \notin \Delta$, contradicting the axiom $\beta \vee \neg\beta$, by (i). □

Lemma 4.14. *Let $\mathcal{L} \in \{\overline{\mathbf{CLaN}}, \mathbf{CLaN0.5}\}$. Let Γ be a set of formulas α-saturated wrt $\mathcal{L}$, and let β, γ be formulas of $For_{\neg\Box}$. Then clauses (i) to (vi) of Lemma 4.12 are valid and, additionally:*

(viii) *$\neg\beta \in \Delta$ implies $\beta \notin \Delta$.*

Proof. If $\neg\beta \in \Delta$ and $\beta \in \Delta$, then, by (i), MP and $\beta \supset (\neg\beta \supset \alpha)$ we would have $\alpha \in \Delta$, which is an absurd. □

Lemma 4.15. *Let $\mathcal{L} \in \{\overline{\mathbf{PC}}, \mathbf{S0.5}^0\}$. Let Γ be a set of formulas α-saturated wrt $\mathcal{L}$, and let β, γ be formulas of $For_{\neg\Box}$. Then clauses (i) to (vi) of Lemma 4.12 are valid and, additionally:*

(ix) *$\neg\beta \in \Delta$ iff $\beta \notin \Delta$.*

Proof. Immediate consequence of Lemma 4.13 and of Lemma 4.14. □

Observe that, given $\mathbf{L} \in \{\mathbf{CLoN}, \mathbf{CLuN}, \mathbf{CLaN}, \mathbf{PC}\}$ and a set $\Delta \cup \{\alpha\} \subseteq For_{\neg\Box}$, the set Δ can be α-saturated either wrt $\overline{\mathbf{L}}$ or wrt $\mathbf{LM}$. The next result shows the relationship between both situations.

Proposition 4.16. *Let $\mathbf{L} \in \{\mathbf{CLoN}, \mathbf{CLuN}, \mathbf{CLaN}, \mathbf{PC}\}$.*

(i) Let $\Delta \cup \{\alpha\}$ be a set of formulas of $For_{\neg\Box}$. If Δ is α-saturated wrt $\mathbf{LM}$ then Δ is α-saturated wrt $\overline{\mathbf{L}}$.

(ii) Let $W_{\overline{\mathbf{L}}}$ and $W_{\mathbf{LM}}$ be the following sets:

$$W_{\overline{\mathbf{L}}} = \{\Delta \subseteq For_{\neg\Box} \ : \ \Delta \ \text{ is } \alpha\text{-saturated wrt } \overline{\mathbf{L}} \text{ for some } \alpha \in For_{\neg\Box}\}, \quad \text{and}$$

$$W_{\mathbf{LM}} = \{\Delta \subseteq For_{\neg\Box} \ : \ \Delta \ \text{ is } \alpha\text{-saturated wrt } \mathbf{LM} \text{ for some } \alpha \in For_{\neg\Box}\}.$$

Then $W_{\mathbf{LM}}$ is a proper subset of $W_{\overline{\mathbf{L}}}$.

Proof. (i) Firstly, we will prove the following
Fact: Let Δ be a closed theory in $\mathbf{LM}$ such that $\Delta \nvdash_{\mathbf{LM}} \alpha$, $\Delta \nvdash_{\mathbf{LM}} \beta$ and $\Delta \cup \{\beta\} \vdash_{\mathbf{LM}} \alpha$. Then $\Delta \cup \{\beta\} \vdash_{\overline{\mathbf{L}}} \alpha$.
Proof of the Fact: By induction on the length n of a derivation of α from $\Delta \cup \{\beta\}$ in $\mathbf{LM}$. If $n = 1$ then $\alpha \in \Delta \cup \{\beta\}$, since, by hypothesis, Δ is a closed theory which does not derive α and so α cannot be an axiom of $\mathbf{LM}$. Then clearly $\Delta \cup \{\beta\} \vdash_{\overline{\mathbf{L}}} \alpha$. Suppose now that the result holds for every derivation in $k \leq n$ steps, and suppose that α is derived from $\Delta \cup \{\beta\}$ in $\mathbf{LM}$ with $n+1$ steps. As observed above, α cannot be an axiom of $\mathbf{LM}$. If $\alpha \in \Delta \cup \{\beta\}$ then $\Delta \cup \{\beta\} \vdash_{\overline{\mathbf{L}}} \alpha$. If $\alpha = \Box\gamma$ is a consequence of $\vdash_{\overline{\mathbf{L}}} \gamma$ by N"$_{\mathbf{L}}$, then $\vdash_{\mathbf{LM}} \alpha$ and so $\Delta \vdash_{\mathbf{LM}} \alpha$, a contradiction; thus, this case is impossible. Finally, suppose that α is obtained from γ and $\gamma \supset \alpha$ by MP. Clearly, one of the formulas γ

and $\gamma \supset \alpha$ cannot be derived from Δ in **LM**, otherwise α would be derived from Δ in **LM** by using MP. Suppose that γ is not a consequence of Δ in **LM**. Since γ is derivable from $\Delta \cup \{\beta\}$ in **LM** with $k \leq n$ steps, then, by induction hypothesis, $\Delta \cup \{\beta\} \vdash_{\overline{\mathbf{L}}} \gamma$. On the other hand, if γ is a consequence of Δ in **LM** then $\gamma \in \Delta$ (since Δ is a closed theory in **LM**), and so it also holds that $\Delta \cup \{\beta\} \vdash_{\overline{\mathbf{L}}} \gamma$. Analogously, it is proved that $\Delta \cup \{\beta\} \vdash_{\overline{\mathbf{L}}} \gamma \supset \alpha$. Then, by MP, it follows that $\Delta \cup \{\beta\} \vdash_{\overline{\mathbf{L}}} \alpha$. This completes the proof of the **Fact**.

Now, suppose that Δ is α-saturated wrt **LM**. Then $\Delta \nvdash_{\mathbf{LM}} \alpha$, and so $\Delta \nvdash_{\overline{\mathbf{L}}} \alpha$ (since clearly $\vdash_{\overline{\mathbf{L}}} \subseteq \vdash_{\mathbf{LM}}$). If $\beta \notin \Delta$, then $\Delta \nvdash_{\mathbf{LM}} \beta$ (since Δ is a closed theory in **LM**) and $\Delta \cup \{\beta\} \vdash_{\mathbf{LM}} \alpha$ (since Δ is α-saturated wrt **LM**). Then, by the **Fact**, $\Delta \cup \{\beta\} \vdash_{\overline{\mathbf{L}}} \alpha$. This shows that Δ is α-saturated wrt $\overline{\mathbf{L}}$.

(ii) Clearly $W_{\mathbf{LM}} \subseteq W_{\overline{\mathbf{L}}}$, by item (i). Now, let $p \in \mathcal{V}$. Observe that, in $\overline{\mathbf{L}}$, the formula $\Box(p \vee \neg p)$ is a propositional variable. Since each $\overline{\mathbf{L}}$ is sound wrt the semantics of classical bivaluations for $\overline{\mathbf{PC}}$ then $\Box(p \vee \neg p) \supset p \nvdash_{\overline{\mathbf{L}}} p$. Thus, there exists a p-saturated set Δ wrt $\overline{\mathbf{L}}$ such that $\Box(p \vee \neg p) \supset p \in \Delta$, by Theorem 4.11. Then, $\Delta \in W_{\overline{\mathbf{L}}}$. Clearly $\Box(p \vee \neg p) \notin \Delta$, otherwise $\Delta \vdash_{\overline{\mathbf{L}}} p$, by MP. But $\vdash_{\mathbf{LM}} \Box(p \vee \neg p)$, and so Δ is not a closed theory in **LM**. Therefore, by lemmas 4.12, 4.13, 4.14 and 4.15, it follows that Δ cannot be an α-saturated set wrt **LM**. This shows that $\Delta \notin W_{\mathbf{LM}}$. From this we conclude that $W_{\mathbf{LM}}$ is a proper subset of $W_{\overline{\mathbf{L}}}$. □

Definition 4.17. Let Γ be a set of formulas of $For_{\neg\Box}$. Then:

$$Den(\Gamma) =_{def} \{\alpha \in For_{\neg\Box} \ : \ \Box\alpha \in \Gamma\}$$ □

Lemma 4.18. *Let* $\mathbf{L} \in \{\mathbf{CLoN}, \mathbf{CLuN}, \mathbf{CLaN}, \mathbf{PC}\}$*. Suppose that* Δ *is a closed theory in* **LM**. *Then,* $Den(\Delta)$ *is a closed theory in* $\overline{\mathbf{L}}$.

Proof. Suppose that $Den(\Delta) \vdash_{\overline{\mathbf{L}}} \beta$. Then $\gamma_1, \ldots, \gamma_n \vdash_{\overline{\mathbf{L}}} \beta$ for some $\gamma_1, \ldots, \gamma_n \in Den(\Delta)$. By the Deduction Theorem 3.3, $\vdash_{\overline{\mathbf{L}}} \gamma_1 \supset (\ldots \supset (\gamma_n \supset \beta)\ldots)$ and then, by N"$_{\mathbf{L}}$, we get $\vdash_{\mathbf{LM}} \Box(\gamma_1 \supset (\ldots \supset (\gamma_n \supset \beta)\ldots))$. By axiom K, MP, and by transitivity of $\supset$, it follows (by induction on n) that $\vdash_{\mathbf{LM}} \Box\gamma_1 \supset (\ldots \supset (\Box\gamma_n \supset \Box\beta)\ldots))$. Using again MP, it follows that $\Box\gamma_1, \ldots, \Box\gamma_n \vdash_{\mathbf{LM}} \Box\beta$ where $\Box\gamma_1, \ldots, \Box\gamma_n \in \Delta$. Then $\Delta \vdash_{\mathbf{LM}} \Box\beta$ and so $\Box\beta \in \Delta$, since Δ is a closed theory in **LM**. Therefore $\beta \in Den(\Delta)$, as required. □

Definition 4.19 (Canonical Model). For $\mathbf{L} \in \{\mathbf{CLoN}, \mathbf{CLuN}, \mathbf{CLaN}, \mathbf{PC}\}$, the canonical model for **LM** is a quadruple $\mathcal{M}_{\mathbf{LM}} = \langle W_{\overline{\mathbf{L}}}, W_{\mathbf{LM}}, R_{\mathbf{LM}}, v_{\mathbf{LM}}\rangle$ in which $R_{\mathbf{LM}} \subseteq W_{\overline{\mathbf{L}}} \times W_{\overline{\mathbf{L}}}$ and $v_{\mathbf{LM}} : For_{\neg\Box} \times W_{\overline{\mathbf{L}}} \to \{0,1\}$ is a function satisfying the following:

1. $\Delta R_{\mathbf{LM}} \Delta'$ iff $\Delta \in W_{\mathbf{LM}}$ and $Den(\Delta) \subseteq \Delta'$;
2. $v_{\mathbf{LM}}(\alpha, \Delta) = 1$ iff $\alpha \in \Delta$, for every $\alpha \in For_{\neg\Box}$. □

Lemma 4.20 (Truth Lemma). *For* $\mathbf{L} \in \{\mathbf{CLoN}, \mathbf{CLuN}, \mathbf{CLaN}, \mathbf{PC}\}$*, the canonical model* $\mathcal{M}_{\mathbf{LM}}$ *is in fact a model for* **LM**.

Proof. We begin by observing that, by Proposition 4.16(ii), $W_{\mathbf{LM}}$ is a proper subset of $W_{\overline{\mathbf{L}}}$, and so we can take $D = W_{\mathbf{LM}}$ as a valid set of normal worlds. From this it is enough to prove that $v_{\mathbf{LM}}$ satisfies the clauses corresponding to each system.

Case **L**=**CLoN**. By Lemma 4.12 it follows that $v_{\mathbf{LM}}$ satisfies clauses *1–5* of Definition 4.1.

With respect to clause *6*, let $\Delta \in D$. Suppose that $v_{\mathbf{LM}}(\Box\beta, \Delta) = 1$. Then $\Box\beta \in \Delta$ and so $\beta \in Den(\Delta)$. Let Δ' such that $\Delta R_{\mathbf{LM}} \Delta'$. Then $\beta \in \Delta'$ and so $v_{\mathbf{LM}}(\beta, \Delta') = 1$. Conversely, suppose that $v_{\mathbf{LM}}(\Box\beta, \Delta) = 0$. Then $\Box\beta \notin \Delta$. By Lemma 4.18, $Den(\Delta) \nvdash_{\overline{\mathbf{L}}} \beta$ (otherwise $\beta \in Den(\Delta)$ and so $\Box\beta \in \Delta$, a contradiction). By Theorem 4.11, there is a set Δ' which is β-saturated wrt $\overline{\mathbf{L}}$ and extends $Den(\Delta)$. This means that there is Δ' in $W_{\overline{\mathbf{L}}}$ such that $\Delta R_{\mathbf{LM}} \Delta'$ and $\beta \notin \Delta'$, that is, $v_{\mathbf{LM}}(\beta, \Delta') = 0$.

For clause *7*, let $\Delta' \in CL$, where CL is the image of $R_{\mathbf{LM}}$. Then there exists $\Delta \in W_{\mathbf{LM}}$ such that $Den(\Delta) \subseteq \Delta'$. Since $\Delta \in W_{\mathbf{LM}}$, then $\Box(\alpha \vee \neg\alpha) \in \Delta$ and $\Box(\alpha \supset (\neg\alpha \supset \beta)) \in \Delta$ for every $\alpha, \beta \in For_{\neg\Box}$. Then $(\alpha \vee \neg\alpha) \in Den(\Delta)$ and $(\alpha \supset (\neg\alpha \supset \beta)) \in Den(\Delta)$, and so $(\alpha \vee \neg\alpha) \in \Delta'$ and $(\alpha \supset (\neg\alpha \supset \beta)) \in \Delta'$, for every $\alpha, \beta \in For_{\neg\Box}$. Therefore $v_{\mathbf{LM}}$ satisfies clause *7*.

Case **L**=**CLuN**. By Lemma 4.13 it follows that $v_{\mathbf{LM}}$ satisfies clauses *1–5* of Definition 4.1. Moreover, using the same proof as above, $v_{\mathbf{LM}}$ satisfies clauses *6–7*. By Lemma 4.13, $v_{\mathbf{LM}}$ satisfies clause *8* of Definition 4.2.

Case **L**=**CLaN**. The proof is similar to the corresponding one for **CLuN**.

Case **L**=**PC**. The proof is a combination of the two previous cases.

This completes the proof. □

Theorem 4.21 (Completeness). *Let* $\Gamma \cup \{\alpha\}$ *be a set of formulas of* $For_{\neg\Box}$. *Then:*

$$\Gamma \vDash_{\mathbf{LM}} \alpha \textit{ implies } \Gamma \vdash_{\mathbf{LM}} \alpha.$$

Proof. Suppose that $\Gamma \nvdash_{\mathbf{LM}} \alpha$. By Theorem 4.11 we can extend Γ to a set Δ which is α-saturated wrt **LM**, and then $\Gamma \subseteq \Delta$ and $\Delta \nvdash_{\mathbf{LM}} \alpha$. Let $\mathcal{M}_{\mathbf{LM}}$ be the canonical model for **LM** (cf. Definition 4.19). By Lemma 4.20 we know that $\mathcal{M}_{\mathbf{LM}}$ is a model for **LM** and $\Delta \in D$ (since $D = W_{\mathbf{LM}}$), such that $v_{\mathbf{LM}}(\gamma, \Delta) = 1$ for every $\gamma \in \Gamma$ and $v_{\mathbf{LM}}(\alpha, \Delta) = 0$, by the definition of $v_{\mathbf{LM}}$. Therefore, $\Gamma \nvDash_{\mathbf{LM}} \alpha$. □

5. □: "is provable", ◇: "is consistent"

We want to draw attention to some phenomena that occur in systems **CLoN0.5**, **CLuN0.5** and **CLaN0.5**. First of all, in those systems it is possible to define a classical negation in the following way:

$$\sim\alpha \equiv_{def} \alpha \supset \bot$$

which allows us to define four dual operators (cf. [6]):

- $\Diamond_1\alpha \equiv_{def} {\sim}\Box{\sim}\alpha$
- $\Diamond_2\alpha \equiv_{def} {\sim}\Box\neg\alpha$
- $\Diamond_3\alpha \equiv_{def} \neg\Box{\sim}\alpha$
- $\Diamond_4\alpha \equiv_{def} \neg\Box\neg\alpha$

In those logics, the partial collapse of these operators occurs, since

$$\Diamond_1\alpha \dashv\vdash \Diamond_2\alpha.$$

In order to prove this, it is enough to see that $\Box{\sim}\alpha \dashv\vdash \Box\neg\alpha$ in **LM**. Using Kripke structures, by clause *7* of Definition 4.1 it follows that every world in CL is classic and so $\sim$ and $\neg$ collapse in those worlds.

Moreover, we can prove that "$\Box\alpha$ is provable in **LM**" means "α is provable in **PC**". Before proving this result, we need to establish the following

Lemma 5.1. *Let* $\bar{t} : For_{\neg\Box} \to For_{\neg}$ *be a function such that* $\bar{t}(\alpha)$ *is the formula of* $For_{\neg}$ *obtained from* α *by eliminating all the occurrences of the symbol* $\Box$ *in* α*. Then, for each* $\alpha \in For_{\neg\Box}$*, if* $\vdash_{\mathbf{LM}} \alpha$ *then* $\vdash_{\mathbf{PC}} \bar{t}(\alpha)$*.*

Proof. By induction on the length of a proof of α in **LM**.
If α is a instance in $For_{\neg\Box}$ of an **L**-axiom, then $\bar{t}(\alpha)$ is an instance in $For_{\neg}$ of an **L**-axiom, therefore $\vdash_{\mathbf{PC}} \bar{t}(\alpha)$.
If α is an instance of axiom $\Box(\beta \supset (\neg\beta \supset \gamma))$ or $\Box(\beta \vee \neg\beta)$, then in both cases $\vdash_{\mathbf{PC}} \bar{t}(\alpha)$.
If α is an instance of axiom K then $\bar{t}(\alpha)$ is a formula of the form $\beta \to \beta$, and therefore $\vdash_{\mathbf{PC}} \bar{t}(\alpha)$.
Suppose that α is derived from $\beta \supset \alpha$ and β by MP. Then $\vdash_{\mathbf{PC}} \bar{t}(\beta) \supset \bar{t}(\alpha)$ and $\vdash_{\mathbf{PC}} \bar{t}(\beta)$, by induction hypothesis and by the definition of $\bar{t}$. From here, $\vdash_{\mathbf{PC}} \bar{t}(\alpha)$, byMP in **PC**.
Finally, suppose that α is of the form $\Box\beta$ and is obtained from $\vdash_{\overline{\mathbf{L}}} \beta$ by rule N"$_{\mathbf{L}}$. This means that β is a theorem of $\overline{\mathbf{L}}$ (and, therefore, of $\overline{\mathbf{PC}}$). By the definition of $\overline{\mathbf{PC}}$, subformulas γ of β of the form $\Box\delta$ are propositional variables (then every subformula of γ is ignored, including subformulas of the form $\Box\varphi$). This being so, $\bar{t}(\beta)$ is obtained from β by substituting every propositional variable γ of the form $\Box\delta$ by $\bar{t}(\gamma)$, which is a formula of $For_{\neg}$. Therefore, by structurality of **PC**, $\bar{t}(\beta)$ is a theorem of **PC**, that is, $\vdash_{\mathbf{PC}} \bar{t}(\beta)$. Thus $\vdash_{\mathbf{PC}} \bar{t}(\alpha)$, because $\bar{t}(\alpha) = \bar{t}(\beta)$. □

Theorem 5.2. *Let* $\alpha \in For_{\neg}$*. Then:*

$$\vdash_{\mathbf{PC}} \alpha \textit{ iff } \vdash_{\mathbf{LM}} \Box\alpha$$

Proof.
($\Rightarrow$) The proof will be done by induction on the length of a derivation of α in **PC**.
If α is an axiom of **L**, then $\vdash_{\overline{\mathbf{L}}} \alpha$. Therefore $\vdash_{\mathbf{LM}} \Box\alpha$, by N"$_{\mathbf{L}}$.
Suppose that $\alpha = \beta \vee \neg\beta$ or $\alpha = \beta \supset (\neg\beta \supset \gamma)$. As we know that $\Box(\beta \vee \neg\beta)$ and $\Box(\beta \supset (\neg\beta \supset \gamma))$ are theorems of **LM**, then $\vdash_{\mathbf{LM}} \Box\alpha$.
Suppose that α is obtained in **PC** from formulas $\beta \supset \alpha$ and β by MP. From

here, $\vdash_{\mathbf{PC}} \beta \supset \alpha$ and $\vdash_{\mathbf{PC}} \beta$. Therefore, by induction hypothesis, $\vdash_{\mathbf{LM}} \Box(\beta \supset \alpha)$ and $\vdash_{\mathbf{LM}} \Box\beta$. So, by K and MP, we have $\vdash_{\mathbf{LM}} \Box\alpha$.
($\Leftarrow$) Suppose that $\vdash_{\mathbf{LM}} \Box\alpha$, for $\alpha \in For_{\neg}$. By Lemma 5.1, $\vdash_{\mathbf{PC}} \bar{t}(\Box\alpha)$. That is, $\vdash_{\mathbf{PC}} \alpha$, because in this case $\bar{t}(\Box\alpha) = \alpha$. □

Another interpretation of the above theorem is the following. Consider the function $t : For_{\neg} \to For_{\neg\Box}$ given by $t(\alpha) = \Box\alpha$. Then t is a translation between **PC** and **LM**. Here, the notion of translation coincides with that of Glivenko in [9], which defines a transformation t from the language of **PC** into the language of intuitionistic propositional logic **IPC**, proving that $\vdash_{\mathbf{PC}} \alpha$ iff $\vdash_{\mathbf{IPC}} t(\alpha)$.

We say that a formula is consistent in **LM** if there is a model $\langle W, D, R, v\rangle$ for **LM** and a normal world $w \in D$ such that $v(\alpha, w) = 1$. If the provability of $\Box\alpha$ in **LM** can be seen as the provability of α in **PC**, its dual "$\Diamond\alpha$ is consistent in **LM**" should be interpreted as "α is consistent in **PC**". The following theorem shows that this interpretation is correct.

Theorem 5.3. *Let $\alpha \in For_{\neg}$. Then: for $i = 1, 2$*

$\Diamond_i\alpha$ is consistent in **LM** *iff α is consistent in* **PC**

Proof. If α is consistent in **PC** then $\nvdash_{\mathbf{PC}} \alpha \supset \bot$, and so $\nvdash_{\mathbf{LM}} \Box(\alpha \supset \bot)$, by Theorem 5.2. By the definition of $\Diamond_i$ and the fact that $\sim\sim\beta$ is equivalent to β in **LM**, it follows that $\nvdash_{\mathbf{LM}} \sim\Diamond_i\alpha$. This means that $v(\Diamond_i\alpha, w) = 1$ for some model $\langle W, D, R, v\rangle$ of **LM** and for some $w \in D$. That is, $\Diamond_i\alpha$ is consistent in **LM**.
Suppose now that α is inconsistent in **PC**. Thus $\vdash_{\mathbf{PC}} \alpha \supset \bot$ and then, by Theorem 5.2, $\vdash_{\mathbf{LM}} \Box(\alpha \supset \bot)$. From here $\vdash_{\mathbf{LM}} \sim\Diamond_i\alpha$. This means that $\Diamond_i\alpha$ is inconsistent in **LM**. □

Consider now the following extensions of **CLoN**, **CluN** and **CLaN**:

- **CLoN0.5*** = **CLoN0.5** + $\neg\Box\alpha \equiv \sim\Box\alpha$
- **CLuN0.5*** = **CLoN0.5** + $\neg\Box\alpha \equiv \sim\Box\alpha$
- **CLaN0.5*** = **CLoN0.5** + $\neg\Box\alpha \equiv \sim\Box\alpha$

From the semantic point of view, we just have to add the following clause to the models:

10. $v(\neg\Box\alpha, w) = 1$ *iff* $v(\Box\alpha, w) = 0$.

It is easy to see that in these logics we have a complete collapse of the four operators:

$$\Diamond_1\alpha \dashv\vdash \Diamond_2\alpha \dashv\vdash \Diamond_3\alpha \dashv\vdash \Diamond_4\alpha$$

Moreover, Theorem 5.3 above can be generalized to $i \in \{1, 2, 3, 4\}$. Verbally expressed, the consistence of $\neg\Box\alpha$ means that "α is not a theorem of **PC**" and the validity of $\neg\Diamond\alpha$ means that "α is inconsistent in **PC**" or "α is contradictory in **PC**". Clearly, if we add $\Box\alpha \supset \alpha$ to those systems, we have an immediate collapse of **CLoN0.5**, **CLoN0.5***, **CluN0.5**, **CLuN0.5***, **CLaN0.5** and **CLaN0.5*** with **S0.5**. This shows that **S0.5**^{0} is the appropriate logic in order to interpret $\Box$ as "is provable in **PC**" and $\Diamond$ as "is consistent in **PC**".

Final Remarks

The above observations intend, on the one hand, to find an application for **S0.5**^{0}, a logic which seems to have been neglected in the literature. On the other hand, the three basic systems adopted here (cf. Section 3) suggest a possible generalization: let **L** be a logic between **CLoN** and **PC** and let α_1, α_2, ..., α_n be axioms which hold in **PC** but do not hold in **L**. Therefore, adding to **L** the axioms $\Box\alpha_1$, $\Box\alpha_2$, ..., $\Box\alpha_n$, K and the rule N"$_{\mathbf{L}}$, we can interpret $\Box$ as "is provable in **PC**" and $\Diamond$ as "is consistent in **PC**" in the sense of theorems 5.2 and 5.3. The fact that general issues of a given logic such as provability and logical consequence can be analyzed from this perspective connects the present research with the fields of Universal Logic and Multimodal Logics.

A wide family of logics in which these concepts could be applied is the class of **LFI**'s – the Logics of Formal Inconsistency – that internalize in the object language the notions of "consistency" and "inconsistency" by means of the operators $\circ$ and $\bullet$ (cf. [4]). For these logics, it would suffice to add $\Box\circ\alpha$ as an axiom, obtaining the same results described above. Another question is how to establish in this interpretation the exact relationship between $\circ$ and $\Diamond$.

The problem of verifying the viability of obtaining the same results in logics in which it is not possible to characterize two negations, such as **IPC** (intuitionistic propositional logic) or fragments of **PC** without $\bot$ (such as **PC**$^{\supset}$ or **PI**, cf. [1]), remains open.

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