# Hilbert-style Presentations of Two Logics Associated to Tetravalent Modal Algebras

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**Abstract**

We analyze the variety of A. Monteiro's tetravalent modal algebras under the perspective of two logic systems naturally associated to it. Taking profit of the contrapositive implication introduced by A. Figallo and P. Landini, sound and complete Hilbert-style calculi for these logics are presented.

## 1 Introduction and Preliminaries

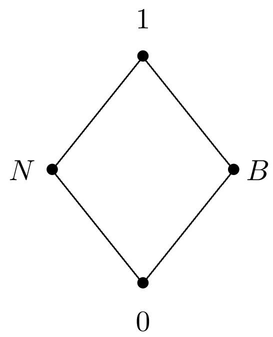
The class **TMA** of tetravalent modal algebras was first considered by Antonio Monteiro, and mainly studied by I. Loureiro, A.V. Figallo, A. Ziliani and P. Landini. Later on, J.M. Font and M. Rius were interested in the logics arising from the algebraic and lattice-theoretical aspects of these algebras. They introduced a sequent calculus (for one of these logics) whose associated propositional logic coincides with the one defined by the matrix formed by the four-element TMA and one of its two prime filters, and to which we refer as *Tetravalent Modal Logic* ($\mathcal{TML}$). Independently, in [4] it was introduced a Hilbert-style propositional calculus for other logic associated to the variety **TMA** called $\mathcal{TML}^N$ (see Section 7). This calculus belongs to the class of standard systems of implicative extensional propositional calculi, but it has the disadvantage of having two implications and so many axioms.

This paper retakes the question of studying the logical aspects of TMAs. By considering the contrapositive implication introduced by A. Figallo and

P. Landini in [7], we introduce Hilbert calculi for two logics naturally associated to TMAs, giving a solution to a problem posed in [7]. Finally, it will be shown that both logics are contained in the propositional classical logic, but they are not maximal sublogics of it. Since $\mathfrak{M}_{4m}$, the four-element TMA described below, generates the variety **TMA**, all the study will be done based on this simple but extremely rich structure. In the rest of this section we recall basic notions of TMAs.

A *De Morgan algebra* is a structure $\mathfrak{A} = \langle A, \wedge, \vee, \neg, 0\rangle$ where $\langle A, \wedge, \vee, 0\rangle$ is a distributive lattice with smallest element 0 and $\neg$ is a De Morgan involution, i.e., an involution that additionally satisfies De Morgan's laws: $\neg\neg x = x$ and $\neg(x \vee y) = \neg x \wedge \neg y$.

A *tetravalent modal algebra* (TMA) is an algebra $\mathfrak{A} = \langle A, \wedge, \vee, \neg, \Box, 0\rangle$ of type $(2, 2, 1, 1, 0)$ such that its non-modal reduct $\mathfrak{A}^- = \langle A, \wedge, \vee, \neg, 0\rangle$ is a De Morgan algebra and the unary operation $\Box$ satisfies the following two axioms: $x \vee \neg\Box x = 1$, and $\Box x \vee \neg x = x \vee \neg x$. The class of all tetravalent modal algebras constitute a variety which is denoted by **TMA**. Besides, **TMA** is generated by the four-element algebra $\mathfrak{M}_{4m} = \langle M_4, \wedge, \vee, \neg, \Box, 0\rangle$ where $M_4 = \{1, N, B, 0\}$ is given by




such that $\neg 1 = 0$, $\neg 0 = 1$ and $\neg x = x$ otherwise. The unary operator $\Box$ is defined as $\Box x = 1$ if $x = 1$, and $\Box x = 0$ otherwise. It is well-known that $\mathfrak{M}_{4m}$ generates the variety **TMA**, i.e., an equation holds in every TMA iff it holds in $\mathfrak{M}_{4m}$. From this, it can be defined a propositional modal logic associated with the class of TMAs or, equivalently, associated to $\mathfrak{M}_{4m}$ (see definitions 1.1 and 1.2 below). It is worth noting that there are several ways to relate a logic to a given class of algebras (cf.[11]). However, in this paper we will concentrate mainly on the study of TMAs under the logical perspective of Definition 1.1. In Section 7 we shall briefly analyze another logic naturally associated to TMAs (cf. Definition 7.1).

From now on, we shall denote by $\mathfrak{Fm} = \langle Fm, \wedge, \vee, \neg, \Box, \bot\rangle$ the absolutely free algebra of type (2,2,1,1,0) generated by a set $Var$ of variables. Consider now the following two logics:

**Definition 1.1.** *The logic of the variety* **TMA** *defined over* $\mathfrak{Fm}$ *is the propositional logic* $\mathbb{L}_{TMA} = \langle Fm, \models_{TMA}\rangle$ *given as follows: for every set* $\Gamma \cup \{\alpha\} \subseteq Fm$, $\Gamma \models_{TMA} \alpha$ *if and only if, there is some finite set* $\Gamma_0 \subseteq \Gamma$ *such that, for every* $\mathfrak{A} \in$ **TMA** *and for every* $h \in Hom(\mathfrak{Fm}, \mathfrak{A})$, $\bigwedge\{h(\gamma) \ : \ \gamma \in \Gamma_0\} \leq h(\alpha)$. *In particular,* $\emptyset \models_{TMA} \alpha$ *if and only if* $h(\alpha) = 1$ *for every* $\mathfrak{A} \in$ **TMA** *and for every* $h \in Hom(\mathfrak{Fm}, \mathfrak{A})$.

**Definition 1.2.** *The tetravalent modal logic* $M_{4m}$ *is the propositional logic* $M_{4m} = \langle Fm, \models_{M_{4m}}\rangle$ *given as follows: for every set* $\Gamma \cup \{\alpha\} \subseteq Fm$, $\Gamma \models_{M_{4m}} \alpha$ *if and only if, there is some finite set* $\Gamma_0 \subseteq \Gamma$ *such that, for every* $h \in Hom(\mathfrak{Fm}, \mathfrak{M}_{4m})$, $\bigwedge\{h(\gamma) \ : \ \gamma \in \Gamma_0\} \leq h(\alpha)$. *In particular,* $\emptyset \models_{M_{4m}} \alpha$ *if and only if* $h(\alpha) = 1$ *for every* $h \in Hom(\mathfrak{Fm}, \mathfrak{M}_{4m})$. *In this case, we say that* $\alpha$ *is valid in* $M_{4m}$.

Since $\mathfrak{M}_{4m}$ generates the variety **TMA**, it is immediate to prove:

**Proposition 1.3.** *The logic* $M_{4m}$ *coincides with the logic* $\mathbb{L}_{TMA}$. *That is: for every set* $\Gamma \cup \{\alpha\} \subseteq Fm$, $\Gamma \models_{M_{4m}} \alpha$ *if and only if* $\Gamma \models_{TMA} \alpha$.

There exist other presentations for the logic $M_{4m}$, namely $\mathbb{L}_{4m}(\mathfrak{Fm})$, $\mathcal{TML}$ and $\mathcal{M}_N$, as we shall see in sections 2 and 3.

## 2 Other presentations for the logic $M_{4m}$

An interesting and deep analysis of the tetravalent modal logics associated to **TMA** in terms of abstract algebraic logic was proposed by J. Font and M. Rius in [9]. Within that framework, the wide notions of *quasi tetravalent modal logic* and *tetravalent modal logic* were introduced. Tetravalent modal logic $M_{4m}$ appears as a particular case, and they showed that it can be characterized as a matrix logic. Specifically, consider the propositional logic $\mathbb{L}_{4m}(\mathfrak{Fm})$ defined by the family of two matrices $\langle \mathfrak{M}_{4m}, \{N, 1\}\rangle$ and $\langle \mathfrak{M}_{4m}, \{B, 1\}\rangle$ over $\mathfrak{M}_{4m}$. Its consequence relation is defined as usual: for every set $\Gamma \cup \{\alpha\} \subseteq Fm$, $\Gamma \models_{\mathbb{L}_{4m}(\mathfrak{Fm})} \alpha$ if and only if there is some finite set $\Gamma_0 \subseteq \Gamma$ such that

1. for every $h \in Hom(\mathfrak{Fm}, \mathfrak{M}_{4m})$, $h(\Gamma_0) \subseteq \{N, 1\}$ implies $h(\alpha) \in \{N, 1\}$;

2. for every $h \in Hom(\mathfrak{Fm}, \mathfrak{M}_{4m})$, $h(\Gamma_0) \subseteq \{B, 1\}$ implies $h(\alpha) \in \{B, 1\}$.

As it was proved in [9], the logic $\mathbb{L}_{4m}(\mathfrak{Fm})$ coincides with $M_{4m}$:

**Theorem 2.1** ([9])**.** *For every set* $\Gamma \cup \{\alpha\} \subseteq Fm$,

$$\Gamma \models_{M_{4m}} \alpha \quad \textit{if and only if} \;\; \Gamma \models_{\mathbb{L}_{4m}(\mathfrak{Fm})} \alpha.$$

This allows a characterization of $M_{4m}$ as a matrix logic defined in terms of two logical matrices. It is worth noting that, since $\langle \mathfrak{M}_{4m}, \{N, 1\} \rangle$ and $\langle \mathfrak{M}_{4m}, \{B, 1\} \rangle$ are isomorphic, $\mathbb{L}_{4m}(\mathfrak{Fm})$ (and therefore $M_{4m}$) can be characterized as a matrix logic in terms of a single logical matrix. In Proposition 3.2 we will give a direct proof of the characterization of $M_{4m}$ by means of a single logical matrix, without using Theorem 2.1.

In order to characterize $M_{4m}$ syntactically, that is, by means of a syntactical deductive system, it was introduced also in [9] a sequent calculus called $\mathfrak{G}$. This generates a propositional logic $\mathcal{TML} = \langle Fm, \vdash_{\mathcal{TML}} \rangle$ defined as follows: for every set $\Gamma \cup \{\varphi\} \subseteq Fm$, $\Gamma \vdash_{\mathcal{TML}} \varphi$ iff there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that the sequent $\Gamma_0 \vdash \varphi$ has a derivation in $\mathfrak{G}$. It was proved that the sequent calculus $\mathfrak{G}$ is sound and complete with respect to the tetravalent modal logic $M_{4m}$, constituting therefore a proof-theoretic counterpart of it (cf. [9]).

## 3 Remarks on $M_{4m}$ as a matrix logic

We start our investigation by obtaining some results about $M_{4m}$ (seen as a matrix logic) that will be useful in the sequel.

**Lemma 3.1.** *Let* $h : \mathfrak{Fm} \to \mathfrak{M}_{4m}$, $\mathcal{V}' \subseteq Var$ *and* $h' : \mathfrak{Fm} \to \mathfrak{M}_{4m}$ *such that, for all* $p \in \mathcal{V}'$,

$$h'(p) = \begin{cases} h(p) & \textit{if } h(p) \in \{0,1\}, \\ N & \textit{if } h(p) = B, \\ B & \textit{if } h(p) = N \end{cases} \quad . \; \textit{Then } h'(\alpha) = \begin{cases} h(\alpha) & \textit{if } h(\alpha) \in \{0,1\}, \\ N & \textit{if } h(\alpha) = B, \\ B & \textit{if } h(\alpha) = N, \end{cases}$$

*for all* $\alpha \in \mathfrak{Fm}$ *whose variables are in* $\mathcal{V}'$.

**Proof.** The result is easily proved by induction on the length of $\alpha$. ■

From this result we can give an easy direct proof of the characterization of the logic $M_{4m}$ through a single logical matrix, without making use of Theorem 2.1.

**Proposition 3.2.** *Let* $\mathcal{M}_N = \langle \mathfrak{M}_{4m}, \{N, 1\}\rangle$ *and* $\mathcal{M}_B = \langle \mathfrak{M}_{4m}, \{B, 1\}\rangle$ *be the logical matrices of Section 2. Then,* $\models_{M_{4m}} = \models_{\mathcal{M}_N} = \models_{\mathcal{M}_B}$.

**Proof.** Since $\models_{M_{4m}}$ is a finitary consequence relation and because of the presence of conjunction (infimum) $\wedge$ in $M_{4m}$, it is enough to consider inferences in $M_{4m}$ of the form $\alpha \models_{M_{4m}} \beta$ (in case $\beta$ is a theorem it is enough to consider $\alpha$ as $\neg\bot$).

Suppose that $\alpha \models_{M_{4m}} \beta$ and let $h \in Hom(\mathfrak{Fm}, \mathfrak{M}_{4m})$ such that $h(\alpha) \in \{1, N\}$. Since $h(\alpha) \leq h(\beta)$, we have that $h(\beta) \in \{1, N\}$. Then, $\alpha \models_{\mathcal{M}_N} \beta$. Conversely, suppose that $\alpha \models_{\mathcal{M}_N} \beta$ and let $h \in Hom(\mathfrak{Fm}, \mathfrak{M}_{4m})$. If $h(\alpha) = 0$ then $h(\alpha) \leq h(\beta)$. If $h(\alpha) = N \in \{1, N\}$ then $h(\beta) \in \{1, N\}$ and, therefore, $h(\alpha) \leq h(\beta)$. If $h(\alpha) = B$, let $h'$ as in the above lemma. Then, $h'(\alpha) = N \in \{1, N\}$ and so $h'(\beta) \in \{1, N\}$. Thus, $h(\beta) \in \{1, B\}$ and so $h(\alpha) \leq h(\beta)$. If $h(\alpha) = 1$ then $h(\beta) \in \{1, N\}$. If $h(\beta) = N$ then, $h'(\alpha) = 1$ and $h'(\beta) = B \notin \{1, N\}$, a contradiction. Therefore, $h(\beta) = 1$ and so $h(\alpha) \leq h(\beta)$. In every case we show that $\alpha \models_{M_{4m}} \beta$. The proof of the other equality is analogous. ■

In a logic, a functionally complete set of logical connectives is one which can be used to express all possible truth-tables over a given sound and complete matrix semantics. We say that a logic is *functionally complete* if it contains a functionally complete set of logical connectives. Next, we shall see that in $M_{4m}$ the set $\{\vee, \wedge, \Box, \neg, \bot\}$ is not functionally complete.

By observing that both $\{0, 1, N\}$ and $\{0, 1, B\}$ are subalgebras of $\mathfrak{M}_{4m}$, the following result is immediate:

**Proposition 3.3.** *In* $\mathfrak{M}_{4m}$ *it is not possible to define the classical (Boolean) negation* $-$ *such that* $-0 = 1$, $-1 = 0$, $-N = B$ *and* $-B = N$.

**Corollary 3.4.** *In* $\mathfrak{M}_{4m}$ *the set* $\{\vee, \wedge, \Box, \neg, \bot\}$ *is not functionally complete.*

**Remark 3.5.** *The De Morgan reduct of* $\mathfrak{M}_{4m}$ *coincides with the reduct of the algebraic structure* $FOUR$ *behind the well-known 4-valued Belnap's logic, namely, the reduct given by the truth-order instead of the knowledge-order (cf. [2, 1]). Moreover,* $M_{4m}$ *presented as* $\mathcal{M}_N$ *is a modal extension of the matrix logic associated to this reduct of* $FOUR$.

*On the other hand, it is immediate that the reduct of* $\mathcal{M}_N$ *to the signature* $\wedge$, $\vee$, $\Box$, $\bot$ *coincides with the four-valued modal logic PM4 introduced in [3]. From Proposition 3.2, it follows that the negation-free reduct of* $M_{4m}$ *coincides with PM4 (as consequence relations).*

*From Proposition 3.3 it follows that $M_{4m}$ does not coincide with Łukasiewicz four-valued modal logic (see [8]), since the latter is a modal extension of classical logic.*

## 4 The contrapositive implication in $M_{4m}$

The original language of the logic of TMAs – in particular, the language of logic $M_{4m}$– does not have an implication as a primitive connective. It is a natural question to ask how to define a deductive implication in TMAs, in terms of the other operators.

Some proposal for an implication operator in TMAs appeared in the literature. I. Loureiro proposed in [13] the following implication for TMAs: $x \to y = \neg\Box x \vee y$. On the other hand, A. Figallo and P. Landini introduced in [7] an interesting implication operator for TMAs that can be defined as follows: $x \succ y = (x \to y) \wedge (\neg y \to \neg x) \wedge ((\neg x \vee y) \to (\Box\neg x \vee y))$.

The main feature of the contrapositive implication is that it internalizes the consequence relation whenever just one premise is considered. Another important aspect of the contrapositive implication is that all the operations of the TMAs can be defined in terms of $\succ$ and 0 (cf. [7]). In fact: $1 = (0 \succ 0)$; $\neg x = (x \succ 0)$; $x \vee y = (x \succ y) \succ y$; $x \wedge y = \neg(\neg x \vee \neg y)$; $\Box x = \neg(x \succ \neg x)$. Additionally, an axiomatization for TMAs was given in [7] in terms of $\succ$ and 0 as follows.

**Proposition 4.1** (cf. [7]). *In every TMA it can be defined a binary operation $\succ$ such that the following holds:*

(C1) $(1 \succ x) = x$,

(C2) $(x \succ 1) = 1$,

(C3) $(x \succ y) \succ y = (y \succ x) \succ x$,

(C4) $x \succ (y \succ z) = 1$ *implies* $y \succ (x \succ z) = 1$,

(C5) $((x \succ (x \succ y)) \succ x) \succ x = 1$,

(C6) $(0 \succ x) = 1$,

(C7) $((x \vee y) \succ z) \succ ((x \succ z) \wedge (y \succ z)) = 1$.

*Conversely, if an algebra with a binary operation* $\succ$ *and an element* $0$ *satisfies (C1)-(C7), in which* $1 =_{def} 0 \succ 0$, $x \vee y =_{def} (x \succ y) \succ y$, $\neg x =_{def} x \succ 0$, $x \wedge y =_{def} \neg(\neg x \vee \neg y)$ *and* $\Box x =_{def} \neg(x \succ \neg x)$, *then the resulting structure is a TMA.*

**Definition 4.2.** *A* contrapositive tetravalent modal algebra *is an algebra* $\langle A, \succ, 0\rangle$ *of type* $(2, 0)$ *that satisfies items (C1)-(C7) of Proposition 4.1 (with the abbreviations defined therein). We shall denote the class of these algebras by* $\mathbf{TMA}^c$.

Let $\mathfrak{Fm}^c = \langle Fm^c, \succ, \bot\rangle$ be the absolutely free algebra of type (2,0) generated by the set of variables $Var$, where $Fm^c$ is the set of formulas. This algebra constitutes the formal language of $\mathbf{TMA}^c$.

Let $\mathfrak{M}^c_{4m} = \langle M_4, \succ, 0\rangle$. Observe that the contrapositive implication $\succ$ has the following truth-table in $\mathfrak{M}^c_{4m}$:

| $\succ$ | 0 | N | B | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| N | N | 1 | B | 1 |
| B | B | N | 1 | 1 |
| 1 | 0 | N | B | 1 |

It is worth noting that the classes $\mathbf{TMA}$ and $\mathbf{TMA}^c$ are termwise equivalent. The main differences reside in the underlying language defining both classes and the fact that the characterization of the latter does not allow to see (because of (C4)) that in fact it is a variety. As we shall see in Section 6, the language $\mathfrak{Fm}^c$ of TMA$^c$s is suitable to define a Hilbert-style presentation of tetravalent modal logic $M_{4m}$.

**Remark 4.3.** *Clearly, the variety* $\mathbf{TMA}^c$ *is generated by the four-element algebra* $\mathfrak{M}^c_{4m}$. *Being so, the logic* $TMA^c$ *of the contrapositive tetravalent modal algebras and the tetravalent modal logic* $M^c_{4m}$, *both defined over the language* $\mathfrak{Fm}^c$ *by analogy with definitions 1.1 and 1.2, coincide. Of course Proposition 3.2 can be adapted to this language and so* $M^c_{4m}$ *coincides with the matrix* $\mathcal{M}^c_N = \langle \mathfrak{M}^c_{4m}, \{N, 1\}\rangle$. *The reader should have in mind that* $M_{4m}$ *(*$= \mathcal{M}_N$*) and* $M^c_{4m}$ *(*$= \mathcal{M}^c_N$*) are the same logic, but presented in different signatures.*

In $M^c_{4m}$ we have a weak version of the *Deduction Metatheorem* with respect to the contrapositive implication.

**Theorem 4.4** ([7])**.** *Let $\alpha, \beta \in Fm^c$. Then: $\alpha \models_{M_{4m}^c} \beta$ iff $\models_{M_{4m}^c} \alpha \succ \beta$.*

The last result shows that the contrapositive implication $\succ$ internalizes the consequence relation of $M_{4m}^c$ whenever just one premise is considered. In algebraic terms, $\succ$ internalizes the partial order $\leq$ of TMAs.

It is worth noting that it is not possible to improve Theorem 4.4 in the following sense:

**Proposition 4.5.** *In $M_{4m}^c$ both directions of the deduction metatheorem, with respect to $\succ$, fail if more than one premise are allowed. Specifically:*

(i) $\alpha, \beta \models_{M_{4m}^c} \gamma$ *does not imply that* $\alpha \models_{M_{4m}^c} \beta \succ \gamma$,

(ii) $\alpha \models_{M_{4m}^c} \beta \succ \gamma$ *does not imply that* $\alpha, \beta \models_{M_{4m}^c} \gamma$.

**Proof.** (i) Consider $\alpha = (\bullet p \wedge \bullet q \wedge \bullet(p \succ q) \wedge p)$, $\beta = q$ and $\gamma = \bot$, where $p, q$ are two different propositional variables, and $\bullet\delta =_{def} \Diamond(\delta \wedge \neg\delta)$ is the *inconsistency* operator (see Section 5 below). Then $h(\alpha \wedge \beta) = 0 = h(\gamma)$ for every $h \in Hom(\mathfrak{Fm}^c, \mathfrak{M}_{4m}^c)$. That is, $\alpha, \beta \models_{M_{4m}^c} \gamma$. Now, let $h$ such that $h(p) = N$ and $h(q) = B$. Then $h(\alpha) = N$ and $h(\beta) = B$ and so $N = h(\alpha) \not\leq h(\beta \succ \gamma) = B \succ 0 = B$. Therefore, $\alpha \not\models_{M_{4m}^c} \beta \succ \gamma$.

(ii) Consider $\alpha = p$, $\beta = \neg p$ and $\gamma = \bot$, where $p$ is a propositional variable. Then $\alpha \models_{M_{4m}^c} \beta \succ \gamma$, since $\alpha \models_{M_{4m}^c} \neg\neg\alpha$. Let $h \in Hom(\mathfrak{Fm}^c, \mathfrak{M}_{4m}^c)$ such that $h(p) = N$. Then $h(\alpha \wedge \beta) = N \not\leq 0 = h(\gamma)$ and so $\alpha, \beta \not\models_{M_{4m}^c} \gamma$. ■

## 5 Some logical aspects of $M_{4m}^c$

In this section some logical aspects of $M_{4m}^c$ seen as a modal, paraconsistent and paracomplete logic, will be briefly discussed.

The logic $M_{4m}^c$, seen as a modal logic, has some nice properties: it satisfies the modal axioms **(K)**, **(4)**, **(T)**, **(B)**, **(D)** and **(.3)** (see [9]). Thus, it satisfies all the modal axioms of the classical modal logic **S5**. Nevertheless, we can not affirm that $M_{4m}^c$ is a normal modal logic since the implication $\succ$ does not satisfy some properties of the classical implication. Additionally, by setting $\Box^0\alpha =_{def} \alpha$ and $\Box^{n+1}\alpha =_{def} \Box^n\Box\alpha$ for any $n \in \mathbb{N}$ (and similarly for $\Diamond$), the following well-known instance of the Lemmon-Scott schemes (cf. [12]) holds in $M_{4m}^c$, for any $n, l, k, m \in \mathbb{N}$: $\models_{M_{4m}^c} \Diamond^k\Box^l\alpha \succ \Box^m\Diamond^n\alpha$. On the other hand, $\not\models_{M_{4m}^c} \Box\Diamond\alpha \succ \Diamond\Box\alpha$, as it can be easily checked.

There exist interesting similarities between Łukasiewicz's Ł$_3$ (seen as a modal logic) and $M^c_{4m}$. In both logics, $\Box\alpha$ and $\Diamond\alpha$ are defined by the same formulas in the respective languages, namely $\neg(\alpha \succ \neg\alpha)$ and $\neg\alpha \succ \alpha$, respectively. Moreover, both implications (Ł$_3$'s implication and contrapositive implication) do not satisfy the contraction law: $\alpha \succ (\alpha \succ \beta)$ is not equivalent to $(\alpha \succ \beta)$. From this, both logics satisfy the following modal principle: $\alpha \succ (\alpha \succ \Box\alpha)$, which is not valid in the classical modal logic **S5**.

It is not hard to see that $M^c_{4m}$ is both a paraconsistent and a paracomplete logic, and so it is non-trivial. In fact, if $p$ and $q$ are different propositional variables then we have that $p, \neg p \not\models_{M^c_{4m}} q$ and $\not\models_{M^c_{4m}} q \vee \neg q$. Indeed, it is enough to take an homomorphism $h$ such that $h(p) = N$ and $h(q) = B$.

As a paraconsistent logic, $M^c_{4m}$ is a *Logic of Formal Inconsistency* (**LFI**, cf. [6]). In fact, it is possible to define a *consistency* operator $\circ\alpha = \Box(\alpha\vee\neg\alpha)$ such that $\circ\alpha, \alpha, \neg\alpha \models_{M^c_{4m}} \beta$ for every $\alpha, \beta \in Fm^c$. In $\mathfrak{M}^c_{4m}$, $\circ x = 1$ if $x \in \{0, 1\}$, and $\circ x = 0$ otherwise. Notice that $\circ\alpha \models_{M_{4m}} \neg(\alpha \wedge \neg\alpha)$ but the converse is not true, and so $M_{4m}$ separates the notions of consistency and non-contradiction. As usual in the framework of **LFI**s, it is possible to define an *inconsistency* operator $\bullet$ on $M^c_{4m}$ as $\bullet\alpha =_{def} \neg\circ\alpha$. Then, $\bullet\alpha$ is equivalent to $\Diamond(\alpha\wedge\neg\alpha)$. It is immediate that, in $\mathfrak{M}^c_{4m}$, $\bullet x = 1$ if $x \in \{N, B\}$, and $\bullet x = 0$ otherwise. Since $\alpha \wedge \neg\alpha \models_{M^c_{4m}} \bullet\alpha$ but $\bullet\alpha \not\models_{M^c_{4m}} \alpha \wedge \neg\alpha$, the notions of inconsistency and contradiction can be separated in $M^c_{4m}$.

Recall from [6] that a logic $\mathbb{L}$ is *boldly* paraconsistent if there is not a formula $\beta(p_1, \ldots, p_n)$ such that: (i) $\not\vdash_L \beta(\gamma_1, \ldots, \gamma_n)$ for some $\gamma_1, \ldots, \gamma_n$, and (ii) $\alpha, \neg\alpha \vdash_{\mathbb{L}} \beta(\gamma_1, \ldots, \gamma_n)$ for every $\alpha, \gamma_1, \ldots, \gamma_n$. Then, it can be proved that $M^c_{4m}$ is boldly paraconsistent.

Concerning paracompleteness, it is interesting to notice that the schemas $(\alpha \succ \beta) \vee (\beta \succ \alpha)$ and $\alpha \vee (\alpha \succ (\beta \vee \neg\beta))$ are valid in $M^c_{4m}$. Both axioms together characterize Smetanich logic **Sm**, the greatest intermediate logic properly included in classical logic (cf. [10]). However, it should be clear that $M_{4m}$ is not an intermediate logic, since the implication connective $\succ$ does not satisfy some basic properties of intuitionistic implication.

## 6 A Hilbert-style presentation for $M^c_{4m}$

Taking profit of Proposition 4.1, which states that the logic $M_{4m}$ can be described as $M^c_{4m}$ just in terms of $\succ$ and $\perp$, we will define in this section a sound and complete Hilbert-style system for tetravalent modal logic $M^c_{4m}$.

Recall from Section 4 that $\mathfrak{Fm}^c = \langle Fm^c, \succ, \bot \rangle$ is the language of $M^c_{4m}$. By $\top$ we mean $\bot \succ \bot$; by $\neg\alpha$ we mean $\alpha \succ \bot$ (thus, $\top$ denotes $\neg\bot$); $\alpha \vee \beta$ denotes $(\alpha \succ \beta) \succ \beta$; $\alpha \wedge \beta$ denotes $\neg(\neg\alpha \vee \neg\beta)$; and $\Box\alpha$ is an abbreviation for $\neg(\alpha \succ \neg\alpha)$.

**Definition 6.1.** *Denote by $\mathcal{H}_{4m} = \langle Fm^c, \vdash_1 \rangle$ the propositional logic defined through the following Hilbert calculus, where $\alpha$, $\beta$, $\gamma \in Fm^c$, and with notations as above.*

**Axioms**

(A1) $\alpha \succ \alpha$,

(A2) $\alpha \succ (\beta \succ \alpha)$

(A3) $(\alpha \vee \beta) \succ (\beta \vee \alpha)$

(A4) $\bot \succ \alpha$

(A5) $(\alpha \succ (\alpha \succ \beta)) \vee \alpha$

(A6) $((\alpha \vee \beta) \succ \gamma) \succ ((\alpha \succ \gamma) \wedge (\beta \succ \gamma))$

(A7) $\Box(\alpha \succ (\beta \succ \gamma)) \succ \Box(\beta \succ (\alpha \succ \gamma))$

(A8) $\Box(\alpha \succ \beta) \succ (\Box(\beta \succ \alpha) \succ \Box((\gamma \succ \alpha) \succ (\gamma \succ \beta)))$

(A9) $\Box(\alpha \succ \beta) \succ \Box((\beta \succ \gamma) \succ (\alpha \succ \gamma))$

(A10) $\Box\alpha \succ \alpha$

**Inference Rules**

(MP) $\dfrac{\alpha \qquad \alpha \succ \beta}{\beta}$ (Conj) $\dfrac{\alpha \qquad \beta}{\alpha \wedge \beta}$ (Nec) $\dfrac{\alpha}{\Box\alpha}$

**Definition 6.2.** *(1) A derivation of a formula $\alpha$ in $\mathcal{H}_{4m}$ is a finite sequence of formulas $\alpha_1 \ldots \alpha_n$ such that $\alpha_n$ is $\alpha$ and every $\alpha_i$ is either an instance of an axiom or it is the consequence of some inference rule whose premises appear in the sequence $\alpha_1 \ldots \alpha_{i-1}$. We say that $\alpha$ is derivable in $\mathcal{H}_{4m}$, and we write $\vdash_1 \alpha$, if there exists a derivation of it in $\mathcal{H}_{4m}$.*
*(2) Let $\Gamma$ be a set of formulas. We say that $\alpha$ is derivable in $\mathcal{H}_{4m}$ from $\Gamma$, and we write $\Gamma \vdash_1 \alpha$, if either $\vdash_1 \alpha$ or there exists a finite, non-empty subset $\{\gamma_1, \ldots, \gamma_n\}$ of $\Gamma$ such that $(\gamma_1 \wedge (\gamma_2 \wedge (\ldots \wedge (\gamma_{n-1} \wedge \gamma_n) \ldots))) \succ \alpha$ is derivable in $\mathcal{H}_{4m}$.*

**Remark 6.3.** *From the last definition, $\emptyset \vdash_1 \alpha$ iff $\vdash_1 \alpha$. It is easy to prove that $\alpha \vdash_1 \beta$ iff $\vdash_1 \alpha \succ \beta$, and so a weak version of the Deduction Metatheorem is forced to be valid, which reflects Theorem 4.4. However, as stated in Proposition 4.5, the general version of the Deduction Metatheorem is not valid in $M_{4m}$ and so this is basically a logic of tautologies (or theorems, from a proof-theoretical perspective). This is the case of the vast majority of modal logics studied in the literature, where a modal logic is simply presented as a set of formulas satisfying certain properties (cf. [5]). This justifies the definition of derivation from premises in the calculus $\mathcal{H}_{4m}$ proposed above, which is similar to that used in the context of modal logics.*

Recall that a structural inference rule is *admissible* in a logic **L** if its conclusion is a theorem of **L** provided that its premises are theorems of **L**.

**Proposition 6.4.** *The following rules are admissible in $\mathcal{H}_{4m}$:*

$$\text{(R1)} \quad \frac{\alpha \succ (\beta \succ \gamma)}{\beta \succ (\alpha \succ \gamma)} \qquad\qquad \text{(R2)} \quad \frac{\alpha \succ \beta \qquad \beta \succ \alpha}{(\gamma \succ \alpha) \succ (\gamma \succ \beta)}$$

$$\text{(R3)} \quad \frac{\alpha \succ \beta}{(\beta \succ \gamma) \succ (\alpha \succ \gamma)} \qquad\qquad \text{(T)} \quad \frac{\alpha \succ \beta \qquad \beta \succ \gamma}{\alpha \succ \gamma}$$

$$\text{(R4)} \quad \frac{\alpha \succ \gamma \qquad \beta \succ \gamma}{(\alpha \vee \beta) \succ \gamma} \qquad\qquad \text{(R5)} \quad \frac{\gamma \succ \alpha \qquad \gamma \succ \beta}{\gamma \succ (\alpha \wedge \beta)}$$

**Proof.**
(R1): Observe that $\vdash_1 \alpha$ iff $\vdash_1 \Box\alpha$, by (A10), (Nec) and (MP). Thus, assuming $\vdash_1 \alpha \succ (\beta \succ \gamma)$ it follows that $\vdash_1 \Box(\alpha \succ (\beta \succ \gamma))$. Then, by (MP) with (A7) it follows that $\vdash_1 \Box(\beta \succ (\alpha \succ \gamma))$. But then, $\vdash_1 \beta \succ (\alpha \succ \gamma)$ and so (R1) is admissible. The admissibility of rules (R2) and (R3) is proved analogously, by using (A8) and (A9), respectively.

(T): It follows from (R3) and (MP).

(R4) The proof is arduous, and is left to the patient reader.

(R5) It follows from (R4), by using basic properties of the negation $\neg$. ■

**Proposition 6.5.** *In $\mathcal{H}_{4m}$ the following hold:*

(i) $\vdash_1 \alpha \succ \top$;

(ii) $\vdash_1 \alpha$ *iff* *both* $\vdash_1 \alpha \succ \top$ *and* $\vdash_1 \top \succ \alpha$.

**Proof.**
(i) By (A2) it follows that $\vdash_1 \bot \succ (\alpha \succ \bot)$. By (R1) (cf. Proposition 6.4) it follows that $\vdash_1 \alpha \succ (\bot \succ \bot)$, that is, $\vdash_1 \alpha \succ \top$.

(ii) Assume that $\vdash_1 \alpha$. Since $\alpha \succ (\top \succ \alpha)$ is an instance of (A2), then $\vdash_1 \top \succ \alpha$ follows by (MP). On the other hand, $\vdash_1 \alpha \succ \top$ follows from (i). Conversely, suppose that $\vdash_1 \top \succ \alpha$. Since $\vdash_1 \top$, by (A1), then $\vdash_1 \alpha$ by (MP). ■

Let $\equiv \subseteq Fm^c \times Fm^c$ defined by $\equiv =_{def} \{(\alpha, \beta) \,:\, \vdash_1 \alpha \succ \beta$ and $\vdash_1 \beta \succ \alpha\}$.

**Lemma 6.6.** *The relation* $\equiv$ *is a congruence on* $\mathfrak{Fm}^c$.

**Proof.** Clearly $\equiv$ is an equivalence relation on $\mathfrak{Fm}^c$. Suppose that $\alpha \equiv \beta$ and $\gamma \equiv \delta$. Then, from the set of assumptions

$$\{\vdash_1 \alpha \succ \beta, \vdash_1 \beta \succ \alpha, \vdash_1 \gamma \succ \delta, \vdash_1 \delta \succ \gamma\}$$

it follows that $\vdash_1 (\alpha \succ \gamma) \succ (\beta \succ \delta)$. Indeed, consider the following syntactical (meta)derivation:

(1) $\vdash_1 \alpha \succ \beta$ [(hyp.)]
(2) $\vdash_1 \beta \succ \alpha$ [(hyp.)]
(3) $\vdash_1 \gamma \succ \delta$ [(hyp.)]
(4) $\vdash_1 \delta \succ \gamma$ [(hyp.)]
(5) $\vdash_1 (\alpha \succ \gamma) \succ (\beta \succ \gamma)$ [(2),(R3)]
(6) $\vdash_1 (\beta \succ \gamma) \succ (\beta \succ \delta)$ [(3),(4),(R2)]
(7) $\vdash_1 (\alpha \succ \gamma) \succ (\beta \succ \delta)$ [(5), (6), (T)]

Analogously, from the same set of assumptions as above, it follows that $\vdash_1 (\beta \succ \delta) \succ (\alpha \succ \gamma)$. That is, $(\alpha \succ \gamma) \equiv (\beta \succ \delta)$. ■

**Theorem 6.7.** *The Lindenbaum algebra* $\mathfrak{Fm}^c/\equiv$ *of* $\mathcal{H}_{4m}$ *is a contrapositive tetravalent modal algebra by defining:* $|\alpha| \succ |\beta| =_{def} |\alpha \succ \beta|$ *and* $0 =_{def} |\bot|$ *where* $|\gamma|$ *denotes the equivalence class of the formula* $\gamma$.

**Proof.** The operations in $\mathfrak{Fm}^c/\equiv$ are well-defined because of Lemma 6.6. Let 1 be $0 \succ 0$, that is, $1 = |\top|$. It is worth noting that $|\alpha| = 1$ iff $\vdash_1 \alpha$, by Proposition 6.5(ii). The fact that $\mathfrak{Fm}^c/\equiv \in \mathbf{TMA}^c$ is an easy consequence of the axioms and rules of $\mathcal{H}_{4m}$, which faithfully reflect the definition of TMA$^c$s (see Definition 4.2). In particular, (A3) represents (C3) and (A5) represents (C5), while (C7) is given by axiom (A6). We leave the details of the proof to the reader. ■

**Theorem 6.8.** *(Soundness and Completeness of $\mathcal{H}_{4m}$) The following conditions are equivalent, for every subset $\Gamma \cup \{\beta\}$ of $Fm^c$:*

(i) $\Gamma \vdash_1 \beta$,

(ii) $\Gamma \models_{M^c_{4m}} \beta$.

**Proof.** (i) $\Rightarrow$ (ii) (Soundness): It is easy to see that every axiom of $\mathcal{H}_{4m}$ is valid in $M^c_{4m}$ (cf. Remark 4.3). On the other hand, if an instance of the premises of an inference rule is valid in $M^c_{4m}$ then the respective conclusion is also valid in $M^c_{4m}$.

Suppose now that $\Gamma \vdash_1 \beta$. If $\vdash_1 \beta$, let $\alpha_1 \dots \alpha_k$ be a derivation of $\beta$ in $\mathcal{H}_{4m}$. By induction on $k$ it is easy to show that $\beta$ is valid in $M^c_{4m}$, by the observations above. On the other hand, if $\nvdash_1 \beta$, there exists a finite subset $\Gamma_0 = \{\gamma_1, \dots, \gamma_n\}$ of $\Gamma$ such that $\vdash_1 \delta$, where $\delta$ is $(\gamma_1 \wedge (\gamma_2 \wedge (\dots \wedge (\gamma_{n-1} \wedge \gamma_n) \dots))) \succ \alpha$. As observed above, $\delta$ is valid in $M^c_{4m}$, and so $\Gamma_0 \models_{M^c_{4m}} \beta$. Therefore, $\Gamma \models_{M^c_{4m}} \beta$.

(ii) $\Rightarrow$ (i) (Completeness): Suppose that $\Gamma \models_{M^c_{4m}} \beta$. By Remark 4.3, there exists a finite subset $\{\gamma_1, \dots, \gamma_n\}$ of $\Gamma$ such that $\{\gamma_1, \dots, \gamma_n\} \models_{M^c_{4m}} \beta$. If $n = 0$, that is, if $\models_{M^c_{4m}} \beta$, then $h(\beta) = 1$ for every $h \in Hom(\mathfrak{Fm}^c, \mathfrak{A})$ and every $\mathfrak{A} \in \mathbf{TMA}^c$. In particular, $h(\beta) = 1$ for every $h \in Hom(\mathfrak{Fm}^c, \mathfrak{Fm}^c/\equiv)$, by Theorem 6.7. Let $h : \mathfrak{Fm}^c \to \mathfrak{Fm}^c/\equiv$ be the canonical map given by $h(\delta) = |\delta|$, for every $\delta$. Then $h \in Hom(\mathfrak{Fm}^c, \mathfrak{Fm}^c/\equiv)$ and so $|\beta| = 1$. By Proposition 6.5(ii) it follows that $\vdash_1 \beta$. Thus, $\Gamma \vdash_1 \beta$, by definition of derivation in $\mathcal{H}_{4m}$. On the other hand, if $n > 0$, let $\gamma$ be $\gamma_1 \wedge \dots \wedge \gamma_n$. Since $h(\alpha \wedge \delta) \in \{1, N\}$ implies that $h(\alpha), h(\delta) \in \{1, N\}$ for every $h \in Hom(\mathfrak{Fm}^c, \mathfrak{M}^c_{4m})$, then $\gamma \models_{M^c_{4m}} \beta$. From this, $\models_{M^c_{4m}} \gamma \succ \beta$. Thus $\vdash_1 \gamma \succ \beta$, by the first part of the proof. Therefore, $\Gamma \vdash_1 \beta$, by definition of derivation in $\mathcal{H}_{4m}$. ■

## 7 Another logic for Tetravalent Modal Algebras

In the paper [7] the following problem was posed: find an axiomatization for the matrix logic $\langle \mathfrak{M}_{4m}, \{1\} \rangle$ in the language $\{\succ, \neg, \top\}$. Observe that this logic, considered over the language $\mathfrak{Fm}$ of TMAs, was briefly studied in [9], where it was called $\mathcal{TML}^N$. However, no Hilbert axiomatization for this logic was presented there. It is worth mentioning that a Hilbert style calculus for $\mathcal{TML}^N$ was proposed in [4], but using two implication connectives and so many axioms.

In this section we shall prove that the Hilbert calculus $\mathcal{H}_{4m}$ for $M_{4m}^c$ is adequate for the matrix logic $\langle \mathfrak{M}_{4m}^c, \{1\}\rangle$ in the language $\{\succ, \bot\}$, provided that the notion of derivation from premises (cf. Definition 6.2(2)) is replaced by the usual one. This gives a simple solution to the problem posed in [7].

**Definition 7.1.** *The normal logic of the variety* $\mathbf{TMA}^c$ *is the logic* $\mathbb{L}_{TMA}^N$ *defined over* $\mathfrak{Fm}^c$ *by the family of matrices* $\langle \mathfrak{U}, \{1\}\rangle$*, for* $\mathfrak{U} \in \mathbf{TMA}^c$*. On the other hand, the normal tetravalent modal logic* $M_{4m}^N$ *is the logic over* $\mathfrak{Fm}^c$ *given by the matrix* $\langle \mathfrak{M}_{4m}^c, \{1\}\rangle$*.*

The analogous of Proposition 1.3 is of course valid:

**Proposition 7.2.** *The logic* $M_{4m}^N$ *coincides with the logic* $\mathbb{L}_{TMA}^N$*.*

Adapting a result from [9], we obtain the following:

**Proposition 7.3.** *Let* $\Gamma \cup \{\alpha\}$ *be a set of formulas in* $Fm^c$*. Then*

$$\Gamma \models_{M_{4m}^N} \alpha \ \text{ iff } \ \Box\Gamma \models_{M_{4m}^c} \alpha.$$

**Remark 7.4.** *It is worth noting that the Deduction Metatheorem is not valid in* $M_{4m}^N$*. For instance,* $\Diamond p, p \models_{M_{4m}^N} \Box p$ *but* $\Diamond p \not\models_{M_{4m}^N} p \succ \Box p$*. Even the weaker form of the Deduction Metatheorem stated in Theorem 4.4 for* $M_{4m}^c$ *does not hold in* $M_{4m}^N$*. For instance,* $p \models_{M_{4m}^N} \Box p$ *but* $\not\models_{M_{4m}^N} p \succ \Box p$*. The logic* $M_{4m}^N$ *is not paraconsistent, since* $\alpha, \neg\alpha \models_{M_{4m}^N} \beta$ *holds for every* $\alpha, \beta \in Fm^c$*. However,* $M_{4m}^N$ *is still paracomplete:* $\not\models_{M_{4m}^N} p \vee \neg p$ *for every* $p \in Var$*. Moreover,* $M_{4m}^c$ *and* $M_{4m}^N$ *have the same valid formulas.*

Recall from Definition 6.1 the Hilbert calculus $\mathcal{H}_{4m}$. The Hilbert calculus $\mathcal{H}_{4m}^N$ is defined by the same axioms and inference rules of $\mathcal{H}_{4m}$, but the notion of derivation from premises is now defined in the usual way (contrast with Definition 6.2(2)):

**Definition 7.5.** *Let* $\Gamma \cup \{\alpha\}$ *be a set of formulas in* $Fm^c$*. We say that* $\alpha$ *is derivable in* $\mathcal{H}_{4m}^N$ *from* $\Gamma$*, and we write* $\Gamma \vdash_2 \alpha$*, if there exists a finite sequence of formulas* $\alpha_1 \ldots \alpha_n$ *such that* $\alpha_n$ *is* $\alpha$ *and every* $\alpha_i$ *is either an instance of an axiom, or* $\alpha_i \in \Gamma$*, or it is the consequence of some inference rule whose premises appear in the sequence* $\alpha_1 \ldots \alpha_{i-1}$*.*

Obviously, $\vdash_1 \alpha$ iff $\vdash_2 \alpha$. On the other hand, the consequence relations are different, as we shall see below.

**Theorem 7.6.** *(Soundness of $\mathcal{H}^N_{4m}$) Let $\Gamma \cup \{\beta\}$ be a subset of $Fm^c$. Then, $\Gamma \vdash_2 \beta$ implies that $\Gamma \models_{M^N_{4m}} \beta$.*

**Proof.** The axioms of $\mathcal{H}^N_{4m}$ are valid in $M^N_{4m}$. On the other hand, it is easy to see that if an homomorphism $h$ assigns the value 1 to the premises of an inference rule of $\mathcal{H}^N_{4m}$ then it must assign the value 1 to the conclusion of the rule. ■

**Proposition 7.7.** *Let $\Gamma \cup \{\alpha\}$ be a set of formulas in $Fm^c$. Then*

$$\Box\Gamma \vdash_1 \alpha \ \text{ implies that } \ \Gamma \vdash_2 \alpha.$$

**Proof.** Let $\Gamma \cup \{\alpha\} \subseteq Fm^c$, and suppose that $\Box\Gamma \vdash_1 \alpha$. In the case that $\vdash_1 \alpha$, the result is immediate. Otherwise, there is a finite, non-empty subset $\{\gamma_1, \ldots, \gamma_k\}$ of $\Gamma$ such that $(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_k) \succ \alpha$ is derivable in $\mathcal{H}_{4m}$, by Definition 6.2. Then $\alpha$ can be derived from $\{\gamma_1, \ldots, \gamma_k\}$ in $\mathcal{H}^N_{4m}$ as follows: from the hypothesis $\gamma_1, \ldots, \gamma_k$, the formulas $\Box\gamma_1, \ldots, \Box\gamma_k$ are obtained by (Nec). Using (Conj) repeatedly we obtain $(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_k)$. Then, the derivation of $(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_k) \succ \alpha$ in $\mathcal{H}_{4m}$ (which is also a derivation in $\mathcal{H}^N_{4m}$) is appended. By (MP) it follows $\alpha$. Therefore, $\Gamma \vdash_2 \alpha$ as desired. ■

**Theorem 7.8.** *(Completeness of $\mathcal{H}^N_{4m}$) Let $\Gamma \cup \{\beta\}$ be a subset of $Fm^c$. Then, $\Gamma \models_{M^N_{4m}} \beta$ implies that $\Gamma \vdash_2 \beta$.*

**Proof.** From Proposition 7.3, Theorem 6.8 and Proposition 7.7. ■

**Proposition 7.9.** *Let $\Gamma \cup \{\alpha\}$ be a subset of $Fm^c$. Then, $\Gamma \vdash_1 \alpha$ implies that $\Gamma \vdash_2 \alpha$.*

**Proof.** By adapting the proof of Proposition 7.7. ■

Notice that the converse of Proposition 7.9 is not true. For instance, $p \vdash_2 \Box p$ but $p \nvdash_1 \Box p$.

## 8 Concluding Remarks

This paper presents some novel results about the logical aspects of tetravalent modal algebras (TMAs). We focuses on the contrapositive implication operator $\succ$ definable in these algebras, showing that this connective allows to define in a simple way Hilbert-style presentations of two logics naturally associated to TMAs that we called $M^c_{4m}$ and $M^N_{4m}$. As it can be appreciated along the paper, the contrapositive implication played an important role in the study of $M^c_{4m}$ and $M^N_{4m}$. Further research on TMAs should take into account the nice properties of this operator.

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