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Splitting Logics

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ABSTRACT. This paper addresses the question of factoring a logic into families of (generally simpler) components, estimating the top–down perspective, splitting, versus the bottom–up, splicing. Three methods are carefully analyzed and compared: possible–translations semantics, nondeterministic semantics and plain fibring (joint with its particularization, direct union of matrices). The possibilities of inter–definability between these methods are also examined. Finally, applications to some well–known logic systems are given and their significance evaluated.

1 Splitting logics, splicing logics and their use

One of fundamental questions in the philosophy of logic, "Why there are so many logics instead of just one?" (or even, instead of none), is naturally counterposed by another: If there are indeed many logics, are they excluding alternatives, or are they compatible? Is it possible to combine them into coherent systems, with the purpose of using them in applications and of taking profit of this composionality capacity to better understand logics? And if we can compose, why not decompose logics?

One of the first, and one of the most general, approaches for the question of combining logics is the concept of fibring introduced by D. Gabbay in [Gabbay, 1996]. Fibring is able to combine logics creating new and expressive systems, in the direction of what we call *splicing logics*.

The other direction is called *splitting logics*. Though, as we shall argue, there is no essential distinction between splicing and splitting, there are important differences with respect to the aims one may have in mind. Splitting as a process for investigating logics has been under–appreciated, and we intend to stress here some results and some views that we believe to be of interest for the sake of splitting in the trade of combining logics. [1]

[1]The process tags "splicing" and "splitting" logics were introduced in [Carnielli and Coniglio, 1999]. As a noun, "splitting" is also used in the literature in a completely different sense, viz., to designate a "logic that splits a class", as e.g. in W.J. Blok, "On the degree of incompleteness of modal logics" (abstract). *Bulletin of the Section of Logic of the Polish Academy of Sciences*, 7(4):167-175, December 1978.

¡book title¿, 1–26.

Possible–translations semantics were proposed in [Carnielli, 1990], and were designed to help solve the problem of assigning semantic interpretations to non–classical logics. The idea behind possible–translations semantics is to build an interpretation for a given logic by taking into account a specific set of translations from its formulas into a class of simpler logics, with known or acceptable semantics. For a certain time it was even called "non–deterministic semantics" (as in [Carnielli and D'Ottaviano, 1997]), due to the apparent ambiguity of having several translations from the same domain.

Such semantics comprise a flexible and widely applicable tool for endowing logics with recursive and palatable semantic interpretation: detailed examples will be given in Section 3, but it is worth mentioning that several paraconsistent logics (as fragments of classical logic) which are not characterizable by finite matrices can be characterized by suitable combinations of many–valued logics. The reader is invited to check details for the case of N. da Costa's hierarchy $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ in [Carnielli, 2000] and [Marcos, 1999].

Examples of possible–translations semantics go in the direction of splitting, illustrating how a complex logic can be analyzed into less complex factors.

We also analyze here the *nondeterministic semantics* (see Section 4) and the *direct union of matrices* and *plain fibring* (see Section 5).

The traditional notion of matrix semantics, due to J. Łukasiewicz and E. Post, is also briefly reviewed in Section 3. Matrix semantics generalize algebraic semantics, as used in algebraic logic. They constitute a method for assigning semantic meaning for logics, as well as a method for defining logical systems.

The fact that possible–translations semantics are a widely applicable tool is witnessed by our results below, which show that both nondeterministic semantics and matrix semantics are particular cases of possible–translations semantics. As the notion of matrix semantics proves to be adequate for any structural deductive system, so are possible–translations semantics.

Another application, better suited for many–valued logics, is the concept of *society semantics* (cf. [Carnielli and Lima-Marques, 1999] and [Fernández and Coniglio, 2003]) that we do not treat here.

Possible–translations semantics (and their particular cases) work not only as general tools for assigning semantics for logics, but also as a tool for splitting logics as well. In the same manner they work for the direct unions of matrices and the plain fibring, which are not reducible to possible–translations semantics.

The particular cases are not to be discounted by any means: on the contrary, they are significant, specially when regarded from the splitting standpoint, in the measure that they provide operative methods for computing factors.

Although there is no fundamental distinction between splitting and splicing logics, as much as there is no fundamental distinction between factoring a number into primes or multiplying primes to compose a number, there is difference of attitudes and expectations, which is reflected in the distinction between what we have and what we wish to obtain.

If we represent the result of a process of factoring logics as $\mathcal{L} = \mathcal{L}_1 \odot \mathcal{L}_2$, there are, by and large, two ways to read this equation:

- If we had started from known logics $\mathcal{L}_1$ and $\mathcal{L}_2$ and $\mathcal{L}$ is our *incognita*, then we have a typical case of splicing $\mathcal{L}_1$ and $\mathcal{L}_2$ to obtain $\mathcal{L}$. Example 47 exemplifies this where the underlying operation is direct union.

- On the other hand, if $\mathcal{L}$ is known, we then have a typical case of splitting $\mathcal{L}$ into (presumably simpler) factors $\mathcal{L}_1$ and $\mathcal{L}_2$. The factors may be new logics (or new fragments of known logics), in which case we have encountered novelty, or they may be known logics, in which case we have found new relations among $\mathcal{L}$, $\mathcal{L}_1$ and $\mathcal{L}_2$ (and this is as much splitting as it is splicing). An instance of the former case is found in Example 45, and of the latter in Example 25. In this sense, Example 41 is also an instance of factoring classical propositional logic into its fragments.

The whole enterprise of splitting and splicing logics has several predecessors, depending upon the particular guise we may have in mind: some ingredients of the possible–translations semantics, even if in incipient form, will also be recognized in some variants of Gabbay's fibring. Still earlier, traces of S. Jaśkowski's discussive logics (cf. [Jaśkowski, 1949]) are recognizable in the general idea of society semantics. Plain fibring of matrices, on their side, have as antecedent both the original (modal) fibring of Gabbay and a certain product of matrices introduced by J. Łukasiewicz in [Łukasiewicz, 1953] to study his four–valued modalities: he used truth-values $\{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$, in such a way that his algebra of truth–functions coincides with $B \times B$ (where B is the two–elements Boolean algebra) and thus his modal-free tautologies coincide with classical tautologies.

2 Basic concepts about signatures and logics

This section briefly describes the basic definitions, notation and facts concerning propositional signatures and logics that will be used throughout the paper.

DEFINITION 1 (*i*) A *set of propositional variables* is a countable set $\mathcal{V}$, which will keep fixed. The elements of $\mathcal{V}$ will be denoted by $p_1, p_2, \ldots$.
(*ii*) A (propositional) *signature* is a family $C = \{C^k\}_{k \in \mathbb{N}}$, where each C^k is a set of *connectives of arity* k. It will be assumed that $C^k \cap C^n = \emptyset = C^k \cap \mathcal{V}$ for every $k \neq n$. The *domain of the signature* C is the set $|C| = \bigcup_{k \in \mathbb{N}} C^k$. Given two signatures C_1 and C_2, we say that C_1 *is included in* C_2 (denoted by $C_1 \subseteq C_2$) if, for every $k \in \mathbb{N}$, $C_1^k \subseteq C_2^k$. The signature $C_1 \uplus C_2$ (the disjoint union of C_1 and C_2) is defined as expected, that is: $(C_1 \uplus C_2)^k = C_1^k \uplus C_2^k$ for every $k \in \mathbb{N}$, where $A \uplus B$ denotes the usual set–theoretic disjoint union of the sets A and B.
(*iii*) A (propositional) *language with signature* C, denoted by $L(C)$, is the algebra of words freely generated by C over $\mathcal{V}$ such that C^k is the set of k–ary operations of $L(C)$. Elements of $L(C)$ are called C*–formulas* (or simply *formulas*).
(*iii*) Given a signature C and $n \in \mathbb{N}$, $L(C)[n]$ is the set of formulas φ such that the set of propositional variables occurring in φ is exactly $\{p_1, \ldots, p_n\}$. ■

Observe that $L(C)[0]$ is the set of formulas without variables. It is worth noting that there may be signatures $C \neq C'$ such that $L(C) = L(C')$. For the sake of simplicity, a signature will be frequently identified with its domain. We now describe the category of signatures.

DEFINITION 2 Let C and C' be signatures. A *signature morphism* f from C to C', denoted $C \xrightarrow{f} C'$, is a mapping $f : |C| \to L(C')$ such that, if $c \in C^n$ then $f(c) \in L(C')[n]$. ■

Given a signature morphism $C \xrightarrow{f} C'$, a mapping $\widehat{f} : L(C) \to L(C')$ can be defined as expected:

1. $\widehat{f}(p) = p$ if $p \in \mathcal{V}$;
2. $\widehat{f}(c) = f(c)$ if $c \in C^0$;
3. $\widehat{f}(c(\varphi_1, \ldots, \varphi_n)) = f(c)(\widehat{f}(\varphi_1), \ldots, \widehat{f}(\varphi_n))$ if $c \in C^n$ and $\varphi_1, \ldots, \varphi_n \in L(C)$.

Clearly the extension $\widehat{f}$ of f is unique. Moreover, if f, f' are signature morphisms such that $\widehat{f} = \widehat{f'}$ then $f = f'$. Additionally, the propositional variables occurring in φ and in $\widehat{f}(\varphi)$ are the same.

DEFINITION 3 Let $C \xrightarrow{f} C'$ and $C' \xrightarrow{g} C''$ be signature morphisms. The *composition* $g \centerdot f$ of f and g is defined to be the signature morphism $C \xrightarrow{g \centerdot f} C''$ given by the mapping $\widehat{g} \circ f : |C| \to L(C'')$. ■

DEFINITION 4 The category **Sig** of (propositional) languages is defined as follows:

- Its objects are propositional signatures (see Definition 1);
- Its morphisms are signature morphisms (see Definition 2);
- The composition of morphisms is as in Definition 3;
- For every signature C, the identity morphism $C \xrightarrow{id_C} C$ is defined by $id_C(c) = c$ (for $c \in C^0$) and $id_C(c) = c(p_1, \ldots, p_n)$ (for $c \in C^n$, $n \geq 1$). ■

The next result was proved in [Bueno-Soler *et al.*, 2005].

THEOREM 5 **Sig** is a category with arbitrary (small) products.

In the category **Sig** all the objects are restricted to sequences of sets, constraining us into considering just small diagrams in the theorem above.

We stipulate below a concept of *propositional logic* which is broad enough to encompass all logics that are usually found, though this does not of course include *all* possible propositional logics.

DEFINITION 6 Let C be a signature. A *consequence relation over the signature* C is a relation $\vdash \subseteq \wp(L(C)) \times L(C)$ satisfying the following properties (as usual, $(\Gamma, \alpha) \in \vdash$ will be denoted by $\Gamma \vdash \alpha$):

- If $\varphi \in \Gamma$ then $\Gamma \vdash \varphi$ (**Reflexivity**).
- If $\Gamma \vdash \varphi$ and $\Sigma \vdash \psi$, for every $\psi \in \Gamma$, then $\Sigma \vdash \varphi$ (**Transitivity**). ■

Observe that, because of Reflexivity and Transitivity, any consequence relation $\vdash$ automatically satisfies the following:

- If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Sigma$ then $\Sigma \vdash \varphi$ (**Monotonicity**).

DEFINITION 7 A *(propositional) logic* is defined to be a pair $\mathcal{L} = \langle C, \vdash \rangle$ such that C is a signature and $\vdash$ is a consequence relation over C. A logic $\mathcal{L}$ is said to be *structural* if, additionally, it satisfies:

- For every substitution σ in C and every $\Gamma \cup \{\varphi\} \subseteq L(C)$: [2]
  If $\Gamma \vdash \varphi$ then $\widehat{\sigma}(\Gamma) \vdash \widehat{\sigma}(\varphi)$ (**Structurality**).

The logic $\mathcal{L}$ is said to be *finitary* if it also satisfies:

- For every $\Gamma \cup \{\varphi\} \subseteq L(C)$:
  If $\Gamma \vdash \varphi$ then $\Gamma' \vdash \varphi$ for some finite set $\Gamma' \subseteq \Gamma$ (**Finitariness**).

The logic $\mathcal{L}$ is said to be *standard* if it is structural and finitary. ■

DEFINITION 8
(i) Let $\mathcal{L} = \langle C, \vdash_{\mathcal{L}} \rangle$ be a logic, and let $C' \subseteq C$. The *C'–fragment of $\mathcal{L}$* is the logic $\mathcal{L}|_{C'} := \langle C', \vdash_{\mathcal{L}|_{C'}} \rangle$ where $\vdash_{\mathcal{L}|_{C'}} = \vdash_{\mathcal{L}} \cap (\wp(L(C')) \times L(C'))$. This means that, for every $\Gamma \cup \{\varphi\} \subseteq L(C')$, $\Gamma \vdash_{\mathcal{L}|_C} \varphi$ iff $\Gamma \vdash_{\mathcal{L}} \varphi$.
(ii) The logic $\mathcal{L}' = \langle C', \vdash_{\mathcal{L}'} \rangle$ is *a strong extension* of $\mathcal{L} = \langle C, \vdash_{\mathcal{L}} \rangle$ if $C \subseteq C'$ and $\vdash_{\mathcal{L}} \subseteq \vdash_{\mathcal{L}'}$.
(iii) The logic $\mathcal{L}' = \langle C', \vdash_{\mathcal{L}'} \rangle$ is *a weak extension* of $\mathcal{L} = \langle C, \vdash_{\mathcal{L}} \rangle$ if $C \subseteq C'$ and $\vdash_{\mathcal{L}} \varphi$ implies that $\vdash_{\mathcal{L}'} \varphi$, for every $\varphi \in L(C)$.
(iv) The logic $\mathcal{L}' = \langle C', \vdash_{\mathcal{L}'} \rangle$ is *a conservative extension* of $\mathcal{L} = \langle C, \vdash_{\mathcal{L}} \rangle$ if $C \subseteq C'$ and $\mathcal{L} = \mathcal{L}'|_C$.
(v) The logic $\mathcal{L}' = \langle C', \vdash_{\mathcal{L}'} \rangle$ is *a conservative weak extension* of $\mathcal{L} = \langle C, \vdash_{\mathcal{L}} \rangle$ if $C \subseteq C'$ and $\vdash_{\mathcal{L}} \varphi$ iff $\vdash_{\mathcal{L}'} \varphi$, for every $\varphi \in L(C)$. ■

From the definitions above the next result is immediate.

THEOREM 9
(i) Each C–fragment of any (structural, finitary, standard) logic is also a (structural, finitary, standard) logic.
(ii) Every logic $\mathcal{L}$ is a conservative extension of any of its C–fragments.

Finally, the category of logics is specified.

DEFINITION 10 Let $\mathcal{L} = \langle C, \vdash_{\mathcal{L}} \rangle$ and $\mathcal{L}' = \langle C', \vdash_{\mathcal{L}'} \rangle$ be logics. A *morphism between logics* from $\mathcal{L}$ to $\mathcal{L}'$, denoted by $\mathcal{L} \xrightarrow{f} \mathcal{L}'$, is a **Sig**–morphism $C \xrightarrow{f} C'$ which satisfies, for every $\Gamma \cup \{\varphi\} \subseteq L(C)$:

$$\Gamma \vdash_{\mathcal{L}} \varphi \ \text{ implies } \ \widehat{f}(\Gamma) \vdash_{\mathcal{L}'} \widehat{f}(\varphi).$$

■

[2] Recall that a substitution in C is any function $\sigma : \mathcal{V} \to L(C)$. Since $L(C)$ is freely generated by C from $\mathcal{V}$, σ can be extended to a unique endomorphism $\widehat{\sigma} : L(C) \to L(C)$.

By defining composition of morphisms and identity morphisms, inheriting from what was done for the case of **Sig**, the category **Log** of (propositional) logics is defined. In this category, logics are presented by means of consequence relations. A fundamental property of **Log** is the following:

THEOREM 11 The category **Log** has arbitrary (small) products.

Proof. The argument can be easily adapted from that in [Bueno-Soler *et al.*, 2005] for the category of standard logics. ∎

3 Possible–translations Semantics

In this section the method of possible–translations semantics (PTSs) is briefly summarized and reviewed, and some examples are addressed. A categorial characterization of the method is also offered.

The concept of PTSs is based on the idea of defining a new global consequence relation by combining other, presumably simpler, consequence relations by means of translations. In this way, as commented in Section 1, PTSs can be seen to work on two opposite directions: as a splitting procedure, and a splicing procedure.

The idea behind possible-translations semantics is to encompass two or more basic semantic models (of the same similarity type) in such a way as to define a new logic which depends upon the basic ones by means of a collection of translations. The basic models can be distinct copies of classical models, or distinct many-valued models, or even Kripke models (for intuitionistic or modal logics).

In [Carnielli and Coniglio, 1999] a somewhat more abstract account of possible–translations semantics was investigated, considering the basic models as organized through sheaf structures. As is well known, sheaves are used in mathematics as a tool for investigating the relationship between local and global phenomena, and seems to be an adequate framework to frame the idea of possible translations.

As mentioned in Section 1, instead of thinking of synthesizing some given logics through a combination process in order to obtain a new logic (as is done with fibring, for instance), a logic can be split into a family of other logics; this question can be examined in terms of categories, resulting in a universal construction. This section outlines a categorial characterization for this process, originally propounded in [Bueno *et al.*, 2004].

DEFINITION 12 Let $\mathcal{L}_i = \langle C_i, \vdash_{\mathcal{L}_i} \rangle$ for $i = 1, 2$ be logics, and let $f : L(C_1) \to L(C_2)$ be a mapping.
(a) f is said to be a *translation* between $\mathcal{L}_1$ and $\mathcal{L}_2$ if it preserves deducibility,

that is, for every $\Gamma \cup \{\varphi\} \subseteq L(C_1)$, $\Gamma \vdash_{\mathcal{L}_1} \varphi$ implies that $f(\Gamma) \vdash_{\mathcal{L}_2} f(\varphi)$.
(b) f is said to be a *conservative translation* between $\mathcal{L}_1$ and $\mathcal{L}_2$ if, for every $\Gamma \cup \{\varphi\} \subseteq L(C_1)$, $\Gamma \vdash_{\mathcal{L}_1} \varphi$ iff $f(\Gamma) \vdash_{\mathcal{L}_2} f(\varphi)$.
(c) A morphism $\mathcal{L}_1 \xrightarrow{f} \mathcal{L}_2$ in **Log** is said to be *conservative* if $\widehat{f} : L(C_1) \to L(C_2)$ is a conservative translation. ■

Observe that each morphism f in **Log** induces a translation between logics $\widehat{f}$ in the sense of the definition above; we call it a *grammatical translation*, in the sense that n–ary connectives are mapped by f into n–ary formula schemas.

We begin by adapting the original definitions of [Carnielli, 2000] in order to make them suitable for categorial formalization.

DEFINITION 13 Let $\mathcal{L} = \langle C, \vdash_{\mathcal{L}} \rangle$ be a logic, and let $\{\mathcal{L}_i\}_{i \in I}$ be a family of logics such that $\mathcal{L}_i = \langle C_i, \vdash_{\mathcal{L}_i} \rangle$ for every $i \in I$. A *possible–translations frame for* $\mathcal{L}$ is a pair $P = \langle \{\mathcal{L}_i\}_{i \in I}, \{f_i\}_{i \in I} \rangle$ such that $f_i : L(C) \to L(C_i)$ is a translation between $\mathcal{L}$ and $\mathcal{L}_i$, for every $i \in I$. We say that $P = \langle \{\mathcal{L}_i\}_{i \in I}, \{f_i\}_{i \in I} \rangle$ is a *possible–translations semantics for* $\mathcal{L}$ (in short, a PTS) if, for every $\Gamma \cup \{\varphi\} \subseteq L(C)$,

$$\Gamma \vdash_{\mathcal{L}} \varphi \ \text{ iff } \ f_i(\Gamma) \vdash_{\mathcal{L}_i} f_i(\varphi) \ \text{ for every } i \in I.$$

A frame $P = \langle \{\mathcal{L}_i\}_{i \in I}, \{f_i\}_{i \in I} \rangle$ is said to be *small* if the class I is a set, and is said to be *grammatical* if f_i is a morphism $\mathcal{L} \xrightarrow{f_i} \mathcal{L}_i$ in **Log**, for every $i \in I$. Analogously, a possible–translations semantics is said to be *small* (respectively, *grammatical*) if it is small (respectively, grammatical) regarded as a frame. ■

REMARK 14 In order to obtain a categorial characterization of PTSs (see Theorem 15 below), possible–translations frames must here be restricted to small grammatical ones. ■

As mentioned above, a PTS for a logic $\mathcal{L}$ can be seen as a way of splitting the logic $\mathcal{L}$ into the family $\{\mathcal{L}_i\}_{i \in I}$ of logics by means of the translations $\{f_i\}_{i \in I}$.

Using Theorem 11, a characterization of PTSs can be given in terms of products and conservative translations. The next result was originally stated in [Bueno *et al.*, 2004] for the category of standard logics.

THEOREM 15 Small grammatical possible–translations semantics for a logic $\mathcal{L}$ are the same as conservative morphisms $\mathcal{L} \xrightarrow{f} \mathcal{L}'$, where $\mathcal{L}'$ is a product in **Log** of some small family of logics.

Proof. Let $\mathcal{L} = \langle C, \vdash_{\mathcal{L}}\rangle$ be a logic and let P be a small grammatical PTS for $\mathcal{L}$. The idea is to define a conservative morphism $\mathcal{L} \xrightarrow{\mathsf{t}(P)} \mathsf{L}(P)$ in **Log**, where $\mathsf{L}(P)$ is a product in **Log** of some family of logics, such that $\mathsf{t}(P)$ encodes P. And, conversely, given a conservative morphism $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ in **Log**, where $\mathcal{L}'$ is a product of logics, a small grammatical PTS for $\mathcal{L}$ encoding f, denoted $\mathsf{PTS}(f)$, can be defined, in such a manner that the assignments t and PTS are one inverse of the other.

Thus, assuming that $P = \langle\{\mathcal{L}_i\}_{i\in I}, \{f_i\}_{i\in I}\rangle$ is a small grammatical PTS for $\mathcal{L}$, consider the product $\langle \mathcal{L}^{\mathcal{F}}, \{\pi_i\}_{i\in I}\rangle$ in **Log** of the small family $\mathcal{F} = \{\mathcal{L}_i\}_{i\in I}$ (cf. Theorem 11). Since each f_i is a morphism in **Log** then, by the universal property of the product, there is a unique morphism $\mathcal{L} \xrightarrow{\mathsf{t}(P)} \mathcal{L}^{\mathcal{F}}$ in **Log** such that $f_i = \pi_i \centerdot \mathsf{t}(P)$ for every $i \in I$. From this, it is not difficult to prove that

$$(*) \qquad \widehat{f_i} = \widehat{\pi_i} \circ \widehat{\mathsf{t}(P)}\,.$$

Using this, it can be proved that $\mathsf{t}(P)$ is a conservative morphism. Clearly, $\mathsf{t}(P)$ together with its codomain $\mathsf{L}(P) := \mathcal{L}^{\mathcal{F}}$ encodes all the information about P: every logic $\mathcal{L}_i$ is obtained as the codomain of π_i, and every morphism f_i is obtained as $f_i = \pi_i \centerdot \mathsf{t}(P)$.

Conversely, let $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ be a conservative morphism in **Log**, such that $\mathcal{L}'$ is a product in **Log** of a small family $\{\mathcal{L}_i\}_{i\in I}$ of logics, with canonical projections π_i for every $i \in I$. For every $i \in I$ consider the morphism $f_i = \pi_i \centerdot f$ in **Log**, and define the small grammatical possible–translations frame $\mathsf{PTS}(f) = \langle\{\mathcal{L}_i\}_{i\in I}, \{f_i\}_{i\in I}\rangle$. Using $(*)$ again, it can be proven that $\mathsf{PTS}(f)$ is a (small and grammatical) PTS for $\mathcal{L}$. Moreover, all the information about f and $\mathcal{L}'$ can be recovered from $\mathsf{PTS}(f)$: in fact $f = \mathsf{t}(\mathsf{PTS}(f))$ and $\mathcal{L}'$ is the product of the family of logics of $\mathsf{PTS}(f)$. It is also clear that, if P is a small grammatical PTS for $\mathcal{L}$, then $\mathsf{PTS}(\mathsf{t}(P)) = P$. ■

We now show that matrix semantics (referred to in Section 1) for propositional logics can be portrayed as a particular case of PTSs (see Theorem 22 below). In order to do this, we briefly recall some basic facts about matrix semantics.

DEFINITION 16 Given a signature C, a *C–matrix* is a pair $M = \langle \mathbf{A}, D\rangle$, where $\mathbf{A} = \langle A, C\rangle$ is an algebra over C, and $D \subseteq A$. The set D is usually referred to as the *set of designated values of* M. The *M–valuations of* $L(C)$ are the C–homomorphisms $v : L(C) \to A$. ■

For simplicity, we sometimes write $M = \langle A, D\rangle$ instead of $M = \langle \mathbf{A}, D\rangle$ in concrete examples. Additionally, the interpretation of a connective c in

M will be frequently written as c^M.

DEFINITION 17 Let C be a signature and let $\mathcal{K}$ be a class of C–matrices. The *matrix semantics for* $L(C)$ *induced by* $\mathcal{K}$ (denoted by $\vdash_{\mathcal{K}}$) is defined by: $\Gamma \vdash_{\mathcal{K}} \varphi$ iff, for every C–matrix $M = \langle \mathbf{A}, D\rangle$ belonging to $\mathcal{K}$ and every M–valuation v of $L(C)$, $v(\Gamma) \subseteq D$ implies that $v(\varphi) \in D$. ■

A logic $\mathcal{L}$ is said to be a *matrix logic* if there exists a class $\mathcal{K}$ of $C_{\mathcal{L}}$–matrices such that $\vdash_{\mathcal{L}} \, = \, \vdash_{\mathcal{K}}$. In this case, we say that $\mathcal{K}$ is *adequate* for $\mathcal{L}$, and that $\mathcal{L}$ is *characterized by* $\mathcal{K}$. As shown in [Wójcicki, 1969], every structural logic is indeed a matrix logic (see Theorem 21 below). When $\mathcal{K}$ $= \{M\}$ is a singleton then $\vdash_M$ will stand for $\vdash_{\{M\}}$.

DEFINITION 18 Let $\mathcal{L}$ be a logic and let M be a $C_{\mathcal{L}}$–matrix. If $\vdash_{\mathcal{L}} \subseteq \vdash_M$ we say that $\vdash_{\mathcal{L}}$ is *sound for* $\vdash_M$, or that M *is a matrix model for* $\mathcal{L}$. We define the class $\mathbf{MatMod}(\mathcal{L})$ as being the class of all the matrix models for $\mathcal{L}$. ■

Clearly, every matrix logic is a logic in the sense of Definition 7. Moreover, the following fundamental result due to J. Łoś and R. Suszko (see [Łoś and Suszko, 1958]) shows that a matrix logic is, in fact, structural:

THEOREM 19 Let $\mathcal{K}$ be a class of C–matrices. Then $\vdash_{\mathcal{K}}$ is a structural consequence relation and $\vdash_{\mathcal{K}} = inf\,\{\vdash_M : \; M \in \mathcal{K}\}$. [3]

Note that $\vdash_{\mathcal{K}}$ do not need to be finitary and, therefore, $\mathcal{L} = \langle C, \vdash_{\mathcal{K}}\rangle$ is not necessarily standard. The following sufficient condition for a matrix logic to be standard was obtained in [Wójcicki, 1973].

THEOREM 20 Every consequence relation induced by a finite class of finite matrices is finitary, and so defines a standard logic.

The next classical result is credited to A. Lindenbaum and R. Wójcicki (see [Wójcicki, 1969; Wójcicki, 1988]).

THEOREM 21 For every structural logic $\mathcal{L}$, the class $\mathbf{MatMod}(\mathcal{L})$ is a complete matrix semantics for $\mathcal{L}$.

It is simple to see that, by just considering identity mappings as translations, the notion of matrix logics is nothing else than a special case of grammatical possible–translation semantics. Indeed:

[3] The infimum is taken with respect to the inclusion ordering $\subseteq$.

THEOREM 22 Let $\mathcal{L} = \langle C, \vdash_{\mathcal{L}} \rangle$ be a matrix logic, and let $\mathcal{K}$ be a class of C–matrices adequate for $\mathcal{L}$. For every $M \in \mathcal{K}$ let $\mathcal{L}_M = \langle C, \vdash_M \rangle$ and let $\mathcal{L} \stackrel{f_M}{\to} \mathcal{L}_M$ be the morphism in **Log** induced by the identity morphism in the signature C. [4] Then the grammatical possible–translations frame

$$\mathsf{PTS}(\mathcal{K}) = \langle \{\mathcal{L}_M\}_{M \in \mathcal{K}}, \{f_M\}_{M \in \mathcal{K}} \rangle$$

is a grammatical possible–translations semantics for $\mathcal{L}$.

Proof. Immediate from Definition 13 and from the notion of adequate class of matrices. ■

The last result can be recast as stating that a logic $\mathcal{L}$ characterized by a class of matrices $\mathcal{K}$ splits over the elements of $\mathcal{K}$. That is, every matrix in $\mathcal{K}$ acts as a legitimate factor of $\mathcal{L}$, and so an adequate matrix semantics works as a particular instance of the splitting method defined by possible–translations semantics. Note that, if $\mathcal{K}$ is a proper class (instead of a set), then $\mathsf{PTS}(\mathcal{K})$ is not small. [5]

As an illustrative example, we prove below that the set of theorems of the propositional intuitionistic logic Int can be characterized by a possible–translations semantics (with identity translations) splitting Int into Heyting algebras.

EXAMPLE 23 It is well–known that theoremhood in Int is characterized by the class of matrices

$$\mathcal{H} = \{\langle \mathbf{H}, \{\top\}\rangle \ : \ \langle \mathbf{H}, \{\top\}\rangle \text{ is a Heyting algebra with top element } \top\}$$

(see, for instance, [Rasiowa and Sikorski, 1968]). For every $H = \langle \mathbf{H}, \{\top\}\rangle$ in $\mathcal{H}$ let $\mathcal{L}_H := \langle C, \vdash_H \rangle$ and let f_H be the morphism in **Log** induced by the identity morphism in the signature of Int. Then the grammatical possible–translations frame

$$\mathsf{PTS}(\mathcal{H}) = \langle \{\mathcal{L}_H\}_{H \in \mathcal{H}}, \{f_H\}_{H \in \mathcal{H}} \rangle$$

characterizes theoremhood for propositional intuitionistic logic Int. That is, given a formula φ, to check whether $\vdash_{\mathsf{Int}} \varphi$ is equivalent to check whether $\vdash_H \varphi$ for every Heyting algebra H. ■

[4]Since $\vdash_{\mathcal{L}} \subseteq \vdash_M$ then f_M is, in fact, a morphism in **Log**.

[5][Marcos, 2004] studies how possible-translations semantics characterize wider classes of propositional logics.

REMARK 24 It is appropriate here to warn the reader that matrix logics do not see the fine distinction between local and global semantics (current in modal and first–order logics): they only convey the notion of global semantics, because they deal with non–ordered algebras. In the example above, the matrix semantics for Int just represents global entailments, which are not the usual ones in algebraic semantics. Thus, given a Heyting algebra H and a set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_H \varphi$ iff, for every homomorphism $v : L(C) \to \mathbf{H}$, $v(\Gamma) \subseteq \{\top\}$ implies $v(\varphi) = \top$. On the other hand, the usual notion of entailment in an algebraic ordered structure is local, inasmuch as it requires that, for every v, $(\bigwedge_{\gamma \in \Gamma} v(\gamma)) \leq v(\varphi)$, where $\bigwedge$ denotes the infimum of a set. However, since Int is finitary and satisfies the Deduction meta–theorem, characterizing theoremhood is equivalent to characterizing the whole deducibility. ■

This section concludes by showing some applications of possible–translations semantics to paraconsistent logics.

EXAMPLE 25 The *Logics of Formal Inconsistency*, **LFI**s, are paraconsistent logics that internalize the metalogic notions of consistency and inconsistency at the object–language level by means of unary connectives $\circ$ for *consistency* and $\bullet$ for *inconsistency* (cf. [Carnielli *et al.*, 2005]), appropriately constrained by specific axioms.

Some interesting **LFI**s are the logics **bC** and **Ci**, as well as its weaker versions **mCi** and **mbC**. In order to obtain PTSs for these systems, which are defined over the signature $C = \{\wedge, \vee, \Rightarrow, \neg, \circ\}$, consider the signature $C_1 = \{\wedge, \vee, \Rightarrow, \neg_1, \neg_2, \neg_3, \circ_1, \circ_2, \circ_3\}$ and the matrix M over C_1 defined by the truth–tables below, where T and t are the designated values.

$\wedge$	T	t	F
T	t	t	F
t	t	t	F
F	F	F	F

$\vee$	T	t	F
T	t	t	t
t	t	t	t
F	t	t	F

$\Rightarrow$	T	t	F
T	t	t	F
t	t	t	F
F	t	t	t

	$\neg_1$	$\neg_2$	$\neg_3$
T	F	F	F
t	F	t	t
F	T	t	T

	$\circ_1$	$\circ_2$	$\circ_3$
T	T	t	F
t	F	F	F
F	T	t	F

Now consider the clauses below for a mapping $f : L(C) \to L(C_1)$.

($tr0$) $f(p) = p$ for $p \in \mathcal{V}$;

($tr1$) $f(\varphi \# \psi) = (f(\varphi) \# f(\psi))$, for $\# \in \{\wedge, \vee, \Rightarrow\}$;

($tr2$) $f(\neg\varphi) \in \{\neg_1 f(\varphi), \neg_2 f(\varphi)\}$;

($tr3$) $f(\neg\varphi) \in \{\neg_1 f(\varphi), \neg_3 f(\varphi)\}$;

($tr4$) $f(\neg^{n+1}\circ\varphi) = \neg_1 f(\neg^n \circ\varphi)$, for $n \in \mathbb{N}$;

($tr5$) $f(\circ\varphi) \in \{\circ_2 f(\varphi), \circ_3 f(\varphi), \circ_2 f(\neg\varphi), \circ_3 f(\neg\varphi)\}$;

($tr6$) $f(\circ\varphi) \in \{\circ_1 f(\varphi), \circ_1 f(\neg\varphi)\}$;

($tr7$) if $f(\neg\varphi) = \neg_1 f(\varphi)$ then $f(\circ\varphi) = \circ_1 f(\neg\varphi)$.

In clause ($tr4$) above, $\neg^n\varphi$ denotes n applications of $\neg$ over formula φ; in particular, $\neg^0\varphi = \varphi$. Now consider the following collections of mappings:
(a) Let $\{f_i^1\}_{i \in I_1}$ be the family of translations $f_i^1 : L(C) \to L(C_1)$ between **mbC** and $\langle C_1, \vdash_M\rangle$ satisfying clauses ($tr0$), ($tr1$), ($tr2$) and ($tr5$) above;
(b) Let $\{f_i^2\}_{i \in I_2}$ be the family of translations $f_i^2 : L(C) \to L(C_1)$ between **mCi** and $\langle C_1, \vdash_M\rangle$ satisfying clauses ($tr0$), ($tr1$), ($tr2$), ($tr4$) and ($tr6$) above;
(c) Let $\{f_i^3\}_{i \in I_3}$ be the family of translations $f_i^3 : L(C) \to L(C_1)$ between **bC** and $\langle C_1, \vdash_M\rangle$ satisfying clauses ($tr0$), ($tr1$), ($tr3$) and ($tr5$) above;
(d) Let $\{f_i^4\}_{i \in I_4}$ be the family of translations $f_i^4 : L(C) \to L(C_1)$ between **Ci** and $\langle C_1, \vdash_M\rangle$ satisfying clauses ($tr0$), ($tr1$), ($tr3$), ($tr6$) and ($tr7$) above.

Let $\mathcal{L}_i = \langle C_1, \vdash_M\rangle$ for $i \in \bigcup_{j=1}^4 I_j$. The next results are found in [Marcos, 2005] (see also [Carnielli *et al.*, 2005]):

(1) $\mathsf{PTS}_1 = \langle\{\mathcal{L}_i\}_{i\in I_1}, \{f_i^1\}_{i\in I_1}\rangle$ is a possible–translations semantics for the logic **mbC**.

(2) $\mathsf{PTS}_2 = \langle\{\mathcal{L}_i\}_{i\in I_2}, \{f_i^2\}_{i\in I_2}\rangle$ is a possible–translations semantics for the logic **mCi**.

(3) $\mathsf{PTS}_3 = \langle\{\mathcal{L}_i\}_{i\in I_3}, \{f_i^3\}_{i\in I_3}\rangle$ is a possible–translations semantics for the logic **bC**.

(4) $\mathsf{PTS}_4 = \langle\{\mathcal{L}_i\}_{i\in I_4}, \{f_i^4\}_{i\in I_4}\rangle$ is a possible–translations semantics for the logic **Ci**.

Examples (1)–(4) above illustrate the fact that the class of grammatical PTSs, even if quite wide, is not enough: for simple logics as the **LFI**s above mentioned no grammatical PTSs are known. There are in the literature other important examples of non–grammatical PTSs, as for instance the one for the hierarchy $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ of da Costa's paraconsistent systems given in [Carnielli, 2000] (see also [Marcos, 1999]). The fact that in the mentioned examples no grammatical *PTS*s are known does not mean, of course, that they would be impossible to find. We conjecture, however, that in all those cases no grammatical *PTS* can be found. ■

4 Nondeterministic semantics and a comparison

This section is devoted to reviewing the nondeterministic semantics introduced in [Avron and Lev, 2001] (see also [Avron and Lev, 2005]), proposing a generalization and comparing them with possible–translation semantics. The basic idea of nondeterministic semantics is to use matrices in which each entry consists of a set of truth–values instead of a single value.

DEFINITION 26 Let C be a signature. A *nondeterministic matrix* for C is a structure $\mathsf{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O}\rangle$ such that $\mathcal{T}$ is a nonempty set (of *truth–values*), $\mathcal{D} \subseteq \mathcal{T}$ is a nonempty set (of *designated values*) and $\mathcal{O}$ is a mapping which assigns to every n–ary connective $c \in C^n$ a mapping $\mathcal{O}(c) : \mathcal{T}^n \to \wp(\mathcal{T}) \setminus \{\emptyset\}$. A *valuation* over M is a mapping $v : L(C) \to \mathcal{T}$ such that $v(c(\varphi_1, \ldots, \varphi_n)) \in \mathcal{O}(c)(v(\varphi_1), \ldots, v(\varphi_n))$ for every $c \in C^n$, $\varphi_i \in L(C)$ $(i = 1, \ldots, n)$ and $n \in \mathbb{N}$. The consequence relation $\models_{\mathsf{M}}$ induced by M is defined as follows: let $\Gamma \cup \{\varphi\} \subseteq L(C)$; then $\Gamma \models_{\mathsf{M}} \varphi$ if, for every valuation v over M, $v(\Gamma) \subseteq \mathcal{D}$ implies $v(\varphi) \in \mathcal{D}$. A logic $\mathcal{L}$ is *sound* (respectively, *complete*) for M if, for every $\Gamma \cup \{\varphi\} \subseteq L(C)$ it holds: $\Gamma \vdash_{\mathcal{L}} \varphi$ only if (respectively, if) $\Gamma \models_{\mathsf{M}} \varphi$. $\mathcal{L}$ is *adequate* for M if it is sound and complete for M. ■

EXAMPLE 27 [Avron and Lev, 2005] Let C be the signature for the logic **Ci** (recall Example 25) and consider the following nondeterministic matrix $\mathsf{M}_{\mathbf{Ci}}$: $\mathcal{T} = \{\top, \bot, u\}$; $\mathcal{D} = \{\top, u\}$; and $\mathcal{O}$ is defined by the tables below.

$\mathcal{O}(\vee)$	$\bot$	u	$\top$
$\bot$	$\{\bot\}$	$\mathcal{D}$	$\mathcal{D}$
u	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
$\top$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$

$\mathcal{O}(\wedge)$	$\bot$	u	$\top$
$\bot$	$\{\bot\}$	$\{\bot\}$	$\{\bot\}$
u	$\{\bot\}$	$\mathcal{D}$	$\mathcal{D}$
$\top$	$\{\bot\}$	$\mathcal{D}$	$\mathcal{D}$

$\mathcal{O}(\Rightarrow)$	$\bot$	u	$\top$
$\bot$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
u	$\{\bot\}$	$\mathcal{D}$	$\mathcal{D}$
$\top$	$\{\bot\}$	$\mathcal{D}$	$\mathcal{D}$

	$\mathcal{O}(\neg)$	$\mathcal{O}(\circ)$
$\bot$	$\{\top\}$	$\{\top\}$
u	$\mathcal{D}$	$\{\bot\}$
$\top$	$\{\bot\}$	$\{\top\}$

Then the logic **Ci** is adequate for $\mathsf{M}_{\mathbf{Ci}}$, as shown in [Avron and Lev, 2005]. ■

In what follows we show that nondeterministic matrices are a particular case of possible–translations semantics. From now on, C will denote a fixed signature.

DEFINITION 28 Let $\mathsf{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be a nondeterministic matrix for C.
(1) Let $c \in C^n$ be a n–ary connective. An *instance of c in* M is a mapping $\mathsf{i} : \mathcal{T}^n \to \mathcal{T}$ such that $\mathsf{i}(\vec{x}) \in \mathcal{O}(c)(\vec{x})$ for every $\vec{x} \in \mathcal{T}^n$. Let $\mathcal{I}_c^{\mathsf{M}}$ be the set of instances of c in M. Note that, if $c \in C^0$ then $\mathcal{I}_c^{\mathsf{M}} = \mathcal{O}(c)$.
(2) For each $c \in |C|$ and every $\mathsf{i} \in \mathcal{I}_c^{\mathsf{M}}$ let $\bar{\mathsf{i}}$ be a new symbol such that $\mathsf{i} \neq \mathsf{j}$ implies that $\bar{\mathsf{i}} \neq \bar{\mathsf{j}}$. The *signature derived from* M is the signature C_{M} such that $C_{\mathsf{M}}^n = \bigcup_{c \in C^n} \{\bar{\mathsf{i}} \ : \ \mathsf{i} \in \mathcal{I}_c^{\mathsf{M}}\}$.
(3) The *matrix derived from* M is the C_{M}–matrix $M(\mathsf{M})$ with domain $\mathcal{T}$ and set of designated values $\mathcal{D}$ such that, for every $n \in \mathbb{N}$ and every $\bar{\mathsf{i}} \in C_{\mathsf{M}}^n$, $\bar{\mathsf{i}}^{M(\mathsf{M})} = \mathsf{i}$. That is, the interpretation of the n–ary connective $\bar{\mathsf{i}}$ in the algebra $M(\mathsf{M})$ is the mapping $\mathsf{i} : \mathcal{T}^n \to \mathcal{T}$. Let $\mathcal{L}_{\mathsf{M}} = \langle C_{\mathsf{M}}, \vdash_{M(\mathsf{M})} \rangle$ be the matrix logic associated to the matrix $M(\mathsf{M})$. ■

EXAMPLE 29 Consider again the nondeterministic matrix $\mathsf{M}_{\mathbf{Ci}}$ (see Example 27). The mappings $\vee_i : \mathcal{T}^2 \to \mathcal{T}$ $(i = 1, 2)$ defined by the tables below are two instances of disjunction $\vee$ in $\mathsf{M}_{\mathbf{Ci}}$.

$\vee_1$	$\bot$	u	$\top$
$\bot$	$\bot$	t	u
u	u	$\top$	u
$\top$	$\top$	$\top$	$\top$

$\vee_2$	$\bot$	u	$\top$
$\bot$	$\bot$	u	u
u	$\top$	$\top$	$\top$
$\top$	u	u	u

On the other hand, there are two instances $\neg_1$ and $\neg_2$ of the negation $\neg$ and just one instance $\circ_1$ of the consistency operator $\circ$ in $\mathsf{M}_{\mathbf{Ci}}$, displayed below.

	$\neg_1$	$\neg_2$	$\circ_1$
$\bot$	$\top$	$\top$	$\top$
u	$\top$	u	$\bot$
$\top$	$\bot$	$\bot$	$\top$

■

Note that the signature C^{MCi} contains 2^8 symbols for disjunction, 2^4 symbols for conjunction, 2^7 symbols for implication, two symbols for negation and one symbol for consistency. In general, if $c \in C^n$ and the set $\mathcal{T}$ has cardinal κ, let $\mathcal{T}^n = \{\vec{x}_\alpha \ : \ \alpha < \kappa^n\}$. Suppose that the set $\mathcal{O}(c)(\vec{x}_\alpha)$ has cardinal κ_α for every $\alpha < \kappa^n$. Then there are $\prod_{\alpha<\kappa^n} \kappa_\alpha$ instances of c in M.

Using the definitions above, we can obtain a possible–translations semantics PTS for $\mathcal{L} = \langle C, \models_{\mathsf{M}} \rangle$. The central idea is to substitute each nondeterministic operation $\mathcal{O}(c)$ by the set of operations $\mathcal{I}_c^{\mathsf{M}}$. This is done by means of a set of translations such that the formula $c(\varphi_1, \ldots, \varphi_n)$ is translated by f as $\bar{\mathsf{i}}(f(\varphi_1), \ldots, f(\varphi_n))$ for some $\mathsf{i} \in \mathcal{I}_c^{\mathsf{M}}$.

DEFINITION 30 Given a nondeterministic matrix M over C, let $\mathcal{F}_{\mathsf{M}}$ be the family $\{f_j\}_{j \in I}$ of all the mappings $f_j : L(C) \to L(C_{\mathsf{M}})$ such that:

($tr0$) $f_j(p) = p$ for $p \in \mathcal{V}$;

($tr1$) $f_j(c) \in \{\bar{\mathsf{i}} \ : \ \mathsf{i} \in \mathcal{I}_c^{\mathsf{M}}\}$, for $c \in C^0$;

($tr2$) $f_j(c(\varphi_1, \ldots, \varphi_n)) \in \{\bar{\mathsf{i}}(f_j(\varphi_1), \ldots, f_j(\varphi_n)) \ : \ \mathsf{i} \in \mathcal{I}_c^{\mathsf{M}}\}$, for $c \in C^n$, $n \geq 1$, $\varphi_1, \ldots, \varphi_n \in L(C)$.

■

LEMMA 31 For every valuation $w : L(C_{\mathsf{M}}) \to \mathcal{T}$ for $\mathcal{L}_{\mathsf{M}}$ and every mapping f_j in $\mathcal{F}_{\mathsf{M}}$ there exists a valuation v over M such that $v(\varphi) = w(f_j(\varphi))$ for every formula $\varphi \in L(C)$.

Proof. Given w and f_j, consider the mapping $v : L(C) \to \mathcal{T}$ such that $v(\varphi) = w(f_j(\varphi))$ for every formula $\varphi \in L(C)$. It is clear by definition of $\mathcal{F}_{\mathsf{M}}$ that v is a valuation over M. ■

COROLLARY 32 Let M be a nondeterministic matrix over C, let $\mathcal{L} = \langle C, \models_{\mathsf{M}} \rangle$ and let $\mathcal{F}_{\mathsf{M}} = \{f_j\}_{j \in I}$ as in Definition 30. Then every mapping f_j in $\mathcal{F}_{\mathsf{M}}$ is in fact a translation between $\mathcal{L}$ and $\mathcal{L}_{\mathsf{M}}$ (recall Definition 28(3)).

DEFINITION 33 Let M be a nondeterministic matrix over C, let $\mathcal{L} = \langle C, \models_{\mathsf{M}} \rangle$ and let $\mathcal{F}_{\mathsf{M}} = \{f_j\}_{j \in I}$ as in Definition 30. The *possible–translations frame for* M is defined as $\mathsf{PTS}(\mathsf{M}) = \langle \{\mathcal{L}_j\}_{j \in I}, \{f_j\}_{j \in I} \rangle$ such that $\mathcal{L}_j = \mathcal{L}_{\mathsf{M}}$ for every $j \in I$. ■

From Corollary 32 it follows that the just defined $\mathsf{PTS}(\mathsf{M})$ is indeed a possible–translations frame.

LEMMA 34 For every valuation v over M there exists a valuation $w : L(C_{\mathsf{M}}) \to \mathcal{T}$ for $\mathcal{L}_{\mathsf{M}}$ and a translation f in $\mathsf{PTS}(\mathsf{M})$ such that $v(\varphi) = w(f(\varphi))$ for every formula $\varphi \in L(C)$.

Proof. Given the valuation v, define $w(p) = v(p)$ for every $p \in \mathcal{V}$, and extend the mapping w to $L(C_{\mathsf{M}})$ homomorphically. The mapping f is recursively defined as follows: $f(p) = p$ if $p \in \mathcal{V}$ and $f(c) = \bar{\mathsf{i}}$ for some element i of $\mathcal{I}_c^{\mathsf{M}}$, if $c \in C^0$. Note that $v(\varphi) = w(f(\varphi))$ for every formula $\varphi \in L(C)$ with complexity 1. Suppose that the mapping f was already defined for every formula φ with complexity $\leq n$ ($n \geq 1$) such that $v(\varphi) = w(f(\varphi))$. Let $c \in C^k$ and $\varphi_1, \ldots, \varphi_k \in L(C)$ (for $k \geq 1$) such that every φ has complexity at most n. Let $\mathsf{i} \in \mathcal{I}_c^{\mathsf{M}}$ such that $\mathsf{i}(v(\varphi_1), \ldots, v(\varphi_k)) = v(c(\varphi_1, \ldots, \varphi_k))$. It is clear that there exists such an instance because $v(c(\varphi_1, \ldots, \varphi_k)) \in \mathcal{O}(c)(v(\varphi_1), \ldots, v(\varphi_k))$. Then let us define $f(c(\varphi_1, \ldots, \varphi_k)) = \bar{\mathsf{i}}(f(\varphi_1), \ldots, f(\varphi_k))$. By induction hypothesis, $v(\varphi_i) = w(f(\varphi_i))$ for $i = 1, \ldots, k$. Then

$$\begin{aligned} v(c(\varphi_1, \ldots, \varphi_k)) = \; & \mathsf{i}(v(\varphi_1), \ldots, v(\varphi_k)) \\ & = \mathsf{i}(w(f(\varphi_1)), \ldots, w(f(\varphi_k))) \\ & = w(\bar{\mathsf{i}}(f(\varphi_1), \ldots, f(\varphi_k))) \\ & = w(f(c(\varphi_1, \ldots, \varphi_k))). \end{aligned}$$

■

THEOREM 35 Nondeterministic semantics can be simulated by possible–translations semantics.

Proof. From Lemmas 34 and 31, it is immediate that the possible–translations frame $\mathsf{PTS}(\mathsf{M})$ is a possible–translations semantics for M. That is, for every $\Gamma \cup \{\varphi\} \subseteq L(C)$ it holds: $\Gamma \models_{\mathsf{M}} \varphi$ if and only if $\Gamma \models_{\mathsf{PTS(M)}} \varphi$. ■

A weaker converse relation (the fact that every possible–translations semantics PTS for a structural logic can be simulated by a nondeterministic matrix) is easily seen to be valid, provided that the logic $\mathcal{L}$ represented by the possible–translations frame PTS is structural and that we expand the definition of a nondeterministic matrix, allowing for *classes* of nondeterministic matrices. Indeed, in formal terms, consider the following:

DEFINITION 36 Let C be a signature. A *multiple nondeterministic matrix* is a class $\mathcal{M}$ of nondeterministic matrices for C. The *multiple nondeterministic matrix semantics for $L(C)$ induced by* $\mathcal{M}$ (denoted by $\models_{\mathcal{M}}$) is defined by: $\Gamma \models_{\mathcal{M}} \varphi$ iff $\Gamma \models_{\mathsf{M}} \varphi$ for every $\mathsf{M} \in M$. A logic $\mathcal{L} = \langle C, \vdash_{\mathcal{L}} \rangle$ over C is *sound* (respectively, *complete*) for $\mathcal{M}$ if $\vdash_{\mathcal{L}} \subseteq \models_{\mathsf{M}}$ (respectively, $\models_{\mathsf{M}} \subseteq \vdash_{\mathcal{L}}$). $\mathcal{L}$ is *adequate* for $\mathcal{M}$ if it is sound and complete for $\mathcal{M}$. ■

THEOREM 37 Possible–translations semantics for structural logics can be simulated by multiple nondeterministic matrix semantics.

Proof. Let PTS be a possible–translations semantics for a structural logic $\mathcal{L}$. Then $\mathcal{L}$ must have a matrix semantics, by Theorem 21. Since matrix semantics are particular cases of multiple nondeterministic matrix semantics, the result follows. ■

As a matter of fact, Theorem 35 can be extended to cover multiple nondeterministic matrix semantics. Indeed:

THEOREM 38 Multiple nondeterministic matrix semantics can be simulated by possible–translations semantics.

Proof. Let $\mathcal{M}$ be a multiple nondeterministic matrix for C. For each $\mathsf{M} \in \mathcal{M}$ let $\mathsf{PTS}(\mathsf{M}) = \langle \{\mathcal{L}_j\}_{j \in I_{\mathsf{M}}}, \{f_j\}_{j \in I_{\mathsf{M}}} \rangle$ be the possible–translations frame for M. Assume, without loss of generality, that $I_{\mathsf{M}} \cap I_{\mathsf{N}} = \emptyset$ if $\mathsf{M} \neq \mathsf{N}$. Let $J = \bigcup_{\mathsf{M} \in \mathcal{M}} I_{\mathsf{M}}$. Then it is immediate that the possible–translations frame $\mathsf{PTS}(\mathcal{M}) = \langle \{\mathcal{L}_j\}_{j \in J}, \{f_j\}_{j \in J} \rangle$ is a possible–translations semantics for $\mathcal{M}$. ■

As a quick assessment of what has been accomplished so far, Theorems 21, 38 and 37 prove that, for structural logics, matrix semantics, possible–translations semantics and multiple nondeterministic matrix semantics are essentially equivalent.

For the original notion of nondeterministic matrices (recall Definition 26), however, the converse of Theorem 35 seems not to be true. Indeed, possible–translations semantics, and even the grammatical ones, are apparently more expressive than nondeterministic matrices, as the following argument intends to endorse.

As we saw above, given a nondeterministic semantics, a valuation v and a formula $c(\varphi_1, \ldots, \varphi_n)$ then, for some appropriate connective c', the value $v(c(\varphi_1, \ldots, \varphi_n))$ is of the form $c'(v(\varphi_1), \ldots, v(\varphi_n))$. This means that, if we think about the values of $v(\varphi)$ as being of the form $w(\widehat{f}(\varphi))$ for some morphism f and some valuation w onto a suitable logic, there is clearly no restriction on such morphisms, contrary to what is expected from the definition of PTSs, where some restrictions should apply to the translations in order to define expressive semantics. More specifically, suppose for simplicity that $\mathsf{PTS} = \langle \{\mathcal{L}_i\}_{i \in I}, \{f_i\}_{i \in I} \rangle$ is a given grammatical possible–translations semantics for $\mathcal{L}$ such that $\mathcal{L}_i = \mathcal{L}_j$ for every $i, j \in I$. Suppose that we want to define a nondeterministic matrix M representing PTS. In order to prove this, it would be necessary to recover Lemmas 34 and 31.

Thus, assuming that there is a given matrix $\langle \mathbf{A}, \mathcal{D}\rangle$ defining every $\mathcal{L}_i$ such that $\mathbf{A}$ is an algebra over signature C_1 with domain $\mathcal{T}$ such that $\mathcal{D} \subseteq \mathcal{T}$ and $\mathcal{L} \xrightarrow{f_i} \mathcal{L}_i$ for every $i \in I$, it seems clear that M should be defined over the set $\mathcal{T}$ with set $\mathcal{D}$ of designated values. Given $c \in C^n$ and $c' \in C_1^n$ we say that they are *compatible with respect to* PTS, written $comp_{\mathsf{PTS}}(c, c')$, if $\widehat{f}(c(\varphi_1, \ldots, \varphi_n)) = c'(\widehat{f}(\varphi_1), \ldots, \widehat{f}(\varphi_n))$ for some morphism f in PTS and some formulas $\varphi_1, \ldots, \varphi_n \in L(C)$. Then, the mapping $\mathcal{O}(c)$ should be defined as follows:

$$\mathcal{O}(c)(\vec{x}) = \{z \in \mathcal{T} \ : \ z = c'(\vec{x}) \text{ for some } c' \text{ such that } comp_{\mathsf{PTS}}(c, c')\}$$

for every $c \in C^n$ and every $\vec{x} \in \mathcal{T}^n$.

Let $\mathsf{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O}\rangle$. It is easy to see that Lemma 31 holds good, that is: if w is a valuation over matrix $\langle \mathbf{A}, \mathcal{D}\rangle$ and f is a morphism in PTS then there exists a valuation v over M such that $v(\varphi) = w(\widehat{f}(\varphi))$ for every formula $\varphi \in L(C)$. But, in order to recover Lemma 34, we found an unbridgeable barrier. In fact, given a valuation v over M, it is clear that the valuation w must be defined as $w(p) = v(p)$ for every $p \in \mathcal{V}$. Assuming that a morphism f was defined for $\varphi_1, \ldots, \varphi_n$ such that $w(\widehat{f}(\varphi_i)) = v(\varphi_i)$ for $i = 1, \ldots, n$, suppose that $v(c(\varphi_1, \ldots, \varphi_n)) = c'(v(\varphi_1), \ldots, v(\varphi_n))$. Then $\widehat{f}(c(\varphi_1, \ldots, \varphi_n))$ must be defined as $c'(\widehat{f}(\varphi_1), \ldots, \widehat{f}(\varphi_n))$. The problem is that there is no guarantee that this definition is possible in PTS!! That is, nothing guarantees that f defined as above is a morphism in PTS.

The results and arguments above provide an answer to a question posed in [Avron, 2005] about the relationship between nondeterministic matrices and possible–translations semantics.

5 Direct unions of matrices

This section discusses two simple mechanisms for combining matrix logics introduced in [Coniglio and Fernández, 2005] and [Fernández, 2005] called *direct union of matrices* and *plain fibring*. The fundamental idea is that, given two matrix logics $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_i$ is characterized by a single matrix M_i with domain A_i and designated values D_i $(i = 1, 2)$, it is possible to extend the original operators of the algebra M_i to the disjoint union $A_1 \uplus A_2$ by means of mappings $f_i : A_j \to A_i$ $(i \neq j)$.

Such mappings f_i allow us to 'transport' the 'foreign' truth–values of the matrix M_j into the truth–values of M_i. This approach follows similar lines as those in the original formulation of fibring (cf. [Gabbay, 1996; Gabbay, 1999]) in which, in order to evaluate, in a Kripke frame F_1, a modal operator $\Box_1$ in a world w_2 belonging to a Kripke frame F_2 (used for evaluating another modal operator $\Box_2$), the world w_2 is 'transported' (by means of a mapping

f) into a world $w_1 = f(w_2)$ of F_1, and vice–versa. The following definitions and results are taken from [Coniglio and Fernández, 2005].

Since the matrix defining the matrix logic is relevant for the operations to be defined below, in the sequel we will write $\langle C, M\rangle$ instead of $\langle C, \vdash_M\rangle$. Let us begin with the simplest case. Given two matrix logics having the same domain and the same sets of designated values, their *direct union* is obtained just by putting together both matrices. Formally:

DEFINITION 39 Let $\mathcal{L}_i = \langle C_i, M_i\rangle$ (with $i = 1, 2$) be two matrix logics, where each $M_i = \langle \mathbf{A}_i, D\rangle$ is a C_i–matrix such that $A_1 = A_2$. [6] Let $A = A_1$. The *direct union of* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ is the logic $\mathcal{L}_1 + \mathcal{L}_2 = \langle C_1 \uplus C_2, \vdash_{M_1+M_2}\rangle$ where $C_1 \uplus C_2$ is the disjoint union of C_1 and C_2 (recall Definition 1) and $\vdash_{M_1+M_2}$ is the consequence relation induced by the $C_1 \uplus C_2$–matrix $M_1 + M_2 = \langle \mathbf{A}, D\rangle$. The matrix $M_1 + M_2$ is defined as follows: if $c \in C_i^k$ and $a_1, \ldots, a_k \in A$, then $c^{M_1+M_2}(a_1, \ldots, a_k) = c^{M_i}(a_1, \ldots, a_k)$ ($k \geq 0$; $i = 1, 2$). ■

THEOREM 40 Let $\mathcal{L} = \langle C, \vdash\rangle$ be a logic characterized by a C–matrix M. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two fragments of $\mathcal{L}$ over C_1 and C_2, respectively, such that $C_1 \uplus C_2 = C$. Then $\mathcal{L}_1 + \mathcal{L}_2 = \mathcal{L}$.

The result above shows that the direct union of logics can be seen as a method for splitting and splicing logics. In particular, a given logic $\mathcal{L}$ can be split into two simpler factors $\mathcal{L}_1$ and $\mathcal{L}_2$ whenever $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$.

EXAMPLE 41 Let $\mathcal{L}_i = \langle C_i, M_i\rangle$ (with $i = 1, 2$) such that C_1 just contains a symbol $\neg$ for negation and C_2 just contains a symbol $\vee$ for disjunction and a symbol $\Rightarrow$ for implication. Suppose that M_1 is the matrix for classical negation, and that M_2 is the matrix for classical disjunction and implication over $A = \{1, 0\}$ where $D = \{1\}$. Then $\mathcal{L}_1 + \mathcal{L}_2$ turns out to be the matrix presentation $\mathcal{L}$ of classical propositional logic over A and D and signature $\{\neg, \vee, \Rightarrow\}$. The logics $\mathcal{L}_1$ and $\mathcal{L}_2$ are two (simpler) factors of $\mathcal{L}$. By its turn, $\mathcal{L}_2$, can of course be split into two elementary logics $\mathcal{L}_2^1$ (the logic of classical disjunction) and $\mathcal{L}_2^2$ (the logic of classical implication), that is, $\mathcal{L}_2 = \mathcal{L}_2^1 + \mathcal{L}_2^2$. Therefore, $\mathcal{L}$ splits into $\mathcal{L}_1$, $\mathcal{L}_2^1$ and $\mathcal{L}_2^2$, and so $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2^1 + \mathcal{L}_2^2$. ■

The possibility of recovering a logic from its components, as in the example above, was already addressed in [Coniglio, 2005], where some limitations of the usual notion of translation were pointed. In that paper it was shown that the operation of fibring as a coproduct (in a category of logics based

[6] It is worth noting that this condition *does not mean* that the operations defined in M_1 and M_2 coincide.

on translations, see, e.g., [Sernadas *et al.*, 1999]) cannot recover, in general, a logic from its fragments; this can be done by means of a stronger notion of translation that preserves meta–properties.

A more attractive case is to combine two matrix logics defined over different domains. In this case, each matrix logic is extended to the disjoint union of the domains by means of a pair of mappings, and then the direct union of the extensions is computed. The set of matrices obtained in this way is the matrix semantics of the so–called *plain fibring* (see Definition 43 below).

DEFINITION 42 Let $\mathcal{L}_i = \langle C_i, M_i \rangle$ (with $i = 1, 2$) be two matrix logics, where each $M_i = \langle \mathbf{A}_i, D_i \rangle$ is a C_i–matrix. Let A_i be the domain of the algebras $\mathbf{A}_i$.
(i) A pair $(f_1, f_2) \in A_1^{A_2} \times A_2^{A_1}$ is said to be *admissible* if it satisfies: $f_i(x) \in D_i$ iff $x \in D_j$, for every $x \in A_j$ $(i \neq j)$.
(ii) Given an admissible pair $\mathsf{a} = (f_1, f_2)$ then the *extension of* M_i *by* a is the C_i–matrix $M_i^{\mathsf{a}} = \langle \mathbf{A}, D_1 \uplus D_2 \rangle$ such that $A = A_1 \uplus A_2$ and, for every $c \in C_i^n$ and every $x_1, \ldots, x_n \in A$, $c^{M_i^{\mathsf{a}}}(x_1, \ldots, x_n) = c^{M_i}(\tilde{x}_1, \ldots, \tilde{x}_n)$ where, for every $k = 1, \ldots, n$:
– If $x_k \in A_i$, then $\tilde{x}_k = x_k$.
– If $x_k \in A_j$, then $\tilde{x}_k = f_i(x_k)$ (for $j \neq i$). ■

It is not hard to prove that the matrix logic $\mathcal{L}_i^{\mathsf{a}} = \langle C_i, M_i^{\mathsf{a}} \rangle$ coincides with $\mathcal{L}_i$, provided that a is admissible: i.e., $\Gamma \vdash_{M_i} \varphi$ iff $\Gamma \vdash_{M_i^{\mathsf{a}}} \varphi$, for every $\Gamma \cup \{\varphi\} \subseteq L(C_i)$ and $i = 1, 2$. This means that, by extending each logic by means of an admissible pair, the logics remain unchanged.

DEFINITION 43 Let $\mathcal{L}_i = \langle C_i, M_i \rangle$ (with $i = 1, 2$) be two matrix logics as in Definition 42. The *plain fibring* of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the pair $\mathcal{L}_1 \odot \mathcal{L}_2 = \langle C_1 \uplus C_2, \vdash_{M_1 \odot M_2} \rangle$ such that $M_1 \odot M_2$ is the set of $C_1 \uplus C_2$–matrices $M_1 \odot M_2 = \{M_1^{\mathsf{a}} + M_2^{\mathsf{a}} \ : \ \mathsf{a} \text{ is admissible}\}$. ■

We say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are *compatible* if there exist admissible pairs in $A_1^{A_2} \times A_2^{A_1}$. Clearly, $\mathcal{L}_1$ and $\mathcal{L}_2$ are compatible iff:
(i) $D_1 \neq \emptyset$ iff $D_2 \neq \emptyset$; and
(ii) $A_1 - D_1 \neq \emptyset$ iff $A_2 - D_2 \neq \emptyset$.

THEOREM 44 Let $\mathcal{L}_i = \langle C_i, M_i \rangle$ (with $i = 1, 2$) be two matrix logics as in Definition 42 such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are compatible. Then $\mathcal{L}_1 \odot \mathcal{L}_2$ is a conservative extension of both $\mathcal{L}_1$ and $\mathcal{L}_2$.

EXAMPLE 45 The 3–valued paraconsistent matrix logic P^1 was introduced in [Sette, 1973] and widely studied afterwards. The signature C_{P^1} of P^1 is such that its domain is $|C_{P^1}| = \{\neg_{P^1}, \Rightarrow_{P^1}\}$, and its matrix $M_{P^1} = \langle \mathbf{A}_{P^1}, \{T, T_1\}\rangle$ is such that $A_{P^1} = \{T, T_1, F\}$. The corresponding operations are displayed in the tables below.

	T	T_1	F
$\neg_{P^1}$	F	T	T

$\Rightarrow_{P^1}$	T	T_1	F
T	T	T	F
T_1	T	T	F
F	T	T	T

It is immediate to see that $P^1 = \mathcal{L}_1 + \mathcal{L}_2$, where $\mathcal{L}_1$ is the logic for $\neg_{P^1}$ and $\mathcal{L}_2$ is the logic for $\Rightarrow_{P^1}$ defined by the corresponding truth–tables. On the other hand, by computing the reduced matrix for $\mathcal{L}_2$ (observing that T and T_1 are congruent) it is clear that $\mathcal{L}_2$ is, in fact, the logic $\mathcal{L}_2^2$ for classical implication over $\{1, 0\}$ with 1 as the only designated value (recall Example 41). This shows that P^1 is obtained by composition of (or, equivalently in this case, can be decomposed into) the simpler logics $\mathcal{L}_1$ (the 3–valued logic of the P^1–negation) and $\mathcal{L}_2^2$ (the 2–valued logic of the classical implication). ■

EXAMPLE 46 The splitting for P^1 exhibited above can be directly obtained as follows: let $\mathcal{L}_1$ the 3–valued logic of the P^1–negation and let M_1 be its matrix (see Example 45). Let $\mathcal{L}_2^2$ be the matrix logic of classical implication given by the matrix M_2 below (recall Example 41).

$\Rightarrow$	1	0
1	1	0
0	1	1

Let $A = \{T, T_1, F, 1, 0\}$ and $D = \{T, T_1, 1\}$, and let $\mathsf{a} = (f_1, f_2)$ such that $f_1(1) = T$, $f_1(0) = F$, $f_2(T) = f_2(T_1) = 1$ and $f_2(F) = 0$. Then a is admissible and M_1^{a} and M_2^{a} are given by the tables below.

	T	T_1	1	F	0
$\neg$	F	T	F	T	T

$\Rightarrow$	T	T_1	1	F	0
T	1	1	1	0	0
T_1	1	1	1	0	0
1	1	1	1	0	0
F	1	1	1	1	1
0	1	1	1	1	1

Let $\mathcal{L}$ be the logic over $\{\neg, \Rightarrow\}$ characterized by the matrix $M_1^{\mathsf{a}} + M_2^{\mathsf{a}}$ given by the two tables above, with $\{T, T_1, 1\}$ as the set of designated values. Since T and 1 are congruent, and F and 0 are also congruent, the reduced matrix for $\mathcal{L}$ produces the 3–valued logic P^1. The details of this construction are left to the reader. ■

EXAMPLE 47 With the same notation as above we can consider, given M_1 and M_2, another admissible pair $\mathsf{a}' = (g_1, f_2)$ such that $g_1(1) = T_1$ and $g_1(0) = F$. It is worth noting that a and a' are the unique admissible pairs. The matrix $M_1^{\mathsf{a}'}$ is displayed below (observe that $M_2^{\mathsf{a}'} = M_2^{\mathsf{a}}$).

	T	T_1	1	F	0
$\neg$	F	T	T	T	T

Therefore the logic $\mathcal{L}_1 \odot \mathcal{L}_2^2$ is characterized by the set of matrices

$$M_1 \odot M_2 = \{M_1^{\mathsf{a}} + M_2^{\mathsf{a}},\ M_1^{\mathsf{a}'} + M_2^{\mathsf{a}'}\}.$$

As we saw in Example 46, the logic characterized by $M_1^{\mathsf{a}} + M_2^{\mathsf{a}}$ is P^1. On the other hand, the logic characterized by $M_1^{\mathsf{a}'} + M_2^{\mathsf{a}'}$ does not satisfy the formula $(p_1 \Rightarrow p_2) \Rightarrow \neg\neg(p_1 \Rightarrow p_2)$. In fact, taking any valuation v over the matrix $M_1^{\mathsf{a}'} + M_2^{\mathsf{a}'}$ such that $v(p_1) = v(p_2)$ then $v(p_1 \Rightarrow p_2) = 1$ and so $v(\neg\neg(p_1 \Rightarrow p_2)) = \neg\neg 1 = \neg T = F$. Consequently $v((p_1 \Rightarrow p_2) \Rightarrow \neg\neg(p_1 \Rightarrow p_2)) = (1 \Rightarrow F) = 0$, a non–designated value. This shows that $(p_1 \Rightarrow p_2) \Rightarrow \neg\neg(p_1 \Rightarrow p_2)$ is not a theorem of the logic $\mathcal{L}_1 \odot \mathcal{L}_2^2$. ■

6 On what is left open

We have analyzed here three methods for providing semantics for logical systems from the point of view of combinations of logics: possible–translations semantics in Section 3, nondeterministic semantics in Section 4 and direct union of matrices and the related plain fibring in Section 5. Such methods make up relevant procedures for combining logics, an area of increasing interest due to the needs of formalization from variate branches of investigation, as linguistics, software engineering, logic programming and others in formal science. In philosophy and logic themselves we will not fail to find appealing possibilities in combining logics, and specially in splitting logics

From the methods investigated, possible–translations semantics have been shown to be conceptually broader, embodying matrix semantics and nondeterministic semantics. There is, however, no known general procedure for deciding whether a given logic can be characterized by possible–translations semantics, nor there seems to be: general semantic tools, like the well–known relational semantics for modal logics, are just this way.

This amply justifies more restrained methods as nondeterministic semantics, direct union of matrices and plain fibring. More so from the outlook of splitting logics, as factors can be effectively computed, according to the examples given.

Attention to combining logics has been delimited to the bottom–up perspective of splicing logics. The top–down perspective of splitting deserves equal attention, as the potentialities for its applications are really significant.

Decomposing a logic $\mathcal{L}$ into simpler components offers optional tools for attacking problems of complexity of algorithms (via the satisfiability problem), questions in proof–theory and in algebraization of logics. We may even be able to define and to characterize which are prime logics, viz,. the ones that cannot be further split (up to a given method). These logics may be really interesting, as stimulating are the posed problems; our outcomes here advance in this direction.

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