Fibring in the Leibniz Hierarchy

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Abstract

This article studies preservation of certain algebraic properties of propositional logics when combined by fibring. The logics analyzed here are classified in protoalgebraic, equivalential and algebraizable. By introducing new categories of algebrizable logics and of deductivizable quasi-varieties, it is stated an isomorphism between these categories. This constitutes an alternative to a similar result found in the literature.

1 Introduction

The method of combination of logics known as *fibring* was introduced by D. Gabbay in the 90's (see [18] and [19]). The method was originally developed for combinations of modal logics, by means of functions between the sets of possible worlds of one logic into the class of Kripke models of the other, and vice-versa.

However, despite its simplicity when applied to modal logics, it is by no means obvious how to extend the method to other kind of logics. This was the main motivation for the categorial version of fibring proposed by

A. Sernadas and his collaborators (see, for instance, [25], [7] and [6]). The present paper follows this approach of fibring. For a comprehensive textbook describing the recent developments on fibring consult [9].

One of the main concerns of fibring is the study of *transference results* from the logics to be combined and the new logic obtained. For instance, the preservation of completeness, soundness and interpolation by fibring are some issues already studied in the literature. An important result of completeness preservation by fibring can be found in [31].

In this paper, we will focus our discussion on the preservation by fibring of certain properties related with algebraic characteristics of the involved logics. Specifically, the preservation of *protoalgebraizability*, *equivalentiality* and *algebraizability* will be analyzed. Such properties, among others, are used to classify propositional logics inside the so-called *Leibniz Hierarchy* of algebraizability. In rough terms, this hierarchy classifies the different logics according to its capability of admitting "better and better" algebraic semantics for them. We assume here that the reader is acquainted with abstract algebraic logic (cf. [3] and [12]). In [17] and [13], for instance, a deep discussion about the Leibniz Hierarchy can be found.

On the algebraic side, one of the consequences of the study proposed here is the construction of new algebraic structures from given ones throughout the process of fibring. The new structures constitute an adequate semantics for the logics obtained by fibring. This semantics is obtained through an isomorphism between the categories of algebraizable logics and deductivizable quasi-varieties (also called Blok-Pigozzi quasi-varieties). The isomorphism and the categories introduced here constitute an alternative to a similar result presented in [21].

This article is based in a number of results appeared in a previous paper [16] of the authors, as well as in the Ph.D Thesis [15] of the first author, under the supervision of the second author.

2 Basic Notions

In this section we briefly introduce the definitions of propositional language and propositional logic, to be used throughout the paper.

The basic step when defining fibring from a categorical point of view is to introduce two categories: the category of signatures (from which the languages of the logics are generated), and the category of logics to be combined by fibring.

Definition 2.1 *Let $\mathcal{V}$ be a denumerable set of* propositional variables*, which will keep fixed from now on. Elements of $\mathcal{V}$ will be denoted by $p_1, p_2, \ldots$ under a given enumeration.*
(1) A signature *is a family $C = \{C_k\}_{k\in\mathbb{N}}$ of sets such that $C_k \cap C_n = \emptyset = C_k \cap \mathcal{V}$ for every $k \neq n$. Elements in C_k are called* connectives of arity k*.*
(2) Given signatures C and C', we say that C is contained in C' *(indicated by $C \subseteq C'$) if, for every $k \in \mathbb{N}$, $C_k \subseteq C'_k$. The signatures $C \cup C'$ and $C \uplus C'$ (the* union *and the* disjoint union *of C and C', respectively) are defined pointwise: $(C \cup C')_k = C_k \cup C'_k$ and $(C \uplus C')_k = C_k \uplus C'_k$ for every $k \in \mathbb{N}$.*
(3) The propositional language generated by the signature C*, denoted by $L(C)$, is the algebra of words freely generated by C over $\mathcal{V}$, such that C_k is the set of k-ary operations of that algebra.*

The following notion is useful when defining logic systems:

Definition 2.2 *Let C be a signature. A function $\sigma : \mathcal{V} \longrightarrow L(C)$ is called* a substitution in C*.*

Since $L(C)$ is freely generated from $\mathcal{V}$, a substitution σ can be extended to a unique endomorphism $\widehat{\sigma} : L(C) \longrightarrow L(C)$ such that $\widehat{\sigma}(p) = \sigma(p)$ for every $p \in \mathcal{V}$, and $\widehat{\sigma}(c(\varphi_1, \ldots, \varphi_k)) = c(\widehat{\sigma}(\varphi_1), \ldots, \widehat{\sigma}(\varphi_k))$ if $c \in C_k$ and $\varphi_1, \ldots, \varphi_k \in L(C)$.

Definition 2.3 *The category **Sig** of signatures is defined as follows:*
(a) Objects: *signatures.*
(b) Morphisms: *a morphism $f : C \longrightarrow C'$ is a family $f = \{f_k\}_{k\in\mathbb{N}}$ such that $f_k : C_k \longrightarrow C'_k$ is a function, for every $k \in \mathbb{N}$.*
(c) Composition: *is defined pointwise, that is: $\{g_k\}_{k\in\mathbb{N}} \circ \{f_k\}_{k\in\mathbb{N}} = \{g_k \circ f_k\}_{k\in\mathbb{N}}$, where $g_k \circ f_k$ is the composition of functions in the category **Set** of sets.*
(d) Identity morphisms: *$id_C : C \longrightarrow C$ is given by $id_C = \{id_{C_k}\}_{k\in\mathbb{N}}$, where $id_{C_k} : C_k \longrightarrow C_k$ is the identity mapping.*

Remark 2.4 *Every morphism $f : C \longrightarrow C'$ in **Sig** generates a unique function $\widehat{f} : L(C) \longrightarrow L(C')$ such that $\widehat{f}(p) = p$ (for $p \in \mathcal{V}$) and*

$$\widehat{f}(c(\varphi_1, \ldots, \varphi_k)) = f(c)(\widehat{f}(\varphi_1), \ldots, \widehat{f}(\varphi_k))$$

*for $c \in C_k$ and $\varphi_1, \ldots, \varphi_k \in L(C)$. If $f : C \longrightarrow C'$ and $g : C' \longrightarrow C''$ are morphisms in **Sig** then $\widehat{g} \circ \widehat{f} = \widehat{g \circ f}$.*

The following result is well-known.

Proposition 2.5 ***Sig*** *is a (small) complete and cocomplete category. In particular,* ***Sig*** *has coproducts and coequalizers.*

The next step is to define a category of logics, based on **Sig**. Recall the following notions from the Polish school of logic:

Definition 2.6 *Given a signature C, a* consequence relation over the signature C *(or* in C*) is a relation* $\vdash \subseteq \wp(L(C)) \times L(C)$ *satisfying the following properties:*

- *If $\varphi \in \Gamma$ then $\Gamma \vdash \varphi$* *(***Extensiveness***).*
- *If $\Gamma \vdash \varphi$ and $\Sigma \vdash \Gamma$, then $\Sigma \vdash \varphi$* *(***Transitivity***).*

Notation 2.7 *In the definition above, $\wp(L(C))$ denotes the powerset of $L(C)$; $\Gamma \vdash \varphi$ stands for $\langle \Gamma, \varphi \rangle \in \vdash$; and $\Sigma \vdash \Gamma$ denotes that $\Sigma \vdash \varphi$ for every $\varphi \in \Gamma$. In particular, $\vdash \Gamma$ means that $\vdash \varphi$, for every $\varphi \in \Gamma$.*

It is worth noting that any consequence relation satisfies the following:

- If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Sigma$ then $\Sigma \vdash \varphi$ (**Monotonicity**).

Definition 2.8 *A* (propositional) logic *is a pair $\mathcal{L} = \langle C, \vdash \rangle$, where C is a signature and $\vdash$ is a consequence relation in C. A logic $\mathcal{L}$ is said to be:*

- Structural *if, for every substitution σ in C and every $\Gamma \cup \{\varphi\} \subseteq L(C)$:*
  If $\Gamma \vdash \varphi$ then $\widehat{\sigma}(\Gamma) \vdash \widehat{\sigma}(\varphi)$ *(***Structurality***).*
- Finitary *if, for every $\Gamma \cup \{\varphi\} \subseteq L(C)$:*
  If $\Gamma \vdash \varphi$ then $\Gamma' \vdash \varphi$ for some finite set $\Gamma' \subseteq \Gamma$ *(***Finitariness***).*
- Standard *if it is structural and finitary.*

Given a signature C, let $Cons_C$, $Stru_C$, Fin_C and $Stan_C$ be the set of consequence relations, structural, finitary and standard, respectively, defined in C. The following results are part of the folklore (see, for instance, [30]):

Proposition 2.9 *Let C be a signature. Then $\mathbf{Cons_C} = \langle Cons_C, \subseteq \rangle$ is a complete lattice.*

Proposition 2.10 *Let C be a signature. Then $\mathbf{Stru}_C = \langle Stru_C, \subseteq \rangle$ is a complete sublattice of $\mathbf{Cons_C}$. Moreover, $\mathbf{Fin}_C = \langle Fin_C, \subseteq \rangle$ and $\mathbf{Stan}_C = \langle Stan_C, \subseteq \rangle$ are complete lattices, but they are not sublattices of $\mathbf{Cons_C}$.*

The natural notion of morphism between logics is the following:

Definition 2.11 *Let $\mathcal{L} = \langle C, \vdash\rangle$ and $\mathcal{L}' = \langle C', \vdash'\rangle$ be two logics. A* translation *from $\mathcal{L}$ to $\mathcal{L}'$ is a function $h : L(C) \longrightarrow L(C')$ such that, for every $\Gamma \cup \{\varphi\} \subseteq L(C)$, $\Gamma \vdash \varphi$ implies $h(\Gamma) \vdash' h(\varphi)$.*

Translations between logics can be characterized as being "continuous functions".

Definition 2.12 *Let $\mathcal{L} = \langle C, \vdash\rangle$ be a logic. A* theory *(or* closed set*) of $\mathcal{L}$ is a set $T \subseteq L(C)$ such that, if $T \vdash_{\mathcal{L}} \varphi$ then $\varphi \in T$.*

The set of all the theories of $\mathcal{L}$ is denoted by $Th_{\mathcal{L}}$. The following classical result can be found, for instance, in [30].

Proposition 2.13
(1) The pair $\mathbf{Th}_{\mathcal{L}} = \langle Th_{\mathcal{L}}, \subseteq \rangle$ is a complete lattice, for every logic $\mathcal{L}$.
(2) Let $\mathcal{L} = \langle C, \vdash\rangle$ and $\mathcal{L}' = \langle C', \vdash'\rangle$ be two logics and let $h : L(C) \longrightarrow L(C')$ be a function. Then h is a translation iff $h^{-1}(T') \in Th_{\mathcal{L}}$ whenever $T' \in Th_{\mathcal{L}'}$.

Finally, a suitable category of (standard) logics is introduced:

Definition 2.14 *The* category **Cons** of (standard) logics based on **Sig** *is defined as follows:*
(a) Objects: *standard logics.*
(b) Morphisms: *a **Cons**-morphism $f : \langle C, \vdash\rangle \longrightarrow \langle C', \vdash'\rangle$ is simply a **Sig**-morphism $f : C \longrightarrow C'$ such that the induced function $\widehat{f} : L(C) \longrightarrow L(C')$ is a translation (cf. Definition 2.11).*
(c) Composition and identity morphisms: *as in **Sig**.*

We will only consider standard logics in **Cons**, because we are interested in combining logics in the Leibniz Hierarchy, which are assumed to be standard. Of course it should make sense to consider a broader category of consequence systems, not necessarily standard, in which **Cons** would appear as a full subcategory.

3 Fibring in the category Cons

The definition of (categorial) fibring in **Cons** is straightforward, following the original ideas from [25]. Thus, it is possible to define two forms of fibring in **Cons**: *unconstrained fibring*, in which there is no sharing of connectives, and *constrained fibring*, in which some connectives are shared in the resulting logic. Of course, unconstrained fibring is a particular case of constrained fibring (the signature to be shared is the empty one).

Both forms of fibring are based on the combination of the underlying signatures of the logics to be combined, and so fibring logics presupposes fibring signatures. The proof of the results in this section can be found, for instance, in [6]. In [15] a similar study was done for logics based on the category **Plan** of signatures, in which the morphisms are slightly improved. We will return to this point in Section 9.

Proposition 3.1 *The category* ***Cons*** *is (small) cocomplete.*

Definition 3.2 *The* (unconstrained) fibring *in* ***Cons*** *of two logics* $\mathcal{L} = \langle C, \vdash \rangle$ *and* $\mathcal{L}' = \langle C', \vdash' \rangle$ *is given by their coproduct* $\mathcal{L} \circledast \mathcal{L}'$ *computed in* ***Cons****.*

It is worth noting that $\mathcal{L} \circledast \mathcal{L}' = \langle C \circledast C', \vdash_{\circledast} \rangle$ such that $C \circledast C'$ is the disjoint union (coproduct in **Sig**) of C and C' (recall Definition 2.1) and $\vdash_{\circledast}$ is the least standard consequence relation over $C \circledast C'$ containing $i(\vdash)$ and $i'(\vdash')$. Here $i : C \longrightarrow C \circledast C'$ and $i' : C' \longrightarrow C \circledast C'$ are the canonical injections of the coproduct, and $i(\vdash) = \{\langle \widehat{i}(\Gamma), \widehat{i}(\varphi) \rangle \ : \ \Gamma \vdash \varphi\}$ (the relation $i'(\vdash')$ is defined similarly). In order to define constrained fibring (that is, fibring sharing connectives) it is necessary to use cocartesian liftings.

Definition 3.3 *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two categories, and let* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *be a functor. A* F-co-structured morphism *with codomain* $d \in \mathcal{D}$ *is a pair* $\langle c, f \rangle$ *such that* c *is a* $\mathcal{C}$*-object and* $f : F(c) \longrightarrow d$ *is a* $\mathcal{D}$*-morphism. A* cocartesian lifting *of a* F*-co-structured morphism* $\langle c, f \rangle$ *is a* $\mathcal{C}$*-morphism* $f^* : c \longrightarrow c'$ *such that* $F(f^*) = f$ *and satisfies the following universal property: for every* $\mathcal{C}$*-morphism* $g : c \longrightarrow c''$*, and every* $\mathcal{D}$*-morphism* $h : d \longrightarrow F(c'')$ *verifying* $h \circ f = F(g)$*, there is a unique* $\mathcal{C}$*-morphism* $h^* : c' \longrightarrow c''$ *with* $F(h^*) = h$ *and* $h^* \circ f^* = g$*. The functor* F *is a* cofibration *if every* F*-co-structured morphism admits a cocartesian lifting.*

Proposition 3.4 *Consider the forgetful functor* $N : \textbf{\textit{Cons}} \longrightarrow \textbf{\textit{Sig}}$*, that is:* $N(\langle C, \vdash \rangle) = C$ *and* $N(f) = f$*. Then* N *is a cofibration.*

Using the last result, it is possible to define constrained fibring in **Cons**. The idea is, given two logics $\mathcal{L} = \langle C, \vdash \rangle$ and $\mathcal{L}' = \langle C', \vdash' \rangle$, to consider a signature $\bar{C}$ and monomorphisms $j : \bar{C} \longrightarrow C$ and $j' : \bar{C} \longrightarrow C'$ in **Sig**. These monomorphisms represent the connectives we want to share through the fibring. Let $\mathcal{L} \circledast \mathcal{L}'$ be the coproduct in **Cons** of $\mathcal{L}$ and $\mathcal{L}'$ with canonical injections $i : \mathcal{L} \longrightarrow \mathcal{L} \circledast \mathcal{L}'$ and $i' : \mathcal{L}' \longrightarrow \mathcal{L} \circledast \mathcal{L}'$, and let $q : C \circledast C' \longrightarrow \widehat{C}$ be the coequalizer of

$$\bar{C} \underset{i' \circ j'}{\overset{i \circ j}{\rightrightarrows}} C \circledast C'$$

in **Sig**. Since $C \circledast C' = N(\mathcal{L} \circledast \mathcal{L}')$ then there exists the cocartesian lifting $q : \mathcal{L} \circledast \mathcal{L}' \longrightarrow \widehat{\mathcal{L}}$ of the N-co-structured morphism $\langle \mathcal{L} \circledast \mathcal{L}', q \rangle$ (note that $N(q) = q$). The logic $\widehat{\mathcal{L}}$ is then defined to be the fibring of $\mathcal{L}$ and $\mathcal{L}'$ constrained to the sharing diagram $C \overset{j}{\longleftarrow} \bar{C} \overset{j'}{\longrightarrow} C'$. This motivates the following definition:

Definition 3.5 *Let $\mathcal{L}$ and $\mathcal{L}'$ be two logics, and let D be a sharing diagram formed by two monomorphisms $j : \bar{C} \longrightarrow N(\mathcal{L})$ and $j' : \bar{C} \longrightarrow N(\mathcal{L}')$ in **Sig**. The* fibring of $\mathcal{L}$ and $\mathcal{L}'$ constrained by the sharing D *is the codomain $\mathcal{L} \circledast_D \mathcal{L}'$ of the cocartesian lifting of the coequalizer $q : N(\mathcal{L} \circledast \mathcal{L}') \longrightarrow \widehat{C}$ in **Sig** of*

$$\bar{C} \underset{i' \circ j'}{\overset{i \circ j}{\rightrightarrows}} N(\mathcal{L} \circledast \mathcal{L}')$$

*where $i : \mathcal{L} \longrightarrow \mathcal{L} \circledast \mathcal{L}'$ and $i' : \mathcal{L}' \longrightarrow \mathcal{L} \circledast \mathcal{L}'$ are the canonical injections of the coproduct $\mathcal{L} \circledast \mathcal{L}'$ in **Cons** of $\mathcal{L}$ and $\mathcal{L}'$.*

Taking into account Definitions 3.2 and 3.5 as well as Propositions 3.1 and 3.4 the first result concerning fibring is obtained:

Theorem 3.6 *The category **Cons** has both unconstrained and constrained fibring.*

4 Fibring protoalgebraic logics

We are now ready to we start our study of fibring within the Leibniz Hierarchy. As mentioned above, this study is based on [15], but using the category of signatures **Sig** instead of **Plan** (see Section 9 below).

In this section we begin with protoalgebraic logics, introduced in [2].

Definition 4.1 *A logic $\mathcal{L} = \langle C, \vdash \rangle$ is said to be* protoalgebraic *if there is a (possibly infinite) non-empty set $\Delta(p_1, p_2)$ of formulas in $L(C)$ depending at most on the propositional variables p_1 and p_2 such that (recall Note 2.7):*

(R) $\quad \vdash \Delta(p_1, p_1)$;

(MP) $\quad p_1, \Delta(p_1, p_2) \vdash p_2$.

The set $\Delta(p_1, p_2)$ will be called a *protoalgebraizator* of $\mathcal{L}$. Clearly, a logic is protoalgebraic if it can express the laws of reflexivity and Modus Ponens by means of a set $\Delta(p_1, p_2)$. From this, any propositional logic with an implication $\rightarrow$ satisfying reflexivity and Modus Ponens is protoalgebraic: it is enough to take

$$\Delta(p_1, p_2) = \{p_1 \rightarrow p_2\}.$$

Note that, if $f : C \longrightarrow C'$ is a morphism in **Sig** and $\varphi \in L(C)$ depends exactly on the propositional variables $p_1, \ldots, p_n$ then $\widehat{f}(\varphi)$ also depends exactly on $p_1, \ldots, p_n$. In particular, if $\Delta(p_1, p_2)$ is a protoalgebraizator of $\mathcal{L} = \langle C, \vdash \rangle$ and $f : \mathcal{L} \longrightarrow \mathcal{L}'$ is a **Cons**-morphism then $\widehat{f}(\Delta)$ is a set of formulas depending at most on p_1 and p_2. This fact, together with Definition 4.1, motivate the following definition:

Definition 4.2 *The* category **Prot** *is defined as follows:*
(a) Objects: *protoalgebraic logics.*
(b) Morphisms: *a morphism $f : \mathcal{L} \longrightarrow \mathcal{L}'$ is a* ***Cons****-morphism such that, for every protoalgebraizator $\Delta(p_1, p_2)$ of $\mathcal{L}$, the set $\widehat{f}(\Delta)(p_1, p_2)$ is a protoalgebraizator of $\mathcal{L}'$.*
(c) Composition and identity morphisms: *as is* ***Cons****.*

The interesting point is that, in fact, every **Cons**-morphisms preserves protoalgebraizators.

Proposition 4.3 ***Prot*** *is a full subcategory of* ***Cons****.*

Proof: Let $f : \mathcal{L} \longrightarrow \mathcal{L}'$ be a **Cons**-morphism and let $\Delta(p_1, p_2)$ be a protoalgebraizator for $\mathcal{L}$. Then

$$p_1,\ \Delta(p_1, p_2) \vdash p_2$$

and so

$$p_1,\ \widehat{f}(\Delta)(p_1, p_2) \vdash' p_2$$

because f is **Cons**-morphism. Analogously it can be proved that $\widehat{f}(\Delta)(p_1, p_2)$ satisfies property (R) of Definition 4.1. Therefore f is a **Prot**-morphism. ■

Despite every morphism in **Cons** preserves protoalgebraizators, the idea behind Definition 4.2 is that a protoalgebraic logic is a logic *plus* an additional structure that the morphisms should preserve. This is the approach we will adopt from now on when defining other categories of logics in the Leibniz Hierarchy.

Categorial fibring in **Prot** can be defined as in **Cons** (using the appropriate forgetful functor, in the case of constrained fibring). However, since **Prot** is a full subcategory of **Cons**, the proof of the following result is immediate:

Theorem 4.4 *The category* ***Prot*** *has both unconstrained and constrained fibring.*

Proof: The proof is a direct consequence of Theorem 3.6 and the following fact: if $\mathcal{L}$ is protoalgebraic and $f : \mathcal{L} \longrightarrow \mathcal{L}'$ is a **Cons**-morphism then $\mathcal{L}'$ is also protoalgebraic. Thus, given two logics in **Prot**, then their fibring (either constrained or unconstrained) as performed in **Cons** will result in a logic in **Prot** which corresponds to the respective fibring in **Prot**. ■

5 Fibring equivalential logics

Equivalential logics were introduced in [24] and, since that, were studied by several authors (see, for instance, [12] and [20]).

Definition 5.1 *A logic* $\mathcal{L} = \langle C, \vdash \rangle$ *is said to be* equivalential *or* congruential *if there exists a protoalgebraizator* $\Delta(p_1, p_2)$ *for* $\mathcal{L}$ *such that*

$$\Delta(p_1, p_{k+1}), \ldots, \Delta(p_k, p_{2k}) \vdash \Delta(c(p_1, \ldots, p_k), c(p_{k+1}, \ldots, p_{2k}))$$

for every $c \in C_k$.

A set $\Delta(p_1, p_2)$ as above is called an *equivalence* in $\mathcal{L}$. If $\mathcal{L}$ admits a finite equivalence then $\mathcal{L}$ is *finitely equivalential*. From now on, "equivalential logic" will stand for "finitely equivalential logic".

Remark 5.2 *The name "equivalential/congruential" is justified by the following fact: if* Δ *is an equivalence in* $\mathcal{L}$ *then*

$$\Delta(p_1, p_2) \vdash \Delta(p_2, p_1)$$

and

$$\Delta(p_1, p_2),\ \Delta(p_2, p_3) \vdash \Delta(p_1, p_3).$$

Therefore, the relation

$$\varphi R \psi \quad \textit{iff} \quad \vdash \Delta(\varphi, \psi)$$

is a congruence in the algebra $L(C)$. It is easy to prove by induction on the complexity of a formula $\varphi(p_1, \ldots, p_k)$ which depends at most on $p_1, \ldots, p_k$ that, if Δ is an equivalence,

$$\Delta(p_1, p_{k+1}), \ldots, \Delta(p_k, p_{2k}) \vdash \Delta(\varphi(p_1, \ldots, p_k), \varphi(p_{k+1}, \ldots, p_{2k})).$$

From now on, $\Delta \dashv\vdash \Gamma$ will stand for $\Delta \vdash \Gamma$ and $\Gamma \vdash \Delta$.

We now proceed to the study of fibring of equivalential logics. Generalizing Definition 4.2 we arrive to the following notion:

Definition 5.3 *The* category **Equiv** *is defined as follows:*
(a) Objects: *equivalential logics.*
(b) Morphisms: *a morphism $f : \mathcal{L} \longrightarrow \mathcal{L}'$ is a **Cons**-morphism such that, for every equivalence $\Delta(p_1, p_2)$ of $\mathcal{L}$, the set $\widehat{f}(\Delta)(p_1, p_2)$ is an equivalence in $\mathcal{L}'$.*
(c) Composition and identity morphisms: *as is **Cons**.*

The following properties of equivalential logics show that Definition 5.3 makes sense:

Proposition 5.4 *Let $\mathcal{L}$ be an equivalential logic, and let $\Delta(p_1, p_2)$ and $\Delta'(p_1, p_2)$ be two equivalences in $\mathcal{L}$. Then Δ and Δ' are interderivable in $\mathcal{L}$, that is:*

$$\Delta(p_1, p_2) \dashv\vdash \Delta'(p_1, p_2).$$

Moreover, If $\Delta(p_1, p_2)$ is an equivalence in $\mathcal{L}$ and $\Delta'(p_1, p_2)$ is a finite non-empty set of formulas such that $\Delta(p_1, p_2) \dashv\vdash \Delta'(p_1, p_2)$ then $\Delta'(p_1, p_2)$ is also an equivalence in $\mathcal{L}$.

Corollary 5.5 *Let $\mathcal{L}$ and $\mathcal{L}'$ be equivalential logics, and let $f : \mathcal{L} \longrightarrow \mathcal{L}'$ be a **Cons**-morphism. If $\widehat{f}(\Delta)$ is an equivalence in $\mathcal{L}'$ for some equivalence Δ in $\mathcal{L}$ then f is a morphism in **Equiv**.*

A proof of these results can be found, for instance, in [15].

Having defined the category **Equiv**, based on the category **Sig**, categorial fibring in **Equiv** can be defined in the same lines as it was done for **Cons** and **Prot** in the previous sections. Thus, unconstrained fibring is given by a coproduct (as in Definition 3.2) and restricted fibring is defined as cocartesian liftings of appropriate morphisms, as in Definition 3.5. In the case of constrained fibring, now it must be considered the forgetful functor from **Equiv** to **Sig**. In order to prove that **Equiv** has (constrained and unconstrained) fibring, we need to state some technical results.

Definition 5.6 *For $j = 1,2$ let $\mathcal{L}_j = \langle C^j, \vdash_j \rangle$ be an equivalential logic with equivalence $\Delta_j = \{\varphi_1^j, \ldots, \varphi_{k_j}^j\}$. Let $C = C^1 \uplus C^2$ be the disjoint union (that is, the coproduct in* ***Sig****) of C^1 and C^2 with canonical injections $i_1 : C^1 \longrightarrow C$ and $i_2 : C^2 \longrightarrow C$. The sequent calculus S over C, with sequents of the form $\Gamma \Longrightarrow \varphi$ such that $\Gamma \cup \{\varphi\}$ is a finite subset of $L(C)$, is defined by the rules below.*

$[Ax1]$ $\widehat{i_1}(\Gamma) \Longrightarrow \widehat{i_1}(\varphi)$ *if* $\Gamma \vdash_1 \varphi$ $\qquad$ $[Ax2]$ $\widehat{i_2}(\Gamma) \Longrightarrow \widehat{i_2}(\varphi)$ *if* $\Gamma \vdash_2 \varphi$

$[Ax3]_j$ $\widehat{i_1}(\Delta_1) \Longrightarrow \widehat{i_2}(\varphi_j^2)$ $\qquad$ $[Ax4]_j$ $\widehat{i_2}(\Delta_2) \Longrightarrow \widehat{i_1}(\varphi_j^1)$

$$[Mon]\ \frac{\Gamma \Longrightarrow \varphi}{\Gamma, \Sigma \Longrightarrow \varphi} \qquad [Cut]\ \frac{\Gamma \Longrightarrow \varphi \quad \varphi, \Sigma \Longrightarrow \psi}{\Gamma, \Sigma \Longrightarrow \psi}$$

$$[Subs]\ \frac{\Gamma \Longrightarrow \varphi}{\widehat{\sigma}(\Gamma) \Longrightarrow \widehat{\sigma}(\varphi)}$$

with the following provisos: in $[Ax1]$ and $[Ax2]$ the set Γ is finite; in $[Ax3]_j$ we have $1 \leq j \leq k_2$; in $[Ax4]_j$ we have $1 \leq j \leq k_1$; in $[Mon]$ the set Σ is finite; in $[Subs]$ the map σ is a substitution in C.

Observe that S depends on $\mathcal{L}_1$, $\mathcal{L}_2$, i_1, i_2, Δ_1 and Δ_2.

Lemma 5.7 *With the same notation as above, let $\vdash \subseteq \wp(L(C)) \times L(C)$ be a relation defined as follows: $\Gamma \vdash \varphi$ iff there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that the sequent $\Gamma_0 \Longrightarrow \varphi$ is provable in S. Then $\mathcal{L} = \langle C, \vdash \rangle$ is an equivalential logic such that $i_j : \mathcal{L}_j \longrightarrow \mathcal{L}$ is a morphism in* ***Equiv****, for $j = 1, 2$.*

Proof: Using the fact that $i_1(p) = i_2(p) = p$ for every propositional variable p, as well as the rules of S, it is easy to check that $\mathcal{L} = \langle C, \vdash \rangle$ is, in fact, a standard logic. For instance, since $p_1 \vdash_1 p_1$ then $p_1 \Longrightarrow p_1$ is derivable in S and so $\varphi \Longrightarrow \varphi$ is derivable in S, for every $\varphi \in L(C)$. From this, and by the very definition, $\mathcal{L}$ satisfies *Extensiveness* (recall Definition 2.6). On the other hand, both sets $\widehat{i_1}(\Delta_1)(p_1, p_2)$ and $\widehat{i_2}(\Delta_1)(p_1, p_2)$ are protoalgebraizators of $\mathcal{L}$, by Definition 4.1 and definition of S (axioms $[Ax1]$ and $[Ax2]$ are used here, as well as the requirement of considering finitely equivalential logics). In order to prove that $\widehat{i_1}(\Delta_1)(p_1, p_2)$ is an equivalence in $\mathcal{L}$ we must show that

$$\widehat{i_1}(\Delta_1)(p_1, p_{k+1}), \ldots, \widehat{i_1}(\Delta_1)(p_k, p_{2k}) \vdash \widehat{i_1}(\Delta_1)(c(p_1, \ldots, p_k), c(p_{k+1}, \ldots, p_{2k}))$$

for every $c \in C_k$. By definition of C, if $c \in C_k$ then either $c = i_1(c_1)$ for $c_1 \in C^1_k$, or $c = i_2(c_1)$ for $c_2 \in C^2_k$. If $c = i_1(c_1)$ for $c_1 \in C^1_k$ then we easily obtain the property above, because Δ_1 is an equivalence in $\mathcal{L}_1$ and by the very definition of $\mathcal{L}$ (axiom $[Ax1]$ is used here). Suppose now that $c = i_2(c_1)$ for $c_2 \in C^2_k$. Then

$$\widehat{i_2}(\Delta_2)(p_1, p_{k+1}), \ldots, \widehat{i_2}(\Delta_2)(p_k, p_{2k}) \vdash \widehat{i_2}(\Delta_2)(c(p_1, \ldots, p_k), c(p_{k+1}, \ldots, p_{2k}))$$

by the same argument as above. But $\widehat{i_1}(\Delta_1)(p_1, p_2) \vdash \widehat{i_2}(\Delta_2)(p_1, p_2)$ (using axiom $[Ax3]_j$, for $j = 1, \ldots, k_2$) and so $\widehat{i_1}(\Delta_1)(\varphi, \psi) \vdash \widehat{i_2}(\Delta_2)(\varphi, \psi)$ for every $\varphi, \psi \in L(C)$. Using this, the result follows by *Transitivity* (recall Definition 2.6). Thus $\widehat{i_1}(\Delta_1)(p_1, p_2)$ is an equivalence in $\mathcal{L}$ and then, by Proposition 5.4, the set $\widehat{i_2}(\Delta_2)(p_1, p_2)$ is also an equivalence in $\mathcal{L}$. Finally, by the very definition of $\mathcal{L}$ and by Corollary 5.5, the **Sig**-morphisms i_1 and i_2 are, in fact, morphisms $i_j : \mathcal{L}_j \longrightarrow \mathcal{L}$ in **Equiv**, for $j = 1, 2$. ■

Proposition 5.8 *With the same notation as above, the logic $\mathcal{L}$ is the coproduct in **Equiv** of $\mathcal{L}_1$ and $\mathcal{L}_2$, with canonical injections i_1 and i_2.*

Proof: Let $f_1 : \mathcal{L}_1 \longrightarrow \mathcal{L}'$ and $f_2 : \mathcal{L}_2 \longrightarrow \mathcal{L}'$ be two morphisms in **Equiv**, where $\mathcal{L}' = \langle C', \vdash' \rangle$. Since $\langle C, i_1, i_2 \rangle$ is the coproduct in **Sig** of $\langle C^1, C^2 \rangle$, there exists a (unique) **Sig**-morphism $h : C \longrightarrow C'$ such that $h \circ i_1 = f_1$ and $h \circ i_2 = f_2$, and so

$$(*) \qquad \widehat{h} \circ \widehat{i_1} = \widehat{f_1} \quad \text{and} \quad \widehat{h} \circ \widehat{i_2} = \widehat{f_2}$$

(recall Remark 2.4). From this, both $\widehat{f_1}(\Delta_1) = \widehat{h}(\widehat{i_1}(\Delta_1))$ and $\widehat{f_2}(\Delta_2) = \widehat{h}(\widehat{i_2}((\Delta_2))$ are equivalences in $\mathcal{L}'$, because Δ_j is an equivalence in $\mathcal{L}_j$ and f_j is a morphism in **Equiv** (for $j = 1, 2$).

Now we will prove that $\widehat{h} : L(C) \longrightarrow L(C')$ is a translation from $\mathcal{L}$ to $\mathcal{L}'$. In order to do this, note that derivations in S can be presented as finite sequences

$$\Gamma_1 \Longrightarrow \varphi_1 \;\cdots\; \Gamma_n \Longrightarrow \varphi_n$$

such that each $\Gamma_i \Longrightarrow \varphi_i$ is either an instance of an axiom of S, or it is obtained from previous sequents in the sequence by applying an inference rule of S.

Thus, suppose that $\Gamma \vdash \varphi$. Then, there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that the sequent $\Gamma_0 \Longrightarrow \varphi$ is provable in S. We will prove, by induction on the length n of a derivation of $\Gamma_0 \Longrightarrow \varphi$ in S, that $\widehat{h}(\Gamma_0) \vdash' \widehat{h}(\varphi)$ (and so $\widehat{h}(\Gamma) \vdash' \widehat{h}(\varphi)$ as required).

If $n = 1$ then $\Gamma_0 \Longrightarrow \varphi$ is an axiom. If $\Gamma_0 \Longrightarrow \varphi$ is $\widehat{i}_1(\Sigma) \Longrightarrow \widehat{i}_1(\psi)$ (an instance of $[Ax1]$) then $\Sigma \vdash_1 \psi$ and so $\widehat{f}_1(\Sigma) \vdash' \widehat{f}_1(\psi)$, because f_1 is a **Equiv**-morphism. Using $(*)$ it follows that $\widehat{h}(\Gamma_0) \vdash' \widehat{h}(\varphi)$. If $\Gamma_0 \Longrightarrow \varphi$ is an instance of $[Ax2]$ the proof is analogous. If $\Gamma_0 \Longrightarrow \varphi$ is $\widehat{i}_1(\Delta_1) \Longrightarrow \widehat{i}_2(\varphi_j^2)$ (that is, $[Ax3]_j$ for some $1 \leq j \leq k_2$) then $\widehat{h}(\Gamma_0) \vdash' \widehat{h}(\varphi)$, because $\widehat{h}(\widehat{i}_1(\Delta_1))$ and $\widehat{h}(\widehat{i}_2((\Delta_2))$ are equivalences in $\mathcal{L}'$ and by Proposition 5.4. If $\Gamma_0 \Longrightarrow \varphi$ is $[Ax4]_j$ for some $1 \leq j \leq k_1$ the proof is analogous.

Suppose that $\widehat{h}(\Gamma_0) \vdash' \widehat{h}(\varphi)$ for any sequent $\Gamma_0 \Longrightarrow \varphi$ which admits a derivation in S in $k \leq n$ steps (induction hypothesis), and let $\Gamma_0 \Longrightarrow \varphi$ be a sequent derivable in S in $n+1$ steps. The unique case which deserves attention is when $\Gamma_0 \Longrightarrow \varphi$ is obtained from a sequent $\Sigma \Longrightarrow \psi$ by the rule $[Subs]$. In this case, there is a substitution σ in C such that $\widehat{\sigma}(\Sigma) = \Gamma_0$ and $\widehat{\sigma}(\psi) = \varphi$. It is easy to prove that the substitution $\sigma' : \mathcal{V} \longrightarrow L(C')$ over C' given by $\sigma'(p) = \widehat{h}(\sigma(p))$ is such that $\widehat{\sigma}' \circ \widehat{h} = \widehat{h} \circ \widehat{\sigma}$. Since $\Sigma \Longrightarrow \psi$ is derivable in S in $k \leq n$ steps then $\widehat{h}(\Sigma) \vdash' \widehat{h}(\psi)$, by induction hypothesis. Since $\mathcal{L}'$ is structural then $\widehat{\sigma}'(\widehat{h}(\Sigma)) \vdash' \widehat{\sigma}'(\widehat{h}(\psi))$, that is, $\widehat{h}(\Gamma_0) \vdash' \widehat{h}(\varphi)$. From this we obtain that h is a morphism $h : \mathcal{L} \longrightarrow \mathcal{L}'$ in **Cons** such that $\widehat{h}(\widehat{i}_1(\Delta_1))$ is an equivalence in $\mathcal{L}'$, for the given equivalence $\widehat{i}_1(\Delta_1)$ in $\mathcal{L}$. From Corollary 5.5 it follows that h is a morphism in **Equiv**. Moreover, $h \circ i_1 = f_1$ and $h \circ i_2 = f_2$ in **Equiv**. The uniqueness of h in **Equiv** follows from the uniqueness of h in **Sig**. This shows that $\mathcal{L}$ is the coproduct of $\mathcal{L}_1$ and $\mathcal{L}_2$ in **Equiv** with canonical injections $i_1 : \mathcal{L}_1 \longrightarrow \mathcal{L}$ and $i_2 : \mathcal{L}_2 \longrightarrow \mathcal{L}$. ■

Remark 5.9 *The last results shows how the derivations look like in the coproduct (unconstrained fibring) of two equivalential logics: all the information is contained in the sequent calculus S. The construction of the calculus S (cf. [15]) was inspired by the notion of inferential basis introduced by J. Łoś and R. Suszko in [23]. However, the latter does not impose any restrictions to the lengths of the formal proofs (in fact, the concept of "inferential basis" is not related to proof theory but to abstract logics). So, the sequent calculus S is in fact a particular case of inferential basis.*

On the other hand, in [15] the following lattice-theoretic characterization of logic $\mathcal{L}$ was given. Recall from Proposition 2.10 that $\mathbf{Stan}_C = \langle Stan_C, \subseteq \rangle$ is a complete lattice. If $X \subseteq Stan_C$ then $\bigwedge_{\mathbf{Stan}_C} X$ will denote the infimum in $\mathbf{Stan}_C$ of the set X. Then, with notation as above, the logic $\mathcal{L}$ is given by $\langle C, \vdash \rangle$ such that $\vdash = \bigwedge_{\mathbf{Stan}_C} X$, where

$$X = \{\vdash \in Stan_C \ : \ i_1(\vdash_1) \cup i_2(\vdash_2) \subseteq \vdash \ \text{ and } \ \widehat{i}_1(\Delta_1) \dashv\vdash \widehat{i}_2(\Delta_2)\}.$$

In order to prove the existence of constrained fibring in **Equiv** we need to prove that the forgetful functor N_e : **Equiv**$\longrightarrow$**Sig** is a cofibration. We begin by introducing some definitions and technical results.

Definition 5.10 *Let $\mathcal{L} = \langle C, \vdash\rangle$ be an equivalential logic with equivalence Δ, and let $f : C \longrightarrow C'$ be a **Sig**-morphism. The sequent calculus S_e over C', with sequents of the form $\Gamma \Longrightarrow \varphi$ such that $\Gamma \cup \{\varphi\}$ is a finite subset of $L(C')$, is defined by the rules below.*

$$[AX1] \quad \widehat{f}(\Gamma) \Longrightarrow \widehat{f}(\varphi) \quad \textit{if}\ \Gamma \vdash \varphi$$

$$[AX2]_c \quad \widehat{f}(\Delta)(p_1, p_{k+1}), \ldots, \widehat{f}(\Delta)(p_k, p_{2k}) \Longrightarrow \widehat{f}(\Delta)(c(p_1, \ldots, p_k), c(p_{k+1}, \ldots, p_{2k}))$$

$$[MON] \quad \frac{\Gamma \Longrightarrow \varphi}{\Gamma, \Sigma \Longrightarrow \varphi} \qquad\qquad [CUT] \quad \frac{\Gamma \Longrightarrow \varphi \quad \varphi, \Sigma \Longrightarrow \psi}{\Gamma, \Sigma \Longrightarrow \psi}$$

$$[SUBS] \quad \frac{\Gamma \Longrightarrow \varphi}{\widehat{\sigma}(\Gamma) \Longrightarrow \widehat{\sigma}(\varphi)}$$

with the following provisos: in $[AX1]$ the set Γ is finite; in $[AX2]_c$ we have $c \in C'_k$ and $k \in \mathbb{N}$; in $[MON]$ the set Σ is finite; in $[SUBS]$ the map σ is a substitution in C'.

Observe that S_e depends on $\mathcal{L}$, f and Δ.

Lemma 5.11 *With the same notation as above, let $\vdash' \subseteq \wp(L(C')) \times L(C')$ be a relation defined as follows: $\Gamma \vdash' \varphi$ iff there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that the sequent $\Gamma_0 \Longrightarrow \varphi$ is provable in S_e. Then $\mathcal{L}' = \langle C', \vdash'\rangle$ is an equivalential logic such that $f : \mathcal{L} \longrightarrow \mathcal{L}'$ is a morphism in **Equiv**.*

Proof: The proof follows the same lines (and is even easier) than the proof of Lemma 5.7. Note that $\widehat{f}(\Delta)$ is an equivalence in $\mathcal{L}'$. The details are left to the reader. ■

Theorem 5.12 *Let be N_e : **Equiv**$\longrightarrow$**Sig** be the forgetful functor, that is: $N_e(\langle C, \vdash\rangle) = C$ and $N_e(f) = f$. Then, N_e is a cofibration.*

Proof: Let be $(\mathcal{L}, f)$ be a co-structured morphism for N_e, where $\mathcal{L} = \langle C, \vdash\rangle$ and $f : C \longrightarrow C'$ is a **Sig**-morphism. Let $\mathcal{L}' = \langle C', \vdash'\rangle$ be the equivalential logic defined as in Lemma 5.11. Then $f : \mathcal{L} \longrightarrow \mathcal{L}'$ is an **Equiv**-morphism such that $N_e(f) = f$. We will show that, in fact $f : \mathcal{L} \longrightarrow \mathcal{L}'$ is a cocartesian lifting of $(\mathcal{L}, f)$. Thus, let $g : \mathcal{L} \longrightarrow \mathcal{L}''$ be a morphism in **Equiv** and let $h : C' \longrightarrow C''$ be a **Sig**-morphism such that $h \circ f = g$ in **Sig**. Thus

$$(*) \qquad \widehat{h} \circ \widehat{f} = \widehat{g}$$

It is enough to prove that, in fact, h is an **Equiv**-morphism $h : \mathcal{L}' \longrightarrow \mathcal{L}''$. The proof will follow the same steps as the proof of Proposition 5.8. Thus, if $\Gamma \vdash' \varphi$ such that $\Gamma_0 \Longrightarrow \varphi$ in provable in S for some finite $\Gamma_0 \subseteq \Gamma$, we will prove by induction on the length n of a derivation of $\Gamma_0 \Longrightarrow \varphi$ in S that $\widehat{h}(\Gamma_0) \vdash'' \widehat{h}(\varphi)$.

If $n = 1$ there are two possibilities: either $\Gamma_0 \Longrightarrow \varphi$ is $\widehat{f}(\Sigma) \Longrightarrow \widehat{f}(\psi)$ (an instance of $[AX1]$) and so the proof is immediate, or $\Gamma_0 \Longrightarrow \varphi$ is

$$\widehat{f}(\Delta)(p_1, p_{k+1}), \ldots, \widehat{f}(\Delta)(p_k, p_{2k}) \Longrightarrow \widehat{f}(\Delta)(c(p_1, \ldots, p_k), c(p_{k+1}, \ldots, p_{2k}))$$

for $c \in C'_k$ (an instance of axiom $[AX2]_c$). Since Δ is an equivalence in $\mathcal{L}$ then $\widehat{g}(\Delta)$ is an equivalence in $\mathcal{L}''$, because g is a morphism in **Equiv**. On the other hand, $h(c) \in C''_k$ and so, using $(*)$,

$$\widehat{h}(\widehat{f}(\Delta))(p_1, p_{k+1}), \ldots, \widehat{h}(\widehat{f}(\Delta))(p_k, p_{2k}) \vdash'' \widehat{h}(\widehat{f}(\Delta))(h(c)\vec{p}, h(c)\vec{q})$$

(where $\vec{p} = (p_1, \ldots, p_k)$ and $\vec{q} = (p_{k+1}, \ldots, p_{2k})$). That is,

$$\widehat{h}(\widehat{f}(\Delta))(p_1, p_{k+1}), \ldots, \widehat{h}(\widehat{f}(\Delta))(p_k, p_{2k}) \vdash'' \widehat{h}(\widehat{f}(\Delta)(c(\vec{p}), c(\vec{q})))$$

and so $\widehat{h}(\Gamma_0) \vdash'' \widehat{h}(\varphi)$ as required. The rest of the proof that h is an **Equiv**-morphism is identical to the proof of Proposition 5.8. Thus, $h : \mathcal{L}' \longrightarrow \mathcal{L}''$ is an **Equiv**-morphism such that $N_e(h) = h$ and $h \circ f = g$ in **Equiv**. The uniqueness of h is obvious, by the very definition of N_e. ■

Remark 5.13 *The sequent calculus S_e constitutes the inferential basis of the logic $\mathcal{L}'$ generated by a cocartesian lifting. As it was observed in Remark 5.9 with respect to the coproduct in **Equiv**, it is possible to characterize $\mathcal{L}'$ in lattice-theoretic terms. Thus, let*

$$X = \{\vdash' \in Stan_{C'} \;:\; f(\vdash) \subseteq \vdash' \text{ and, for every } k \in \mathbb{N} \text{ and every } c \in C'_k,$$

$$\widehat{f}(\Delta)(p_1, p_{k+1}), \ldots, \widehat{f}(\Delta)(p_k, p_{2k}) \vdash' \widehat{f}(\Delta)(c(p_1, \ldots, p_k), c(p_{k+1}, \ldots, p_{2k}))\}.$$

Then $\vdash' = \bigwedge_{\mathbf{Stan}_C} X$ (see [15]).

From Proposition 5.8 and Theorem 5.12, and taking into account that fibring in **Equiv** is defined in the same lines as in **Cons**, we arrive to the following result:

Theorem 5.14 *The category **Equiv** has both unconstrained and constrained fibring.*

6 Fibring algebraizable logics

The results obtained in the last section can be applied to define fibring within a more sophisticate class of logics inside the Leibniz Hierarchy: the so-called algebraizable logics.

We briefly recall the definition of algebraizable logics, under a convenient presentation:

Definition 6.1 *An equivalential logic $\mathcal{L} = \langle C, \vdash \rangle$ is said to be* algebraizable *if there exists an equivalence $\Delta(p_1, p_2)$ for $\mathcal{L}$ as well as a finite (non-empty) set*

$$\langle \bar{\delta}, \bar{\varepsilon} \rangle = \{ \langle \delta_1(p_1), \varepsilon_1(p_1) \rangle, \ldots, \langle \delta_n(p_1), \varepsilon_n(p_1) \rangle \}$$

of pairs of formulas in $L(C)$ such that $\delta_j(p_1)$ and $\varepsilon_j(p_1)$ depend at most on variable p_1 (for $j = 1, \ldots, n$), satisfying the following property:

$$p_1 \dashv\vdash \Delta(\bar{\delta}(p_1), \bar{\varepsilon}(p_1)).$$

The pair $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ is called an algebraizator *for $\mathcal{L}$.*

In the definition above, $\Delta(\bar{\delta}(p_1), \bar{\varepsilon}(p_1))$ stands for

$$\{ \varphi(\delta(p_1), \varepsilon(p_1)) \ : \ \varphi \in \Delta \ \text{ and } \ \langle \delta(p_1), \varepsilon(p_1) \rangle \in \langle \bar{\delta}, \bar{\varepsilon} \rangle \}.$$

With the definitions and results above, the category of algebraizable logics is defined in a natural way.

Definition 6.2 *The* category **Alge** *is the full subcategory of **Equiv** whose objects are algebraizable logics.*

The following useful result justifies the last definition:

Proposition 6.3 *Let $\mathcal{L}$ be an algebraizable logic with algebraizator $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$, and let $f : \mathcal{L} \longrightarrow \mathcal{L}'$ be a a morphism in **Equiv**. Then $\mathcal{L}'$ is also algebraizable with algebraizator $\langle \widehat{f}(\Delta), \langle \widehat{f}(\bar{\delta}), \widehat{f}(\bar{\varepsilon}) \rangle \rangle$ such that*

$$\langle \widehat{f}(\bar{\delta}), \widehat{f}(\bar{\varepsilon}) \rangle = \{ \langle \widehat{f}(\delta), \widehat{f}(\varepsilon) \rangle \ : \ \langle \delta, \varepsilon \rangle \in \langle \bar{\delta}, \bar{\varepsilon} \rangle \}$$

*and so f is an **Alge**-morphism.*

Proof: Immediate, from the definitions. ■

Using the proposition above, the fundamental result obtained for equivalential logics (that is, Theorem 5.14) is also extended to algebraizable logics:

Theorem 6.4 *The category* ***Alge*** *has both unconstrained and constrained fibring. That is,* ***Alge*** *has coproducts and the forgetful functor* $N_a : \mathbf{Alge} \longrightarrow \mathbf{Sig}$ *is a cofibration.*

Proof: The proof is an immediate consequence of Theorem 5.14, Proposition 6.3 and the fact that **Alge** is a full subcategory of **Equiv**. ■

From the results above, when fibring in **Alge** (without sharing connectives) two logics $\mathcal{L}_1$ and $\mathcal{L}_2$ with algebraizators $\langle \Delta_1, \langle \bar{\delta}_1, \bar{\varepsilon}_1 \rangle \rangle$ and $\langle \Delta_2, \langle \bar{\delta}_2, \bar{\varepsilon}_2 \rangle \rangle$, respectively, then the resulting logic $\mathcal{L}_1 \circledast \mathcal{L}_2$, which is the coproduct in **Alge** of $\mathcal{L}_1$ and $\mathcal{L}_2$, is algebraizable with either

$$\langle \widehat{i_1}(\Delta_1), \langle \widehat{i_1}(\bar{\delta}_1), \widehat{i_1}(\bar{\varepsilon}_1) \rangle \rangle \quad \text{or} \quad \langle \widehat{i_2}(\Delta_2), \langle \widehat{i_2}(\bar{\delta}_2), \widehat{i_2}(\bar{\varepsilon}_2) \rangle \rangle$$

where $i_1 : \mathcal{L}_1 \longrightarrow \mathcal{L}$ and $i_2 : \mathcal{L}_2 \longrightarrow \mathcal{L}$ are the canonical injections.

On the other hand, given $\mathcal{L}_1$ and $\mathcal{L}_2$ as above, and a sharing diagram D formed by two monomorphisms $j_1 : \bar{C} \longrightarrow N_a(\mathcal{L}_1)$ and $j_2 : \bar{C} \longrightarrow N_a(\mathcal{L}_2)$ in **Sig**, the fibring in **Alge** of $\mathcal{L}_1$ and $\mathcal{L}_2$ constrained by the sharing D is a logic $\widehat{\mathcal{L}}$ algebraizable with either

$$\langle \widehat{q}(\widehat{i_1}(\Delta_1)), \langle \widehat{q}(\widehat{i_1}(\bar{\delta}_1)), \widehat{q}(\widehat{i_1}(\bar{\varepsilon}_1)) \rangle \rangle \quad \text{or} \quad \langle \widehat{q}(\widehat{i_2}(\Delta_2)), \langle \widehat{q}(\widehat{i_2}(\bar{\delta}_2)), \widehat{q}(\widehat{i_2}(\bar{\varepsilon}_2)) \rangle \rangle$$

where $q : \mathcal{L}_1 \circledast \mathcal{L}_2 \longrightarrow \widehat{\mathcal{L}}$ is the cocartesian lifting of the coequalizer

$$q : N_a(\mathcal{L}_1 \circledast \mathcal{L}_2) \longrightarrow \widehat{C}$$

in **Sig** of the diagram below.

$$\bar{C} \underset{i_2 \circ j_2}{\overset{i_1 \circ j_1}{\rightrightarrows}} N(\mathcal{L}_1 \circledast \mathcal{L}_2)$$

7 Fibring algebraic semantics

In this section, and based on [21], we will give the semantical counterpart of fibring algebraizable logics. That is, taking into account that algebraizable logics can be presented in terms of quasi-varieties, it is natural to recast the category **Alge** in algebraic terms. To begin with, we briefly recall the characterization of algebraizable logics in terms of algebraic semantics.

Definition 7.1 *Given a propositional signature C, the* equational language *$LEq(C)$ is the first-order language such that $\mathcal{V}$ is the set of individual variables and C_k is the set of k-ary function symbols (for $k \geq 0$); no predicate symbols are included, besides the logical symbol $\approx$ for equality (which acts as a binary predicate). The other logical symbols to be considered in the (classical) first-order logic defined over $LEq(C)$ are the following: $\sim$ (negation), $\bar{\wedge}$ (conjunction), $\veebar$ (disjunction), $\Rightarrow$ (implication) and $\forall$ (universal quantifier).*

The (first-order) classical connectives $\sim$, $\bar{\wedge}$, $\veebar$ and $\Rightarrow$ considered in the definition above should not be confused with the propositional ones: the usual (propositional) connectives $\neg$, $\wedge$, $\vee$ and $\rightarrow$ appear in $LEq(C)$ as function symbols. In general, the set $L(C)$ of propositional formulas coincides with the set of terms of $LEq(C)$ (this property is essential, as we shall see). Thus, the atomic formulas of $LEq(C)$ are of the form $(\varphi \approx \psi)$, with $\varphi, \psi \in L(C)$; these formulas are called *equations in $LEq(C)$*, or *C-equations.* The set of C-equations will be denoted by $Eq(C)$.

It is worth noting that a first-order structure for $LEq(C)$ is formed simply by a C-algebra $\mathbf{A}$. On the other hand, a variable assignment for a structure $\mathbf{A}$ is a map $I : \mathcal{V} \longrightarrow A$, where A is the domain of $\mathbf{A}$. Thus, an attribution map $[\![\cdot]\!]_I^{\mathbf{A}} : L(C) \longrightarrow A$ is defined recursively as usual, such that $[\![p]\!]_I^{\mathbf{A}} = I(p)$ for $p \in \mathcal{V}$ and $[\![c(\varphi_1, \ldots, \varphi_k)]\!]_I^{\mathbf{A}} = c^{\mathbf{A}}([\![\varphi_1]\!]_I^{\mathbf{A}}, \ldots, [\![\varphi_k]\!]_I^{\mathbf{A}})$.

A consequence relation between the equations in $LEq(C)$ is defined as expected, using classical (first-order) model theory.

Definition 7.2 *Let $\mathcal{K}$ be a class of C-algebras. The* equational consequence relation generated by $\mathcal{K}$ *is the relation $\models_{\mathcal{K}} \subseteq \wp(Eq(C)) \times Eq(C)$ such that: $\Upsilon \models_{\mathcal{K}} (\varphi \approx \psi)$ if and only if, for every $\mathbf{A}$ in $\mathcal{K}$ and for every variable assignment $I : \mathcal{V} \longrightarrow A$, if $[\![\gamma]\!]_I^{\mathbf{A}} = [\![\gamma']\!]_I^{\mathbf{A}}$ for every $(\gamma \approx \gamma') \in \Upsilon$ then $[\![\varphi]\!]_I^{\mathbf{A}} = [\![\psi]\!]_I^{\mathbf{A}}$.*

If Υ and Υ' are sets of C-equations, then $\Upsilon \models_{\mathcal{K}} \Upsilon'$ means that $\Upsilon \models_{\mathcal{K}} (\varphi' \approx \psi')$ for every $(\varphi' \approx \psi') \in \Upsilon'$; and the expression $\Upsilon \dashv\models_{\mathcal{K}} \Upsilon'$ means that $\Upsilon \models_{\mathcal{K}} \Upsilon'$ and $\Upsilon' \models_{\mathcal{K}} \Upsilon$.

In analogy to what was done in Section 2, we can define an *equational theory* (for $\mathcal{K}$) as being a set Υ of C-equations such that: $\Upsilon \models_{\mathcal{K}} (\varphi \approx \psi)$ implies that $(\varphi \approx \psi) \in \mathcal{K}$. The set $Th_{\mathcal{K}}$ of theories for $\mathcal{K}$, ordered by inclusion, is a complete lattice denoted by $\mathbf{Th}_{\mathcal{K}}$.

Definition 7.3 *A* quasi-equation *(in the language $LEq(C)$) is a sentence* *of* $LEq(C)$ *of the form*

$$(\forall)((\varphi_1 \approx \psi_1) \mathbin{\bar{\wedge}} \ldots \mathbin{\bar{\wedge}} (\varphi_n \approx \psi_n) \Rightarrow (\varphi \approx \psi))$$

where $(\forall)\Psi$ *denotes the universal closure of the formula* Ψ *of* $LEq(C)$*. A* quasi-variety over C *is a class* $\mathcal{K}$ *of* C*-algebras axiomatized (in the sense of classical first-order model theory) by a set* Υ *of quasi-equations in* $LEq(C)$*.*

If a class $\mathcal{K}$ of C-algebras is closed under ultraproducts, the consequence relation $\models_{\mathcal{K}}$ is finitary. In particular, if $\mathcal{K}$ is a quasi-variety then $\models_{\mathcal{K}}$ is finitary. On the other hand, $\models_{\mathcal{K}}$ always satisfies *Structurality* (recall Definition 2.6), if we consider the following definition: a *substitution* σ *in* $Eq(C)$ is a substitution $\sigma : \mathcal{V} \longrightarrow L(C)$ such that $\widehat{\sigma}(\varphi \approx \psi)$ is the equation $(\widehat{\sigma}(\varphi) \approx \widehat{\sigma}(\psi))$. Thus, if $\mathcal{K}$ is a quasi-variety then the pair $\langle Eq(C), \models_{\mathcal{K}} \rangle$ is a standard logic.

Notation 7.4 *From now on, given a signature* C*,* $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ *will denote a pair such that:*

(1) Δ *is a finite non-empty set* $\{\varphi_1(p_1, p_2), \ldots, \varphi_k(p_1, p_2)\}$ *of formulas in* $L(C)$ *depending at most on* p_1, p_2*;*

(2) $\langle \bar{\delta}, \bar{\varepsilon} \rangle$ *is a finite non-empty set* $\{\langle \delta_1(p_1), \varepsilon_1(p_1) \rangle \ldots \langle \delta_n(p_1), \varepsilon_n(p_1) \rangle\}$ *of pairs of formulas of* $L(C)$ *depending at most on* p_1*.*

Given a pair as above, we will use the following standard notation:

$$(\bar{\delta}(\psi) \approx \bar{\varepsilon}(\psi))$$

will stand for $\{(\delta(\psi) \approx \varepsilon(\psi)) \ : \ \langle \delta, \varepsilon \rangle \in \langle \bar{\delta}, \bar{\varepsilon} \rangle\}$*;*

$$\{(\bar{\delta}(\psi) \approx \bar{\varepsilon}(\psi)) \ : \ \psi \in \Gamma\}$$

will stand for $\{(\delta(\psi) \approx \varepsilon(\psi)) \ : \ \psi \in \Gamma$ *and* $\langle \delta, \varepsilon \rangle \in \langle \bar{\delta}, \bar{\varepsilon} \rangle\}$*; and*

$$(\bar{\delta}(\Delta(\gamma, \psi)) \approx \bar{\varepsilon}(\Delta(\gamma, \psi)))$$

will stand for $\{(\delta(\varphi(\gamma, \psi)) \approx \varepsilon(\varphi(\gamma, \psi))) \ : \ \varphi \in \Delta$ *and* $\langle \delta, \varepsilon \rangle \in \langle \bar{\delta}, \bar{\varepsilon} \rangle\}$*.*

The following characterization of algebraizable logics is due to W. Blok and D. Pigozzi (see [3]):

Theorem 7.5 *A logic $\mathcal{L} = \langle C, \vdash \rangle$ is algebraizable iff there exists a unique quasi-variety $\mathcal{K}$ over C, and a (not necessarily unique) pair $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ such that:*

(1) For every $\Gamma \cup \{\varphi\} \subseteq L(C)$,

$$\Gamma \vdash \varphi \quad \textit{iff} \quad \{(\bar{\delta}(\psi) \approx \bar{\varepsilon}(\psi)) \ : \ \psi \in \Gamma\} \models_{\mathcal{K}} (\bar{\delta}(\varphi) \approx \bar{\varepsilon}(\varphi));$$

(2) $(p_1 \approx p_2) \dashv\models_{\mathcal{K}} (\bar{\delta}(\Delta(p_1, p_2)) \approx \bar{\varepsilon}(\Delta(p_1, p_2)))$.

Of course a pair $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ as in Theorem 7.5 is an algebraizator for $\mathcal{L}$. From now on, $\mathcal{K}$ will be called *the quasi-variety of $\mathcal{L}$*.

In [21] it was introduced the interesting notion of *deductivizator* of a quasi-variety, by duality with the notion of algebraizator for logics. The concept of deductivizator is central in order to obtain an isomorphism between the category of algebraizable logics and deductivizable quasi-varieties. From the above mentioned paper we reproduce the following two definitions.

Definition 7.6 *Let C be a signature, and let $\mathcal{K}$ be a quasi-variety over C. Let $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ be a pair as in Note 7.4. We say that $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ is a* deductivizator of $\mathcal{K}$ *if $\models_{\mathcal{K}}$ verifies:*

$$(p_1 \approx p_2) \dashv\models_{\mathcal{K}} (\bar{\delta}(\Delta(p_1, p_2)) \approx \bar{\varepsilon}(\Delta(p_1, p_2))).$$

We say that a quasi-variety $\mathcal{K}$ is deductivizable *(or a* Blok-Pigozzi quasi-variety*, cf. [13]) if it has a deductivizator.*

As proved in [21], deductivizable quasi-varieties are the algebraic counterpart of algebraizable logics. In the same way as not every logic is algebraizable, it is the case that not every quasi-variety is deductivizable. By defining an appropriate notion of morphism, we shall prove below that deductivizable quasi-varieties form a category isomorphic to **Alge**.

Definition 7.7 *Given a deductivizable quasi-variety K, the equivalence relation $\simeq_{\mathcal{K}}$ between deductivizators of $\mathcal{K}$ is defined as follows:*

$$\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle \simeq_{\mathcal{K}} \langle \Delta', \langle \bar{\delta}', \bar{\varepsilon}' \rangle \rangle \quad \textit{iff} \quad (\bar{\delta}(p_1) \approx \bar{\varepsilon}(p_1)) \dashv\models_{\mathcal{K}} (\bar{\delta}'(p_1) \approx \bar{\varepsilon}'(p_1)).$$

The equivalence class of $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ w.r.t $\simeq_{\mathcal{K}}$ will be denoted by $[\bar{\delta}, \bar{\varepsilon}, \Delta]_{\mathcal{K}}$.

Now we can define the envisaged category of equivalent algebraic semantics:

Definition 7.8 *The* category **Asem** *is defined as follows:*
(a) Objects: *triples* $\mathcal{A} = \langle C, \mathcal{K}, [\bar{\delta}, \bar{\varepsilon}, \Delta]_{\mathcal{K}} \rangle$ *such that* C *is a signature,* $\mathcal{K}$ *is a deductivizable quasi-variety over* C*, and* $[\bar{\delta}, \bar{\varepsilon}, \Delta]_{\mathcal{K}}$ *is the equivalence class of a deductivizator* $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ *of* $\mathcal{K}$*.*
(b) Morphisms: *given* $\mathcal{A} = \langle C, \mathcal{K}, [\bar{\delta}, \bar{\varepsilon}, \Delta]_{\mathcal{K}} \rangle$ *and* $\mathcal{A}' = \langle C', \mathcal{K}', [\bar{\delta}', \bar{\varepsilon}', \Delta']_{\mathcal{K}'} \rangle$*, a morphism* $f : \mathcal{A} \longrightarrow \mathcal{A}'$ *is a* ***Sig****-morphism* $f : C \longrightarrow C'$ *such that:*

(1) For every set $\Upsilon \cup \{\varphi \approx \psi\}$ *of* C*-equations, if* $\Upsilon \models_{\mathcal{K}} (\varphi \approx \psi)$ *then*

$$\{(\widehat{f}(\psi_1) \approx \widehat{f}(\psi_2)) \ : \ (\psi_1 \approx \psi_2) \in \Upsilon\} \models_{\mathcal{K}'} (\widehat{f}(\varphi) \approx \widehat{f}(\psi));$$

(2) $(\bar{\delta}'(p_1) \approx \bar{\varepsilon}'(p_1)) \mathrel{=\!\mid\models}_{\mathcal{K}'} (\widehat{f}(\bar{\delta}(p_1)) \approx \widehat{f}(\bar{\varepsilon}(p_1)))$*.*

(c) Composition and identity morphisms: *as is* ***Sig****.*

Remark 7.9 *Firstly, it should be noted that, using condition (1) above and definition of* $\simeq_{\mathcal{K}}$ *and* $\simeq_{\mathcal{K}'}$*, condition (2) is well-defined.*

Next, observe that, for every ***Sig****-morphism* $f : C \longrightarrow C'$ *and every substitution* σ *in* C*, the substitution* σ' *in* C' *given by* $\sigma'(p) = \widehat{f}(\sigma(p))$ *is such that* $\widehat{\sigma}' \circ \widehat{f} = \widehat{f} \circ \widehat{\sigma}$*. Using this and* Structurality *of* $\models_{\mathcal{K}'}$*, condition (2) above is equivalent to*

$$(\bar{\delta}'(\widehat{f}(\varphi)) \approx \bar{\varepsilon}'(\widehat{f}(\varphi))) \mathrel{=\!\mid\models}_{\mathcal{K}'} (\widehat{f}(\bar{\delta}(\varphi)) \approx \widehat{f}(\bar{\varepsilon}(\varphi)))$$

for every $\varphi \in L(C)$*.*

On the other hand, condition (1) is strictly weaker than the following condition:

$$(*) \quad \textit{for every formula } \Psi \textit{ of } LEq(C), \textit{ if } \models_{\mathcal{K}} \Psi \textit{ then } \models_{\mathcal{K}'} \bar{f}(\Psi)$$

where $\bar{f} : LEq(C) \longrightarrow LEq(C')$ *is given by* $\bar{f}(\varphi \approx \psi) = (\widehat{f}(\varphi) \approx \widehat{f}(\psi))$*;* $\bar{f}({\sim}\Psi) = {\sim}\bar{f}(\Psi)$*;* $\bar{f}(\Psi \star \Psi') = (\bar{f}(\Psi) \star \bar{f}(\Psi'))$*;* $\bar{f}(\forall p \Psi) = \forall p \bar{f}(\Psi)$ *for* $\varphi, \psi \in L(C)$*,* $\star \in \{\bar{\wedge}, \bar{\vee}, \Rightarrow\}$*,* $\Psi, \Psi' \in LEq(C_1)$ *and* $p \in \mathcal{V}$*. Condition* $(*)$*, which is required (together with (2)) in Definition 3.9 of [21], obviously implies condition (1) above. But conditions (1) plus (2) do not imply* $(*)$*: it is enough to consider* C' *as being the terminal object in* ***Sig*** *(defined by* $C'_k = \{c_k\}$ *for every* $k \in \mathbb{N}$*) and* $\mathcal{K}' = \{\mathbf{A}'\}$ *such that* A' *(the domain of* $\mathbf{A}'$*) is a singleton. If* $\mathcal{K}$ *contains a quasi-variety* $\mathbf{A}$ *such that* $[\![\varphi]\!]^{\mathbf{A}}_I \neq [\![\psi]\!]^{\mathbf{A}}_I$ *for some* $\varphi, \psi \in L(C)$ *and some assignment* I*, then the (unique)* ***Sig****-morphism* $f : C \longrightarrow C'$ *satisfies (1) and (2), however* $(*)$ *is not satisfied: it is the case that* $\models_{\mathcal{K}} {\sim}(\varphi \approx \psi)$ *but* $\not\models_{\mathcal{K}'} \bar{f}({\sim}(\varphi \approx \psi))$*. Condition (1) in Definition 7.8 is natural, as we shall see in Proposition 8.2 below.*

In order to prove that **Alge** and **Asem** are isomorphic we need some technical results:

Lemma 7.10 *Let* $\mathcal{L} = \langle C, \vdash \rangle$ *be a logic with algebraizator* $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ *and let* $\mathcal{K}$ *be the respective quasi-variety. Let* $\Upsilon \cup \{(\varphi \approx \psi)\}$ *be a set of* C*-equations. Then*

$$\Upsilon \models_{\mathcal{K}} (\varphi \approx \psi) \;\; \textit{iff} \;\; \{\Delta(\psi_1, \psi_2) \;:\; (\psi_1 \approx \psi_2) \in \Upsilon\} \vdash \Delta(\varphi, \psi)$$

where $\{\Delta(\psi_1, \psi_2) \;:\; (\psi_1 \approx \psi_2) \in \Upsilon\}$ *stands for*

$$\{\varphi(\psi_1, \psi_2) \;:\; \varphi \in \Delta \;\; \textit{and} \;\; (\psi_1 \approx \psi_2) \in \Upsilon\}.$$

Proof: See [3]. ■

Lemma 7.11 *Let* $\mathcal{L} = \langle C, \vdash \rangle$ *and* $\mathcal{L}' = \langle C', \vdash' \rangle$ *be algebraizable logics with algebraizators* $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ *and* $\langle \Delta', \langle \bar{\delta}', \bar{\varepsilon}' \rangle \rangle$*, respectively, and let* $\mathcal{K}$ *and* $\mathcal{K}'$ *be the respective quasi-varieties. Let* $f : C \longrightarrow C'$ *be a* ***Sig****-morphism. The following conditions are equivalent:*
(a) f *is a morphism* $f : \mathcal{L} \longrightarrow \mathcal{L}'$ *in* ***Alge****;*
(b) f *is a morphism* $f : \langle C, \mathcal{K}, [\bar{\delta}, \bar{\varepsilon}, \Delta]_{\mathcal{K}} \rangle \longrightarrow \langle C', \mathcal{K}', [\bar{\delta}', \bar{\varepsilon}', \Delta']_{\mathcal{K}'} \rangle$ *in* ***Asem****.*

Proof: We only prove that (a) implies (b), because the other direction is proved similarly. Thus, assume that $f : \mathcal{L} \longrightarrow \mathcal{L}'$ is an **Alge**-morphism.

Let $\Upsilon \cup \{\varphi \approx \psi\}$ be a set of C-equations such that $\Upsilon \models_{\mathcal{K}} (\varphi \approx \psi)$. Then, by Lemma 7.10, $\{\Delta(\psi_1, \psi_2) \;:\; (\psi_1 \approx \psi_2) \in \Upsilon\} \vdash \Delta(\varphi, \psi)$ and so

$$\{\widehat{f}(\Delta)(\widehat{f}(\psi_1), \widehat{f}(\psi_2)) \;:\; (\psi_1 \approx \psi_2) \in \Upsilon\} \vdash' \widehat{f}(\Delta)(\widehat{f}(\varphi), \widehat{f}(\psi)).$$

Therefore $\{\Delta'(\widehat{f}(\psi_1), \widehat{f}(\psi_2)) \;:\; (\psi_1 \approx \psi_2) \in \Upsilon\} \vdash' \Delta'(\widehat{f}(\varphi), \widehat{f}(\psi))$, by Proposition 5.4. Using again Lemma 7.10, it follows that

$$\{(\widehat{f}(\psi_1) \approx \widehat{f}(\psi_2)) \;:\; (\psi_1 \approx \psi_2) \in \Upsilon\} \models_{\mathcal{K}'} (\widehat{f}(\varphi) \approx \widehat{f}(\psi))$$

and so f satisfies condition (1) of **Asem**-morphism given in Definition 7.8.

On the other hand, by Definition 6.1, $p_1 \vdash \Delta(\bar{\delta}(p_1), \bar{\varepsilon}(p_1))$ and so

$$p_1 \vdash' \widehat{f}(\Delta)(\widehat{f}(\bar{\delta}(p_1)), \widehat{f}(\bar{\varepsilon}(p_1))).$$

Since, by Proposition 5.4, $\widehat{f}(\Delta)(\widehat{f}(\bar{\delta}(p_1)), \widehat{f}(\bar{\varepsilon}(p_1))) \vdash' \Delta'(\widehat{f}(\bar{\delta}(p_1)), \widehat{f}(\bar{\varepsilon}(p_1)))$, it follows that $p_1 \vdash' \Delta'(\widehat{f}(\bar{\delta}(p_1)), \widehat{f}(\bar{\varepsilon}(p_1)))$. But

$$\Delta'(\bar{\delta}'(p_1), \bar{\varepsilon}'(p_1)) \vdash' p_1$$

by Definition 6.1, and so $\Delta'(\bar{\delta}'(p_1), \bar{\varepsilon}'(p_1)) \vdash' \Delta'(\widehat{f}(\bar{\delta}(p_1)), \widehat{f}(\bar{\varepsilon}(p_1)))$. Using Lemma 7.10 it follows that $(\bar{\delta}'(p_1) \approx \bar{\varepsilon}'(p_1)) \models_{\mathcal{K}'} (\widehat{f}(\bar{\delta}(p_1)) \approx \widehat{f}(\bar{\varepsilon}(p_1)))$. Analogously, it can be proven that $(\widehat{f}(\bar{\delta}(p_1)) \approx \widehat{f}(\bar{\varepsilon}(p_1))) \models_{\mathcal{K}'} (\bar{\delta}'(p_1) \approx \bar{\varepsilon}'(p_1))$ and so f satisfies condition (2) of **Asem**-morphism. ∎

Lemma 7.12 *Let $\mathcal{L}$ be an algebraizable logic with quasi-variety $\mathcal{K}$. If $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ and $\langle \Delta', \langle \bar{\delta}', \bar{\varepsilon}' \rangle \rangle$ are two algebraizators for $\mathcal{L}$ then $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle \simeq_{\mathcal{K}} \langle \Delta', \langle \bar{\delta}', \bar{\varepsilon}' \rangle \rangle$.*

Proof: Let $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ and $\langle \Delta', \langle \bar{\delta}', \bar{\varepsilon}' \rangle \rangle$ be two algebraizators for $\mathcal{L}$. By Definition 6.1,

$$\Delta(\bar{\delta}(p_1), \bar{\varepsilon}(p_1)) \vdash p_1 \quad \text{and} \quad p_1 \vdash \Delta'(\bar{\delta}'(p_1), \bar{\varepsilon}'(p_1))$$

and so $\Delta(\bar{\delta}(p_1), \bar{\varepsilon}(p_1)) \vdash \Delta'(\bar{\delta}'(p_1), \bar{\varepsilon}'(p_1))$. But

$$\Delta'(\bar{\delta}'(p_1), \bar{\varepsilon}'(p_1)) \vdash \Delta(\bar{\delta}'(p_1), \bar{\varepsilon}'(p_1))$$

by Proposition 5.4, so $\Delta(\bar{\delta}(p_1), \bar{\varepsilon}(p_1)) \vdash \Delta(\bar{\delta}'(p_1), \bar{\varepsilon}'(p_1))$. Using Lemma 7.10 it follows that $(\bar{\delta}(p_1) \approx \bar{\varepsilon}(p_1)) \models_{\mathcal{K}} (\bar{\delta}'(p_1) \approx \bar{\varepsilon}'(p_1))$. Analogously it can be proven that $(\bar{\delta}'(p_1) \approx \bar{\varepsilon}'(p_1)) \models_{\mathcal{K}} (\bar{\delta}(p_1) \approx \bar{\varepsilon}(p_1))$ and so $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle \simeq_{\mathcal{K}} \langle \Delta', \langle \bar{\delta}', \bar{\varepsilon}' \rangle \rangle$. ∎

Lemma 7.13 *Let $\mathcal{L}$ and $\mathcal{L}'$ be two logics defined over the same signature C, which are algebraizable with the same quasi-variety $\mathcal{K}$ over C. Suppose that $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ and $\langle \Delta', \langle \bar{\delta}', \bar{\varepsilon}' \rangle \rangle$ are algebraizators for $\mathcal{L}$ and $\mathcal{L}'$, respectively, such that $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle \simeq_{\mathcal{K}} \langle \Delta', \langle \bar{\delta}', \bar{\varepsilon}' \rangle \rangle$. Then $\mathcal{L} = \mathcal{L}'$.*

Proof: Assume that $\mathcal{L} = \langle C, \vdash \rangle$, $\mathcal{L}' = \langle C, \vdash' \rangle$ and $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle \simeq_{\mathcal{K}} \langle \Delta', \langle \bar{\delta}', \bar{\varepsilon}' \rangle \rangle$. Let $\Gamma \cup \{\varphi\} \subseteq L(C)$. Then:

$$\begin{aligned} \Gamma \vdash \varphi \quad & \text{iff} \quad \{(\bar{\delta}(\psi) \approx \bar{\varepsilon}(\psi)) \ : \ \psi \in \Gamma\} \models_{\mathcal{K}} (\bar{\delta}(\varphi) \approx \bar{\varepsilon}(\varphi)) \\ & \text{iff} \quad \{(\bar{\delta}'(\psi) \approx \bar{\varepsilon}'(\psi)) \ : \ \psi \in \Gamma\} \models_{\mathcal{K}} (\bar{\delta}'(\varphi) \approx \bar{\varepsilon}'(\varphi)) \\ & \text{iff} \quad \Gamma \vdash' \varphi. \end{aligned}$$

Therefore $\mathcal{L} = \mathcal{L}'$. ∎

Lemma 7.14 *Let $\mathcal{A} = \langle C, \mathcal{K}, [\bar{\delta}, \bar{\varepsilon}, \Delta]_{\mathcal{K}} \rangle$ be an object of **Asem**. Then the pair $\mathcal{L} = \langle C, \vdash \rangle$ such that*

$$\Gamma \vdash \varphi \ \ \textit{iff} \ \ \{(\bar{\delta}(\psi) \approx \bar{\varepsilon}(\psi)) \ : \ \psi \in \Gamma\} \models_{\mathcal{K}} (\bar{\delta}(\varphi) \approx \bar{\varepsilon}(\varphi))$$

is an algebraic logic with algebraizator $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ and quasi-variety $\mathcal{K}$.

Proof: Observe that, by definition of $\simeq_{\mathcal{K}}$, the relation $\vdash$ is well-defined. From the definitions above, it is immediate that $\mathcal{L}$ is an algebraizable logic with algebraizator $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ and quasi-variety $\mathcal{K}$. ■

Theorem 7.15 ***Alge*** *and* ***Asem*** *are isomorphic categories.*

Proof: Consider the functors $G_1 : \mathbf{Alge} \longrightarrow \mathbf{Asem}$ and $G_2 : \mathbf{Asem} \longrightarrow \mathbf{Alge}$ defined as follows:

- $G_1(\langle C, \vdash \rangle) = \langle C, \mathcal{K}, [\bar{\delta}, \bar{\varepsilon}, \Delta]_{\mathcal{K}} \rangle$, where $\mathcal{K}$ is the quasi-variety of $\langle C, \vdash \rangle$ (cf. Theorem 7.5) and $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ is an algebraizator for $\langle C, \vdash \rangle$;
- $G_1(f) = f$ if $f : \mathcal{L} \longrightarrow \mathcal{L}'$ is an **Alge**-morphism;
- $G_2(\langle C, \mathcal{K}, [\bar{\delta}, \bar{\varepsilon}, \Delta]_{\mathcal{K}} \rangle) = \langle C, \vdash \rangle$ where $\vdash$ is defined as in Lemma 7.14;
- $G_2(f) = f$ if $f : \mathcal{A} \longrightarrow \mathcal{A}'$ is an **Asem**-morphism.

Given $\mathcal{L}$ with quasivariety $\mathcal{K}$, any algebraizator of $\mathcal{L}$ (seen as a deductivizator of $\mathcal{K}$) determines a unique equivalence class, by Lemma 7.12, and so $G_1(\mathcal{L})$ is well-defined. On the other hand, $G_2(\mathcal{A})$ is well-defined, by Lemma 7.14. The definitions of $G_1(f)$ and $G_2(f)$ are right, because of Lemma 7.11. Clearly, both G_1 and G_2 are functors.

Let $\mathcal{L} = \langle C, \vdash \rangle$ in **Alge** with algebraizator $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ and quasi-variety $\mathcal{K}$. Then $G_1(\mathcal{L}) = \langle C, \mathcal{K}, [\bar{\delta}, \bar{\varepsilon}, \Delta]_{\mathcal{K}} \rangle$ and so $G_2(G_1(\mathcal{L}))$ is a logic $\mathcal{L}' = \langle C, \vdash' \rangle$ which coincides with $\mathcal{L}$, by Lemmas 7.14 and 7.13. Therefore $G_2(G_1(\mathcal{L})) = \mathcal{L}$.

On the other hand it is immediate that $G_1(G_2(\mathcal{A})) = \mathcal{A}$ for every object $\mathcal{A}$ in **Asem**, using Lemmas 7.12 and 7.14.

Finally, $G_2(G_1(f)) = f$ and $G_1(G_2(f)) = f$, by the very definitions. ■

As an immediate consequence of the last result we obtain the following:

Corollary 7.16 *The category* ***Asem*** *has coproducts, and the forgetful functor* $N_{as} : \boldsymbol{Asem} \longrightarrow \boldsymbol{Sig}$ *is a cofibration. Thus,* ***Asem*** *has both unconstrained and constrained fibring.*

The isomorphism between **Alge** and **Asem** defined in Theorem 7.15 helps to understand the relationship between algebraizable logics and deductivizable quasi-varieties, while using natural notions of morphisms (in contrast to the unusual and restricted notion of morphisms used in [21]).

In particular, it is interesting to observe the following phenomenon: in [3] and [1] it was shown that there exists different algebraizable logics, say $\mathcal{L}$ and $\mathcal{L}'$, defined over the same signature and with the same quasivariety $\mathcal{K}$. As a consequence of Lemma 7.13, the respective algebraizators of $\mathcal{L}$ and $\mathcal{L}'$ cannot be equivalent as deductivizators of $\mathcal{K}$, in the sense of Definition 7.7. For instance, in [1] it was shown that Łukasiewicz's logic Ł$_3$ and paraconsistent logic J_3 (cf. [14]), when defined over the same signature, are algebraizable with the same quasi-variety, Moisil's 3-valued MV-algebras (see Example 7.20 below). Since Ł$_3$ and J_3 are different, the respective algebraizators are inequivalent, seen as deductivizators of their quasivariety. Consequently, both (different) logics are represented by different objects in the category **Asem**: despite having the same signature and the same quasi-variety, the (different) equivalence classes of the respective deductivizators separate both objects.

It is possible to characterize logics which have the same quasi-variety: they are *equivalent* in the following sense (cf. [1]):

Definition 7.17 *Let* $\mathcal{L} = \langle C, \vdash\rangle$ *and* $\mathcal{L}' = \langle C, \vdash'\rangle$ *be two logics over the same signature* C*. An* equivalence from $\mathcal{L}'$ to $\mathcal{L}$ *is a pair* $\langle\bar{\tau}, \bar{\eta}\rangle$ *such that* $\bar{\tau} = \{\tau^i(p_1) \ : \ 1 \leq i \leq r\}$ *and* $\bar{\eta} = \{\eta^j(p_1) \ : \ 1 \leq j \leq s\}$ *are finite sets of formulas in* $L(C)$ *depending at most on* p_1*, satisfying the following:*
(1) $\Gamma \vdash' \varphi$ *iff* $\{\bar{\tau}(\psi) \ : \ \psi \in \Gamma\} \vdash \bar{\tau}(\varphi)$*, for every* $\Gamma \cup \{\varphi\} \subseteq L(C)$*;*
(2) $p_1 \dashv\vdash \bar{\tau}(\bar{\eta}(p_1))$*.*

As usual, in the definition above, $\bar{\tau}(\varphi)$ stands for $\{\tau(\varphi) \ : \ \tau \in \bar{\tau}\}$ and $\{\bar{\tau}(\psi) \ : \ \psi \in \Gamma\}$ stands for $\{\tau(\psi) \ : \ \tau \in \bar{\tau} \text{ and } \psi \in \Gamma\}$. On the other hand, $\bar{\tau}(\bar{\eta}(p_1))$ stands for $\{\tau(\eta(p_1)) \ : \ \tau \in \bar{\tau} \text{ and } \eta \in \bar{\eta}\}$.

From [1] it follows that, if $\langle\bar{\tau}, \bar{\eta}\rangle$ is an equivalence from $\mathcal{L}'$ to $\mathcal{L}$, then $\langle\bar{\eta}, \bar{\tau}\rangle$ is an equivalence from $\mathcal{L}$ to $\mathcal{L}'$, and so the relation introduced in the definition above is indeed an equivalence relation between logics defined over C. Thus, we say that $\mathcal{L}$ *is equivalent to* $\mathcal{L}'$ if there exists an equivalence from $\mathcal{L}$ to $\mathcal{L}'$.

Proposition 7.18 *Let* $\mathcal{L}$ *be an algebraizable logic with algebraizator* $\langle\Delta, \langle\bar{\delta}, \bar{\varepsilon}\rangle\rangle$ *and quasi-variety* $\mathcal{K}$*. Let* $\langle\bar{\tau}, \bar{\eta}\rangle$ *be an equivalence from* $\mathcal{L}'$ *to* $\mathcal{L}$*. Then* $\mathcal{L}'$ *is algebraizable with algebraizator* $\langle\bar{\eta} \circ \Delta, \langle\bar{\delta} \circ \bar{\tau}, \bar{\varepsilon} \circ \bar{\tau}\rangle\rangle$ *and quasi-variety* $\mathcal{K}$*, where* $\bar{\eta} \circ \Delta = \{\eta(\varphi(p_1, p_2)) \ : \ \eta \in \bar{\eta} \text{ and } \varphi \in \Delta\}$ *and* $\langle\bar{\delta} \circ \bar{\tau}, \bar{\varepsilon} \circ \bar{\tau}\rangle = \{\langle\delta(\tau(p_1)), \varepsilon(\tau(p_1))\rangle \ : \ \langle\delta, \varepsilon\rangle \in \langle\bar{\delta}, \bar{\varepsilon}\rangle \text{ and } \tau \in \bar{\tau}\}$*.*

Proof: The proof uses repeatedly Theorem 7.5. Let $\Gamma \cup \{\varphi\} \subseteq L(C)$. Then

$$(*) \quad \Gamma \vdash' \varphi \quad \text{iff } \{\bar{\tau}(\psi) \ : \ \psi \in \Gamma\} \vdash \bar{\tau}(\varphi)$$
$$\text{iff } \{(\bar{\delta}(\bar{\tau}(\psi)) \approx \bar{\varepsilon}(\bar{\tau}(\psi))) \ : \ \psi \in \Gamma\} \models_{\mathcal{K}} (\bar{\delta}(\bar{\tau}(\varphi)) \approx \bar{\varepsilon}(\bar{\tau}(\varphi))).$$

On the other hand, $\bar{\tau}(\bar{\eta}(\Delta(p_1, p_2))) \dashv\vdash \Delta(p_1, p_2)$ and so

$$(\bar{\delta}(\bar{\tau}(\bar{\eta}(\Delta(p_1, p_2)))) \approx \bar{\varepsilon}(\bar{\tau}(\bar{\eta}(\Delta(p_1, p_2))))) \;=\!\!|\!\models_{\mathcal{K}} (\bar{\delta}(\Delta(p_1, p_2)) \approx \bar{\varepsilon}(\Delta(p_1, p_2))).$$

But $(\bar{\delta}(\Delta(p_1, p_2)) \approx \bar{\varepsilon}(\Delta(p_1, p_2))) \;=\!\!|\!\models_{\mathcal{K}} (p_1 \approx p_2)$, hence

$$(**) \quad (\bar{\delta}(\bar{\tau}(\bar{\eta}(\Delta(p_1, p_2)))) \approx \bar{\varepsilon}(\bar{\tau}(\bar{\eta}(\Delta(p_1, p_2))))) \;=\!\!|\!\models_{\mathcal{K}} (p_1 \approx p_2).$$

From $(*)$ and $(**)$, and using again Theorem 7.5, it follows that $\mathcal{L}'$ is algebraizable with algebraizator $\langle \bar{\eta} \circ \Delta, \langle \bar{\delta} \circ \bar{\tau}, \bar{\varepsilon} \circ \bar{\tau} \rangle \rangle$ and quasi-variety $\mathcal{K}$. ■

As an immediate consequence of Lemma 7.13 we obtain the following:

Corollary 7.19 *With notation as above, if* $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle \simeq_{\mathcal{K}} \langle \bar{\eta} \circ \Delta, \langle \bar{\delta} \circ \bar{\tau}, \bar{\varepsilon} \circ \bar{\tau} \rangle \rangle$ *then* $\mathcal{L} = \mathcal{L}'$.

Example 7.20 *Consider Łukasiewicz's logic* $Ł_3$ *and D'Ottaviano and da Costa's paraconsistent logic* J_3 *(cf. [14]), defined over the signature* C *such that* $C_1 = \{\neg\}$, $C_2 = \{\rightarrow\}$ *and* $C_k = \emptyset$ *in any other case. Both logics are defined as matrix logics with truth-tables as below.*

$\rightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

	$\neg$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

The only difference between them is that the set of designated values of $Ł_3$ *is* $D_1 = \{1\}$ *whereas the set of designated values of* J_3 *is* $D_2 = \{1, \frac{1}{2}\}$. *The logic* $Ł_3$ *is algebraizable with* $\langle \Delta, \langle \bar{\delta}, \bar{\varepsilon} \rangle \rangle$ *such that* $\Delta = \{p_1 \rightarrow p_2, p_2 \rightarrow p_1\}$ *and* $\langle \bar{\delta}, \bar{\varepsilon} \rangle = \{\langle p_1, p_1 \rightarrow p_1 \rangle\}$. *On the other hand,* $\langle \bar{\tau}, \bar{\eta} \rangle$ *such that* $\bar{\tau} = \{\neg p_1 \rightarrow p_1\}$ *and* $\bar{\eta} = \{\neg(\neg\neg p_1 \rightarrow \neg p_1)\}$ *is an equivalence from* J_3 *to* $Ł_3$ *(cf. [1]). Thus, by Proposition 7.18,* J_3 *is algebraizable with* $\langle \Delta', \langle \bar{\delta}', \bar{\varepsilon}' \rangle \rangle$ *such that*

$$\Delta' = \{\neg(\neg\neg(p_1 \rightarrow p_2) \rightarrow \neg(p_1 \rightarrow p_2)), \neg(\neg\neg(p_2 \rightarrow p_1) \rightarrow \neg(p_2 \rightarrow p_1))\}$$

and

$$\langle \bar{\delta}', \bar{\varepsilon}' \rangle = \{\langle \neg p_1 \rightarrow p_1, (\neg p_1 \rightarrow p_1) \rightarrow (\neg p_1 \rightarrow p_1) \rangle\}.$$

Consider now Sette's paraconsistent logic P^1 *(cf. [26]), defined over the same signature* C*, given by the matrix below.*

$\to$	1	$\frac{1}{2}$	0
1	1	1	0
$\frac{1}{2}$	1	1	0
0	1	1	1

	$\neg$
1	0
$\frac{1}{2}$	1
0	1

The set of designated values of P^1 *is* $D_3 = \{1, \frac{1}{2}\}$*. The logic* P^1 *is algebraizable with* $\langle \Delta'', \langle \bar{\delta}'', \bar{\varepsilon}'' \rangle\rangle$ *such that* $\Delta'' = \{p_1 \to p_2, p_2 \to p_1, \neg p_1 \to \neg p_2, \neg p_2 \to \neg p_1\}$ *and* $\langle \bar{\delta}'', \bar{\varepsilon}'' \rangle = \{\langle (p_1 \to p_1) \to p_1, p_1 \to p_1 \rangle\}$ *(cf. [22]). Consider now the coproduct signature* $C \uplus C$*. Then* $(C \uplus C)_1 = \{\neg_1, \neg_2\}$*,* $(C \uplus C)_2 = \{\to_1, \to_2\}$ *and* $(C \uplus C)_k = \emptyset$ *in any other case. Let* $\mathcal{L} = \langle C \uplus C, \vdash \rangle$ *be the matrix logic defined by the truth-tables below.*

$\to_1$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

$\to_2$	1	$\frac{1}{2}$	0
1	1	1	0
$\frac{1}{2}$	1	1	0
0	1	1	1

	$\neg_1$	$\neg_2$
1	0	0
$\frac{1}{2}$	$\frac{1}{2}$	1
0	1	1

The set of designated values of $\mathcal{L}$ *is* $D = \{1, \frac{1}{2}\}$*. Let* $i_1 : C \longrightarrow C \uplus C$ *and* $i_2 : C \longrightarrow C \uplus C$ *be the canonical injections. Then* $i_j(\neg) = \neg_j$ *and* $i_j(\to) = \to_j$ *for* $j = 1, 2$*. Clearly, the reducts of the matrix above along* i_1 *and* i_2 *correspond to the truth-tables of* J_3 *and* P^1*, respectively. Thus, the valuations for* $\mathcal{L}$ *are homomorphisms* $v : L(C \uplus C) \longrightarrow \{1, \frac{1}{2}, 0\}$ *which, restricted to* $\{\neg_1, \to_1\}$ *are valuations for* J_3*, and restricted to* $\{\neg_2, \to_2\}$ *are valuations for* P^1*. From this, it follows that* $\mathcal{L}$ *is, in fact, the* direct union *of* J_3 *and* P^1 *(cf. [11]).*

On the other hand, if v *is a valuation for* $\mathcal{L}$ *then* $v(\widehat{i_1}(\Delta'(p_1, p_2))) \subseteq D$ *iff* $v(p_1) = v(p_2)$ *iff* $v(\widehat{i_2}(\Delta''(p_1, p_2))) \subseteq D$*, and so*

$$\widehat{i_1}(\Delta') \dashv\vdash \widehat{i_2}(\Delta'').$$

From this, $\langle \widehat{i_1}(\Delta'), \langle \widehat{i_1}(\bar{\delta}'), \widehat{i_1}(\bar{\varepsilon}') \rangle\rangle$ *is an algebraizator for* $\mathcal{L}$*, and* $i_1 : J_3 \longrightarrow \mathcal{L}$ *is an* ***Alge****-morphism. Similarly,* $\langle \widehat{i_2}(\Delta''), \langle \widehat{i_2}(\bar{\delta}''), \widehat{i_2}(\bar{\varepsilon}'') \rangle\rangle$ *is an algebraizator for* $\mathcal{L}$ *and* $i_2 : P^1 \longrightarrow \mathcal{L}$ *is also an* ***Alge****-morphism. Therefore, the fibring* $J_3 \circledast P^1 = \langle C \uplus C, \vdash_\circledast \rangle$ *of* J_3 *and* P^1 *in* ***Alge*** *is sound for* $\mathcal{L}$*, that is: if* $\Gamma \vdash_\circledast \varphi$ *then* $\Gamma \vdash \varphi$*. From the analysis in [10] about fibring, there is no guarantee that* $J_3 \circledast P^1$ *is complete for* $\mathcal{L}$*: some interactions between the connectives of* J_3 *and* P^1 *could be present in* $\mathcal{L}$ *but not in* $J_3 \circledast P^1$*.*

8 Morphisms in Asem and the Leibniz operator

All the logics already studied here belong to the so called *Leibniz Hierarchy*. This hierarchy classifies logics according to the properties of the *Leibniz operator* Ω. In this section a brief characterization of the morphisms in **Asem** in terms of Ω will be given.

Recall from Proposition 2.13 that $Th_{\mathcal{L}}$, the set of theories of a logic $\mathcal{L}$, is a complete lattice ordered by inclusion, denoted by $\mathbf{Th}_{\mathcal{L}}$. On the other hand, the set $Th_{\mathcal{K}}$ of equational theories for a quasi-variety $\mathcal{K}$, ordered by inclusion, is also a complete lattice denoted by $\mathbf{Th}_{\mathcal{K}}$, as it was observed after Definition 7.2. We begin by briefly recalling the characterization of algebraizable logics in terms of the Leibniz operator Ω given in [3].

Theorem 8.1 *Let $\mathcal{L}$ be an algebraizable logic defined over a signature C, with equivalence $\Delta(p_1, p_2)$, and let $\mathcal{K}$ be a quasi-variety over C. Then, $\mathcal{K}$ is the quasi-variety of $\mathcal{L}$ if and only if the mapping*

$$\Omega_{\mathcal{K}}T = \{(\varphi \approx \psi) \ : \ \Delta(\varphi, \psi) \subseteq T\}$$

is a complete lattice isomorphism $\Omega_{\mathcal{K}} : \mathbf{Th}_{\mathcal{L}} \longrightarrow \mathbf{Th}_{\mathcal{K}}$ that commutes with substitutions over C.

The mapping $\Omega_{\mathcal{K}}$ is the *Leibniz operator related to $\mathcal{K}$*. Observe that the definition of $\Omega_{\mathcal{K}}$ does not depend on the particular choice of Δ, because of Proposition 5.4. In a certain sense, the (class of) mapping(s) Ω relates the category **Alge** with **Asem**: the lattice of theories of each algebraizable logic is isomorphically mapped into the lattice of equational theories of the corresponding quasi-variety; we will write $\mathbf{Alge} \xrightarrow{\ \Omega\ } \mathbf{Asem}$ to refer to this situation (see figure at the end of this section). We stress that this notation is informal, because Ω *is not* a functor from **Alge** to **Asem**.

Recall from Remark 7.9 that any **Sig**-morphism $f : C \longrightarrow C'$ induces a mapping $\bar{f} : LEq(C) \longrightarrow LEq(C')$ such that $\bar{f}(\varphi \approx \psi) = (\widehat{f}(\varphi) \approx \widehat{f}(\psi))$. Using this, translations between equational languages can be characterized as "closed functions", in the sense of Proposition 2.13(2).

Proposition 8.2 *Let $\mathcal{K}$, $\mathcal{K}'$ be two quasi-varieties over C and C', respectively, and let $f : C \longrightarrow C'$ be a **Sig**-morphism. Then f satisfies condition (b)(1) of Definition 7.8 if and only if $\bar{f}^{-1}(\Upsilon) \in Th_{\mathcal{K}}$, for every $\Upsilon \in Th_{\mathcal{K}'}$.*

Proof: See [27]. ■

The key result in order to characterize **Asem**-morphisms is the following:

Proposition 8.3 *Let $\mathcal{L} = \langle C, \vdash \rangle$ and $\mathcal{L}' = \langle C', \vdash' \rangle$ be algebraizable logics with quasi-varieties $\mathcal{K}$ and $\mathcal{K}'$, respectively, and let $f : \mathcal{L} \longrightarrow \mathcal{L}'$ be a* ***Cons****-morphism with associated mappings $\widehat{f} : L(C) \longrightarrow L(C')$ and $\bar{f} : LEq(C) \longrightarrow LEq(C')$. Then, the following conditions are equivalent:*
(a) f is an ***Equiv****-morphism (that is, an* ***Alge****-morphism);*
(b) $\Omega_{\mathcal{K}}(\widehat{f}^{-1}(T)) = \bar{f}^{-1}(\Omega_{\mathcal{K}'}(T))$, for every theory T in $Th_{\mathcal{L}'}$.

Proof: Firstly, observe that condition (b) makes sense because of Proposition 2.13(2): if $T \in Th_{\mathcal{L}'}$ then $\widehat{f}^{-1}(T) \in Th_{\mathcal{L}}$, provided that f is a **Cons**-morphism. Thus $\Omega_{\mathcal{K}}(\widehat{f}^{-1}(T))$ is a well-defined set.

(a) implies (b):
Let $\Delta(p_1, p_2)$ be an equivalence in $\mathcal{L}$, and suppose that $(\varphi \approx \psi) \in \Omega_{\mathcal{K}}(\widehat{f}^{-1}(T))$. Then $\Delta(\varphi, \psi) \subseteq \widehat{f}^{-1}(T)$, from definition of $\Omega_{\mathcal{K}}$ (cf. Theorem 8.1). Hence

$$\widehat{f}(\Delta)(\widehat{f}(\varphi), \widehat{f}(\psi)) \subseteq T$$

and so $(\widehat{f}(\varphi) \approx \widehat{f}(\psi)) \in \Omega_{\mathcal{K}'}(T)$, by definition of $\Omega_{\mathcal{K}'}$ and because $\widehat{f}(\Delta)$ is an equivalence in $\mathcal{L}'$. But $(\widehat{f}(\varphi) \approx \widehat{f}(\psi)) = \bar{f}(\varphi \approx \psi)$, whence $(\varphi \approx \psi) \in \bar{f}^{-1}(\Omega_{\mathcal{K}'}(T))$. Thus, $\Omega_{\mathcal{K}}(\widehat{f}^{-1}(T)) \subseteq \bar{f}^{-1}(\Omega_{\mathcal{K}'}(T))$. The other inclusion is proved similarly.

(b) implies (a):
Let $\Delta(p_1, p_2)$ and $\Delta'(p_1, p_2)$ be equivalences in $\mathcal{L}$ and $\mathcal{L}'$, respectively. We begin by proving that $\widehat{f}(\Delta)(p_1, p_2) \vdash' \Delta'(p_1, p_2)$. Thus, let

$$T = \{\varphi \in L(C') \ : \ \widehat{f}(\Delta)(p_1, p_2) \vdash' \varphi\}.$$

Clearly $T \in Th_{\mathcal{L}'}$ and $\widehat{f}(\Delta)(p_1, p_2) \subseteq T$, therefore $\Delta(p_1, p_2) \subseteq \widehat{f}^{-1}(T)$. Then, using (b), $(p_1 \approx p_2) \in \Omega_{\mathcal{K}}(\widehat{f}^{-1}(T)) = \bar{f}^{-1}(\Omega_{\mathcal{K}'}(T))$. From this, $\bar{f}(p_1 \approx p_2) \in \Omega_{\mathcal{K}'}(T)$, that is, $(p_1 \approx p_2) \in \Omega_{\mathcal{K}'}(T)$. Hence $\Delta'(p_1, p_2) \subseteq T$, by definition of $\Omega_{\mathcal{K}'}$. That is, $\widehat{f}(\Delta)(p_1, p_2) \vdash' \Delta'(p_1, p_2)$.

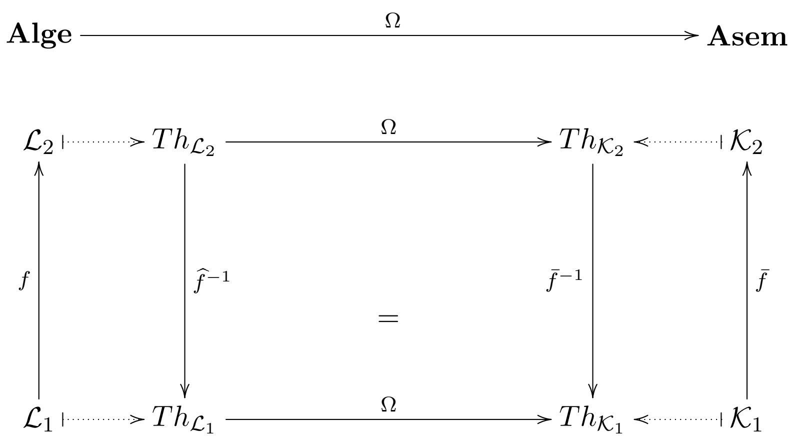
Analogously, it can be proven that $\Delta'(p_1, p_2) \vdash' \widehat{f}(\Delta)(p_1, p_2)$. Thus, the set of formulas $\widehat{f}(\Delta)(p_1, p_2)$ is an equivalence in $\mathcal{L}'$, by Proposition 5.4, and so f is an **Equiv**-morphism. ■

Now we arrive to the desired characterization of morphisms in **Alge** using the Leibniz operator Ω:

Theorem 8.4 *Let $\mathcal{L} = \langle C, \vdash \rangle$ and $\mathcal{L}' = \langle C', \vdash' \rangle$ be algebraizable logics with quasi-varieties $\mathcal{K}$ and $\mathcal{K}'$, respectively, and let $f : C \longrightarrow C'$ be a **Sig**-morphism with associated mappings $\widehat{f} : L(C) \longrightarrow L(C')$ and $\bar{f} : LEq(C) \longrightarrow LEq(C')$. Then f is an **Alge**-morphism if and only if it verifies the following:*
*(1) $f : \mathcal{L} \longrightarrow \mathcal{L}'$ is a **Cons**-morphism;*
(2) $\Omega_{\mathcal{K}}(\widehat{f}^{-1}(T)) = \bar{f}^{-1}(\Omega_{\mathcal{K}'}(T))$, for every theory T in $Th_{\mathcal{L}'}$.

Proof: Immediate, from Proposition 8.3. ■

From the results above, if $\mathcal{L}$ and $\mathcal{L}'$ are algebraizable with $\mathcal{K}$ and $\mathcal{K}'$, respectively, and $f : \mathcal{L} \longrightarrow \mathcal{L}'$ is an **Alge**-morphism, then f induces three mappings: $\widehat{f}^{-1} : Th_{\mathcal{L}'} \longrightarrow Th_{\mathcal{L}}$ (from Proposition 2.13(2)); $\bar{f} : \mathcal{K} \longrightarrow \mathcal{K}'$ (from Lemma 7.11 and Definition 7.8) and $\bar{f}^{-1} : Th_{\mathcal{K}'} \longrightarrow Th_{\mathcal{K}}$ (from Proposition 8.2). Thus, the characterization of **Alge**-morphisms obtained in Theorem 8.4 can be visualized through the figure below.



Observe that the inner square is a commutative diagram in the category **Set** of sets; moreover, it is a commutative diagram in the category of complete lattices. Since it would be reasonable to expect the commutativity of such diagram, Theorem 8.4 shows that the definition of morphisms in **Alge** is as natural as one could think.

9 Concluding remarks

In this paper we address the problem of combining, by fibring, some kind of logics belonging to the Leibniz Hierarchy. Specifically, we shown how to obtain (unconstrained and constrained) fibring of protoalgebraic logics, equivalential logics and algebraizable logics, by considering appropriate categories of logics in each case. Finally, it was proven that the category of algebraizable logics defined here is isomorphic to a category of algebraic semantics. As a consequence of this, the fibring properties of the former can be automatically transferred to the latter.

Most of the material here was adapted from [15], where a slightly different category of signatures, called **Plan**, was used. This category, which have a strong connection with that used in [21], has the same objects than **Sig**, but a morphism $f : C \longrightarrow C'$ sends any k-ary connective $c \in C_k$ into a formula $\varphi_c(p_1, \ldots, p_k) \in L(C')$ having at most the variables $p_1, \ldots, p_k$. This broader notion of morphism encompasses more cases of translations between logics, occurring in 'real' examples, than the tight notion of morphism in **Sig**. This approach was also used, for instance, in [4] and [5] for combining logics through products of (algebraizable) logics, using a categorial version of the technique of *Possible-Translations Semantics* proposed in [8]. When combining logics by fibring, colimits are used instead of limits; but the category **Plan** is not well-behaved for colimits, and then constrained fibring is only possible when just share connectives (as in **Sig**), instead of using monomorphism in general. Thus, fibring in **Plan** or in **Sig** are the same. This is why we decide to consider here **Sig** instead of **Plan** as the linguistic basis for the categories of logics.

Our approach is related (and inspired, to some extent) with the proposal in [21]. However, some differences should be pointed out:

- In the above mentioned article, the morphisms between logics are equivalence classes of **Plan**-morphisms, by identifying **Plan**-morphisms whose images are interderivable in the target logic. This move allows to prove that the resulting category is (small) cocomplete. Moreover, using the notions introduced later on in [25], constrained fibring could be computed in this category, without any restrictions for the sharing diagrams. However, as we already observe above, this kind of morphism is somewhat unnatural and hard to handle with.

- The definition of the category of algebraic semantics proposed in [21] follow the same lines than the category of logics systems (that is, to consider equivalence classes of certain class of mappings) and then the same criticism as above can be applied. Moreover, the equivalence relation is defined over certain class of mappings which are strictly stronger than the morphisms we propose here for the category **Asem** (see Remark 7.9). On the other hand, our choice for morphisms in **Asem**, despite being simpler, is as natural as expected, if we look at Proposition 8.2.

- From the observations above, the isomorphism between **Alge** and **Asem** obtained in Theorem 7.15 improves, in our opinion, the pioneering result obtained in [21]. The characterization of **Alge**-morphisms in terms of the Leibniz operator Ω given in Theorem 8.4 give us additional support for this claim.

There are still several topics of this brief study that deserve future research.

The most obvious remark is that the results obtained here could be extended to another classes of logics belonging to the Leibniz Hierarchy. So, it would be interesting to analyze fibring of strongly algebraizable logics, Fregean logics, implicative logics and so on. However, things are not so easy as they seem and new techniques could be required, mainly when finitariness is lost.

Another line of research is a broader study of the semantical counterpart of fibring. Specifically, what is the impact of fibring quasi-varieties? How fibring in **Asem** can be algebraically characterized? What is the relationship between the quasi-variety of the logic obtained by (unconstrained and constrained) fibring with the quasi-variety of the given logics, in algebraic (and not merely categorial) terms?

It should be mentioned that there exist in the literature another categorial approach to abstract algebraic logic, developed by G. Voutsadakis (see [28] and [29]). Despite that approach is not devoted to fibring, its categorial treatment of abstract algebraic logic can be considered broader than ours, because it is based on the very general notion of π-institutions. The adaptation of our approach to that setting deserves future research.

Finally, it would be interesting to extend our results to categories of logic systems based on different categories of signatures than **Sig** or **Plan**.

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