A Hilbert-style axiomatization of higher-order intuitionistic logic

Marcelo E. Coniglio[1] Cristina Sernadas[2]

[1] GTAL, Departamento de Filosofia, Universidade Estadual de Campinas, Brazil
[2] CLC/CMA, Departamento de Matemática, IST, Portugal

Abstract

Two Hilbert calculi for higher-order intuitionistic logic (or theory of types) are introduced. The first is defined in a language that uses just exponential types of power type, and corresponds to Bell's local set theory. The second one is defined in a language with arbitrary functional types and correspond to Church's simple type theory. We show that both systems are sound and complete with respect to usual topos semantics.

Introduction

Higher-order logic (or theory of types) is defined in a very rich language which permits to express most of mathematics reasoning. Theory of types was introduced by Russell in 1908 as a solution to paradoxes in set theory (see [12]) and was reformulated by Church in [3].

The basic idea of theory of types is to consider objects of different kind, or *types*. A type can be seen as a given range of values, all of them conforming a certain *specie*. The universe of things within a structure is then supposed to be classified by species or types. Thus, it is natural (and useful) to think about natural numbers, real numbers, boolean values, and strings, for instance, as been different types of data. The distinction between types is particularly useful in computer science because the different requirements for data storage. Conceptually, it is also very natural to describe mathematical structures using different types of individuals. For instance, the standard definition of vector spaces uses two kind of individuals: Scalars and vectors.

If x is an individual of type θ (written $x : \theta$) and x belongs to a collection A (or if x has the property A) then the individual A cannot be of sort θ; instead A is an individual of sort "collections of individuals of sort θ" or, in short, $A : P(\theta)$, where $P(\theta)$ denotes the "power" type of the type θ. This kind of distinction between "element of a given type" and "collections of objects of a given type" allows Russell to avoid the paradox discovered by himself in 1901, namely, $A = \{x \mid x \notin x\}$. In fact, if it would possible to have $x \in x$ for some x then $A \in A$ if and only if $A \notin A$ (where A is as above). Therefore, according to Russell's type theory, the statement "$x \in x$" is senseless (and not contradictory, as appears according to Zermelo-Fraenkel's set theory), and it is forbidden by allowing types for the individuals.

Of course, a type $P(\theta)$ is of higher-order than θ, then the resulting logic is called higher-order logic. For instance, in second-order logic we have just two types: Individuals θ and properties $P(\theta)$. Then, using lowercase letters for variables of type θ and uppercase letters for variables of type $P(\theta)$ (or second-order variables) we can express the second-order Peano's Induction axiom as follows:

$$\forall Y[Y(0) \wedge \forall x(Y(x) \Rightarrow Y(S(x))) \Rightarrow \forall x Y(x)].$$

This axiom, together with first-order Peano's axiom for arithmetic, characterizes with a single second-order sentence Φ the standard structure $\langle \mathbb{N}, +, \cdot, 0, 1\rangle$.

We see that theory of types permits, together with the definition of basic types, to *construct* recursively more complex types from the given ones. For instance, given types θ_1 and θ_2, it is possible to define the functional type $(\theta_1 \to \theta_2)$ of maps from θ_1 to θ_2. Moreover, if $\mathbf{\Omega}$ denotes the "truth-values" type, then $P(\theta)$ is obtained as the functional type $(\theta \to \mathbf{\Omega})$ (considering collections of individuals of type θ as being characteristic maps). We can also define the product type $\theta_1 \times \theta_2$ formed by all the ordered pairs of individuals of type θ_1 and θ_2, respectively, as well as the type $\theta_1 \coprod \theta_2$ (the disjoint union of θ_1 and θ_2), etcetera.

The standard set-theoretic semantics for higher-order logic is obtained by straightforward generalization of the semantics of first-order logic (cf. [14]). For instance, if θ_1 and θ_2 are interpreted by sets A_1 and A_2, respectively, then the functional type $(\theta_1 \to \theta_2)$ is interpreted as the set $A_2^{A_1}$ of all the maps from A_1 to A_2.

By Gödel's second theorem, it is immediate to show that there is no (reasonable) proof system complete for the standard (set) semantics of higher-order logic. In fact, if Φ is the second-order Peano's arithmetic sentence mentioned above then

$$\langle \mathbb{N}, +, \cdot, 0, 1\rangle \models \varphi \;\text{ iff }\; (\Phi \Rightarrow \varphi) \;\text{ is second-order valid}$$

for every first-order sentence φ. By Tarski's theorem, the left-hand side is not arithmetically definable, then the set of second-order validities cannot be arithmetical either. Thus, there is no effective and complete axiomatization of second-order validity (cf. [14]).

On the other hand, Henkin proves in [5] that it is possible to give an axiomatization of higher-order logic sound and complete w.r.t. a wider class of models, called *general models*, in which types of the form $(\theta_1 \to \theta_2)$ are interpreted as subsets of the set of maps from (the interpretation of) θ_1 to (the interpretation of) θ_2. The trick consists of enlarging adequately the class of models, reducing therefore the set of validities, which can be now captured by proof-theoretic methods.

From the works of Lawvere (see for example [8, 9]) it was proved that the usual proof-methods for higher-order intuitionistic logic (from now on denoted as *hol*) are sound and complete w.r.t. an extremely elegant topos semantics. The discover of Lawvere that category theory is able to interpret logical languages in a natural way, opens the possibilities to consider topoi as a large class of new mathematical universes of discourse. The basic idea is to substitute sets (interpreting types) by arbitrary objects in a given topos. Function symbols are interpreted as morphisms, cartesian products are categorial products, relation symbols are interpreted as subobjects, functional types are interpreted using exponentials, and so on (see, for instance, [7, 2, 10, 6, 11]). The fact that categorial semantics uses topoi guarantees the minimum amount of categorial operations needed to interpret the logical symbols of higher-order languages.

Bell proposes in his book [2] a sequent calculus-style axiomatization of *hol* called *local set theory*, which permits to describe syntactically formal properties of topoi. The language proposed by Bell uses product types and power types as type constructors, as well as a distinguished type **1** for singleton (the terminal object). It is well-known that it suffices to describe functional types (which correspond to exponentiation in the topos semantics).

The choice of a sequent calculus for local set theory is not surprising: The proof-methods for *hol* one can found in the literature are, in general, expressed as sequent-calculus or natural-deduction systems. On the other hand, the Hilbert-calculus presentations of *hol* contain complicated rules of inference (cf. [5, 1]).

The goal of this article is to introduce two very simple and natural Hilbert-style axiomatizations of *hol*, which are sound and complete w.r.t. topos semantics. The first one is obtained by adapting the sequent calculus for local set theory mentioned above, defined in a language with power types but without arbitrary functional types. The second one, originally introduced in [4], is an extension of the former to a language with arbitrary functional types, corresponding to Church's simple type theory. The notions of signature with schema variables and of Hilbert calculus with provisos, as well as the notion of local and global entailment used here, are taken and adapted from [4], and this article should be seen as a companion to that paper.

The organization of this article is as follows: In the first section we give an account of the higher-order languages to be considered. The main characteristic is the use of symbols for *arbitrary* terms (called schema terms, introduced in [13]) as well as schema terms of the form ${}^{x}_{\xi'}\xi$, denoting the substitution of every free occurrence of variable x in ξ for ξ'. Another remarkable feature of our definition is the formalization of provisos in the rules. These features are useful for express rules in higher-order languages, and are specially profitable for fibring logics (cf. [13, 15, 4]). In Section 2 we briefly describe topos semantics, and we use, according to [13, 15, 4], two notions of semantic entailment: Local and global. Local entailment is the usual one, stating, roughly speaking, that the object interpreting the meet of the premises is contained in the object interpreting the conclusion. The global entailment is a weaker notion, stating that the conclusion is true (in a given interpretation) provided that the premises are also true in that model. In third section we introduce the notion of Hilbert calculus, which, again, consider two different notions of entailment, one (global) weaker than the other (local). Of course each syntactical notion of entailment corresponds to the semantical one. Since in categorial logic it is allowed to use "empty domains" interpreting types, the cut rule is no longer valid in semantical terms (whenever some variable occurring free in the cut formula does not occur free in the result). This forces us to consider a weaker notion of soundness.

Section 4 describes briefly local set theory. In Section 5 we introduce the first Hilbert-style axiomatization for *hol* we propose, obtaining the main result: The equivalence with local set theory. Finally, in Section 6 we introduce the second axiomatization of *hol* expressed in the language of simple type theory, and prove the completeness theorem w.r.t. standard topos semantics.

Throughout this paper, the symbol △ will be used to finish Definitions and Remarks, and the symbol QED will be used to finish proofs (of Propositions, Lemmas, Theorems and Corollaries).

1 Higher-Order Languages

In this section we recall the notion of signature introduced in [4]. This is a simplified version, enough for our purposes.

Definition 1.1 Given a set S with distinguished element $\mathbf{1}$, we denote by $\Theta(S)$ the set inductively defined as follows: (i) $S \subseteq \Theta(S)$; (ii) if $\theta_1, \ldots, \theta_n \in \Theta(S)$ for integer $n \geq 2$ then $(\theta_1 \times \cdots \times \theta_n) \in \Theta(S)$; (iii) if $\theta \in \Theta(S)$ then $P(\theta) \in \Theta(S)$. △

As usual, we write θ^n for the n-th power of θ (the product of θ with itself n times) and by convention θ^0 is $\mathbf{1}$ and θ^1 is θ.

Definition 1.2 A *signature* is a tuple $\Sigma = \langle S, \mathbf{1}, \Xi, X, F \rangle$ where:

- S is a set with distinguished element $\mathbf{1}$;
- $\Xi = \{\Xi_\theta\}_{\theta \in \Theta(S)}$ where each Ξ_θ is a denumerable set $\Xi_\theta = \{\xi_k^\theta \mid k \in \mathbb{N}\}$;
- $X = \{X_\theta\}_{\theta \in \Theta(S)}$ where each X_θ is a denumerable set $X_\theta = \{x_k^\theta \mid k \in \mathbb{N}\}$;
- $F = \{F_{\theta\theta'}\}_{\theta,\theta' \in \Theta(S)}$ where each $F_{\theta\theta'}$ is a set. △

The elements of S are known as *sorts* or *ground types*. The elements of $\Theta(S)$ are known as *types* over S. Ground type $\mathbf{1}$ is called the *unit sort*. The type $P(\mathbf{1})$, denoted by $\mathbf{\Omega}$, is called the *truth value type*. The elements of each Ξ_θ and X_θ are called *schema variables* and *variables* of type θ, respectively. The elements of each $F_{\theta\theta'}$ are called *function symbols* of type $\theta\theta'$.

Definition 1.3 The family $ST(\Sigma) = \{ST(\Sigma)_\theta\}_{\theta \in \Theta(S)}$ is inductively defined as follows:

- $\Xi_\theta \cup X_\theta \subseteq ST(\Sigma)_\theta$;
- if $x \in X_{\theta'}$, $\xi' \in \Xi_{\theta'}$ and $\xi \in \Xi_\theta$ then ${}^{x}_{\xi'}\xi \in ST(\Sigma)_\theta$;
- if $f \in F_{\theta\theta'}$ and $t \in ST(\Sigma)_\theta$ then $(f\, t) \in ST(\Sigma)_{\theta'}$;
- $\langle\rangle \in ST(\Sigma)_{\mathbf{1}}$;
- if $t_i \in ST(\Sigma)_{\theta_i}$ for $1 \leq i \leq n$ with $n \geq 2$ then $\langle t_1, \ldots, t_n \rangle \in ST(\Sigma)_{(\theta_1 \times \cdots \times \theta_n)}$;

- if $t \in ST(\Sigma)_{(\theta_1 \times \cdots \times \theta_n)}$, $n \geq 2$ and $1 \leq i \leq n$ then $(t)_i \in ST(\Sigma)_{\theta_i}$;
- if $t_1, t_2 \in ST(\Sigma)_\theta$ then $(=_\theta \langle t_1, t_2 \rangle) \in ST(\Sigma)_{\mathbf{\Omega}}$;
- if $t_1 \in ST(\Sigma)_\theta$ and $t_2 \in ST(\Sigma)_{P(\theta)}$ then $(\in_\theta \langle t_1, t_2 \rangle) \in ST(\Sigma)_{\mathbf{\Omega}}$;
- if $x \in X_\theta$ and $t \in ST(\Sigma)_{\mathbf{\Omega}}$ then $(\mathbf{set}_\theta x\, t) \in ST(\Sigma)_{P(\theta)}$. △

The elements of each $ST(\Sigma)_\theta$ are called *schema terms* of type θ. Schema terms of type $\mathbf{\Omega}$ are also known as *schema formulae*. Schema terms without occurrences of schema variables are called *terms*: $T(\Sigma)_\theta$ denotes the set of terms of type θ. Note that schema terms with occurrences of ${}^{x}_{\xi'}\xi$ are not terms. Schema formulae without schema variables are called *formulae*. We write $SL(\Sigma)$ and $L(\Sigma)$ for $ST(\Sigma)_{\mathbf{\Omega}}$ and $T(\Sigma)_{\mathbf{\Omega}}$, respectively. As we shall see in Section 3, schema variables are used in Hilbert calculi to express arbitrary terms within rules. Thus, with respect to semantics we are just interested in terms, and schema terms will be useful just as a tool for Hilbert calculi.

Every occurrence of a variable x in a schema term $(\mathbf{set}_\theta x\, \delta)$ or in ${}^{x}_{\xi'}\xi$, inside a schema term t, is said to be *bound* in t. Any other occurrence of x in a schema term t is said to be *free* in t. In particular, the unique bound occurrences of a variable x in a term t are in the scope of a term $(\mathbf{set}_\theta x\, \varphi)$ occurring in t. If t, t' are schema terms and x is a variable of the same type that t then $t^x_{t'}$ denotes the schema term obtained from t by substituting every free occurrence of x in t by t'. We say that a term $t' \in ST(\Sigma)_\theta$ *is free for a variable* $x \in X_\theta$ *in a term* t if, for every variable y occurring free in t', every occurrence of y in $t^x_{t'}$ not already in t is free.

Frequently we will omit the types attached to the symbols. As usual, we will adopt infix notation, writing for example $(t_1 = t_2)$ instead of $(=_\theta \langle t_1, t_2 \rangle)$. We also write $\{x : \gamma\}$ for $(\mathbf{set}_\theta x\, \gamma)$, and $t_1 \in_\theta t_2$ (or even $t_1 \in t_2$) instead of $(\in_\theta \langle t_1, t_2 \rangle)$.

Other logical operations can be introduced through abbreviations (cf. [2]):

- Equivalence: $(\delta_1 \Leftrightarrow \delta_2)$ for $(\delta_1 =_{\mathbf{\Omega}} \delta_2)$.
- True: $\mathbf{t}$ for $(\langle\rangle =_{\mathbf{1}} \langle\rangle)$.
- Conjunction: $(\delta_1 \wedge \delta_2)$ for $(\langle \delta_1, \delta_2 \rangle =_{(\mathbf{\Omega} \times \mathbf{\Omega})} \langle \mathbf{t}, \mathbf{t} \rangle)$.
- Implication: $(\delta_1 \Rightarrow \delta_2)$ for $((\delta_1 \wedge \delta_2) \Leftrightarrow \delta_1)$.
- Universal quantification: $(\forall_\theta x^\theta_k\, \delta)$ for $(\{x^\theta_k : \delta\} =_{P(\theta)} \{x^\theta_k : \mathbf{t}\})$.
- False: $\mathbf{f}$ for $(\forall_{\mathbf{\Omega}} x^{\mathbf{\Omega}}_1\, x^{\mathbf{\Omega}}_1)$.
- Negation: $(\neg\, \delta)$ for $(\delta \Rightarrow \mathbf{f})$.
- Disjunction: $(\delta_1 \vee \delta_2)$ for

  $$(\forall_{\mathbf{\Omega}} x^{\mathbf{\Omega}}_i\, (((\delta_1 \Rightarrow x^{\mathbf{\Omega}}_i) \wedge (\delta_2 \Rightarrow x^{\mathbf{\Omega}}_i)) \Rightarrow x^{\mathbf{\Omega}}_i)),$$

  where $x^{\mathbf{\Omega}}_i$ is the first variable of type $\mathbf{\Omega}$ not occurring free in $\langle \delta_1, \delta_2 \rangle$.

- Existential quantification: $(\exists_\theta x_k^\theta\, \delta)$ for

$$(\forall_{\mathbf{\Omega}} x_i^{\mathbf{\Omega}}\, (\forall_\theta x_k^\theta((\delta \Rightarrow x_i^{\mathbf{\Omega}}) \Rightarrow x_i^{\mathbf{\Omega}}))),$$

  where $x_i^{\mathbf{\Omega}}$ is the first variable of type $\mathbf{\Omega}$ not occurring free in δ.

2 Topos Semantics

Higher-order languages can be interpreted in any topos (see, for instance, [7, 2, 10, 6, 11]). In order to interpret Σ-terms in a given topos we need to introduce the notion of context.

By a Σ-*context* we mean a finite sequence $\vec{x} = x_1 \ldots x_n$ of distinct variables. We denote by $[]$ the *empty context*. Given a context $\vec{x} = x_1 \ldots x_n$ where the variables $x_1, \ldots, x_n$ are of type $\theta_1, \ldots, \theta_n$, respectively, we write $\theta_{\vec{x}}$ for $\theta_1 \times \cdots \times \theta_n$ and say that $\theta_{\vec{x}}$ is the type of the context $\vec{x}$. By definition $\theta_{[]}$ is $\mathbf{1}$.

The set $ST(\Sigma, \vec{x})_\theta$ is composed by all Σ-schema terms t of type θ such that every variable occurring free in t appears in the context $\vec{x}$. The sets $ST(\Sigma, \vec{x})$, $SL(\Sigma, \vec{x})$, $T(\Sigma, \vec{x})$ and $L(\Sigma, \vec{x})$ are defined analogously.

Given a finite set Γ of terms we may refer to its *canonical context* formed exclusively by the variables occurring free in some term t of Γ (this canonical context is unique once we fix a total ordering of the variables).

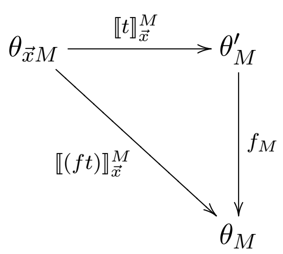
Definition 2.1 Let Σ be a signature. A Σ-*structure* is a pair $M = \langle \mathcal{E}, \cdot_M \rangle$ such that $\mathcal{E}$ is a (non-degenerate) topos and $\cdot_M$ is a map such that:

- θ_M is an object of $\mathcal{E}$ for all $\theta \in \Theta(S)$ such that $\mathbf{1}_M$ is terminal 1, $(\theta_1 \times \cdots \times \theta_n)_M$ is $\theta_{1M} \times \cdots \times \theta_{nM}$ and $P(\theta)_M$ is the exponential Ω^{θ_M} (thus, $\mathbf{\Omega}_M$ is identified with the subobject classifier Ω);
- if $f \in F_{\theta\theta'}$ then $f_M : \theta_M \to \theta'_M$ in $\mathcal{E}$. $\triangle$

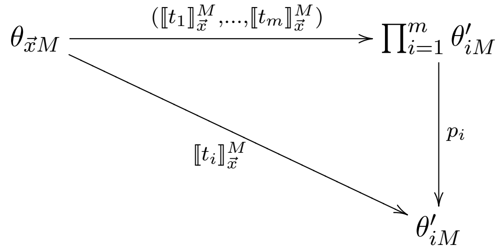
Given a Σ-structure M and a context $\vec{x}$ of type $\theta_{\vec{x}}$ let $\theta_{\vec{x}M}$ be $\theta_{1M} \times \cdots \times \theta_{nM}$.

Definition 2.2 If $t \in T(\Sigma, \vec{x})_\theta$ and M is a Σ-structure then we define inductively a morphism $[\![t]\!]_{\vec{x}}^M : \theta_{\vec{x}M} \to \theta_M$ as follows:

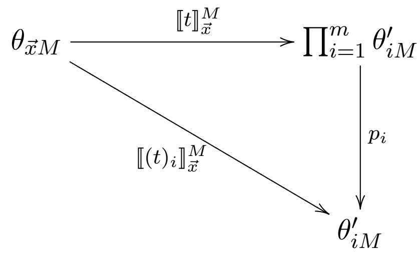
- $[\![x_i]\!]_{\vec{x}}^M$ is the canonical projection over θ_{iM};
- $[\![\langle\rangle]\!]_{\vec{x}}^M$ is the unique map from $\theta_{\vec{x}M}$ to 1;
- $[\![(ft)]\!]_{\vec{x}}^M$ is the composite $f_M \circ [\![t]\!]_{\vec{x}}^M$;



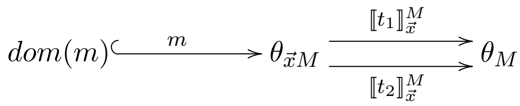
- $[\![\langle t_1,\dots,t_m\rangle]\!]^M_{\vec{x}}$ is $([\![t_1]\!]^M_{\vec{x}},\dots,[\![t_m]\!]^M_{\vec{x}})$;



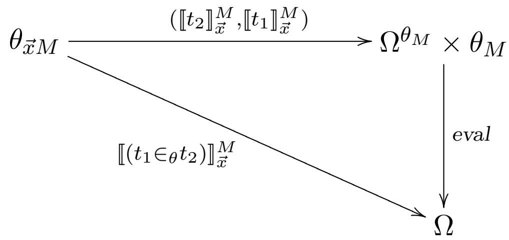
- $[\![(t)_i]\!]^M_{\vec{x}}$ is $p_i \circ [\![t]\!]^M_{\vec{x}}$, where t is of type $(\theta'_1 \times \cdots \times \theta'_m)$ and p_i is the canonical projection over θ'_{iM};



- $[\![(t_1 =_\theta t_2)]\!]^M_{\vec{x}}$ is the characteristic map of $m : dom(m) \hookrightarrow \theta_{\vec{x}M}$, the monomorphism obtained from the equalizer of $\{[\![t_i]\!]^M_{\vec{x}} : \theta_{\vec{x}M} \to \theta_M\}_{i=1,2}$;



- $[\![(t_1 \in_\theta t_2)]\!]^M_{\vec{x}}$ is $eval \circ ([\![t_2]\!]^M_{\vec{x}}, [\![t_1]\!]^M_{\vec{x}})$, where $eval : \Omega^{\theta_M} \times \theta_M \to \Omega$ is the evaluation map associated to the exponential Ω^{θ_M};



- $[\![\{x : \varphi\}]\!]^M_{\vec{x}}$ is the exponential transpose of $[\![\varphi^x_y]\!]^M_{\vec{x}y} : \theta_{\vec{x}M} \times \theta_M \to \Omega$ with respect to θ_M, where y is the first variable free for x in φ not occurring in $\vec{x}$.

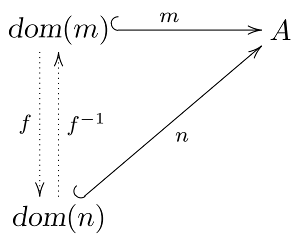
$$[\![\varphi^x_y]\!]^M_{\vec{x}y} : \theta_{\vec{x}M} \times \theta_M \longrightarrow \Omega$$
$$[\![\{x : \varphi\}]\!]^M_{\vec{x}} : \theta_{\vec{x}M} \longrightarrow \Omega^{\theta_M}$$

$\triangle$

Definition 2.3 An *interpretation system* is a pair $\mathcal{S} = \langle \Sigma, \mathcal{M} \rangle$ where Σ is a signature and $\mathcal{M}$ is a class of Σ-structures. $\triangle$

Using our abbreviations, it can be proved that quantifiers and connectives are interpreted in any topos in the usual way (see for instance [10, 11]). As mentioned in Section 1, schema variables are just used for express rules in Hilbert calculi and we are not interested in interpret them.

In order to define semantic entailment we need to introduce the following notation: Given an object A in a topos $\mathcal{E}$, then $true_A : A \to \Omega$ is the characteristic map of the monomorphism $id_A : A \to A$. Recall that $Sub(A)$ is the collection of equivalence classes $[m]$ of monomorphisms $m : dom(m) \hookrightarrow A$, where $m \sim n$ iff there exists a (necessarily unique) isomorphism $f : dom(m) \to dom(n)$ such that $m = n \circ f$.



Given $[m_i] \in Sub(A)$ $(i = 1, 2)$ we say that $[m_1] \leq [m_2]$ iff there exists a morphism $f : dom(m_1) \to dom(m_2)$ such that $m_1 = m_2 \circ f$. Then $\langle Sub(A), \leq \rangle$ is a Heyting algebra. If X is a finite subset of $Sub(A)$ then $\bigwedge X$ will denote the infimum of X w.r.t. the Heyting algebra-structure of $Sub(A)$. Usually, monomorphisms are identified with their equivalence classes (see, for instance, [10]).

Definition 2.4 Let $\mathcal{S}$ be an interpretation system. Given a finite subset $\Psi \cup \{\varphi\}$ of $L(\Sigma, \vec{x})$ we say:

- Ψ *globally entails* φ within $\mathcal{S}$ and $\vec{x}$, written $\Psi \vDash^{\mathcal{S}}_{\mathrm{p}\vec{x}} \varphi$, iff, for every $M \in \mathcal{M}$: $\bigwedge_{\psi \in \Psi} [\![\psi]\!]^M_{\vec{x}} = true_{\theta_{\vec{x}M}}$ implies $[\![\varphi]\!]^M_{\vec{x}} = true_{\theta_{\vec{x}M}}$;
- Ψ *locally entails* φ within $\mathcal{S}$ and $\vec{x}$, written $\Psi \vDash^{\mathcal{S}}_{\mathrm{d}\vec{x}} \varphi$, iff, for every $M \in \mathcal{M}$, $\bigwedge_{\psi \in \Psi} [\![\psi]\!]^M_{\vec{x}} \leq [\![\varphi]\!]^M_{\vec{x}}$. $\triangle$

If $\Psi \cup \{\varphi\} \subseteq L(\Sigma, \vec{x})$ is finite and $\vec{y}$ is the canonical context of $\Psi \cup \{\varphi\}$ then we write $\Psi \vDash^{\mathcal{S}}_{\mathrm{o}} \varphi$ instead of $\Psi \vDash^{\mathcal{S}}_{\mathrm{o}\vec{y}} \varphi$, for $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$. It is easy to prove that $\Psi \vDash^{\mathcal{S}}_{\mathrm{o}} \varphi$ implies $\Psi \vDash^{\mathcal{S}}_{\mathrm{o}\vec{x}} \varphi$ (and the converse is not necessarily true, because the possibly empty domains used in the interpretation of types of Σ).

The usual notion of semantic entailment considered in categorial semantics (and in set-theoretic semantics for first-order logic) is the local one. On the other hand, we will define two different notions of syntactical inference, one for each concept of semantic entailment. In several contexts (for example, modal logic and predicate logic) it is useful to maintain the distinction between the two notions of (semantic and syntactical) inferences (cf. [13, 15, 4]). Consider, for instance, the necessitation rule for normal modal logic:

$$\frac{\alpha}{\Box\alpha}\ .$$

The meaning of that rule is *global*: If α is true (is a theorem) then $\Box\alpha$ is true (is a theorem). On the other hand, the stronger (*local*) version of the rule is not valid: $\alpha \Rightarrow \Box\alpha$ is not a theorem. The same happens with *Generalization* rule in first-order logic:

$$\frac{\alpha}{\forall x\alpha}\ .$$

If α is true (is a theorem) then $\forall x\alpha$ is true (is a theorem). Clearly, the stronger (*local*) version of the rule is not valid: $\alpha \Rightarrow \forall x\alpha$ is not a theorem. This shows that there are two kinds of inference rules, corresponding to each notion of semantic entailments, as we will see in next section.

3 Hilbert Calculi

In this section we recall (a simplified version of) the notion of Hilbert calculus introduced in [4]. In order to represent arbitrary terms in rules of Hilbert calculi we will use schema variables. Moreover, some rules will have provisos which control their range of application. Thus we need to introduce the following concepts.

By a Σ-*substitution* ρ we mean a $\Theta(S)$-indexed family of maps from Ξ_θ to $T(\Sigma)_\theta$. As usual we write $\xi\rho$ instead of $\rho_\theta(\xi)$. Any Σ-substitution ρ induces a map $\widehat{\rho} : ST(\Sigma) \to T(\Sigma)$ defined inductively as usual, with: $\widehat{\rho}(^x_{\xi'}\xi) = (\xi\rho)^x_{\xi'\rho}$, where the right-side expression is the Σ-term obtained from $\xi\rho$ by substituting every free occurrence of x by $\xi'\rho$. Note that $\widehat{\rho}(\delta) \in T(\Sigma)_\theta$ if $\delta \in ST(\Sigma)_\theta$. We denote $\widehat{\rho}(\delta)$ by $\delta\rho$. Let $Sbs(\Sigma)$ be the set of all Σ-substitutions.

By a Σ-*proviso* we mean a map $\pi : Sbs(\Sigma) \to 2$. Intuitively, $\pi(\rho) = 1$ iff the Σ-substitution ρ is allowed. (In [4] it is introduced a different notion of proviso which is suitable to perform fibring of deduction systems.) Provisos are very common in rules of logics. For instance, it is well known that a substitution instance $\xi\rho\Rightarrow\forall x\, \xi\rho$ of the schema formula $\xi\Rightarrow\forall x\, \xi$ is valid in first-order predicate logic *provided that* x is not free in $\xi\rho$; in this case we have $\pi(\rho) = 1$ iff x is not free in $\xi\rho$. We denote by $Prov(\Sigma)$ the set of all Σ-provisos. The *unit proviso* $\mathbf{u}$ maps every Σ-substitution to 1. Binary product of provisos $\pi \sqcap \pi'$ is defined as expected: $(\pi \sqcap \pi')(\rho) = \pi(\rho) \sqcap \pi'(\rho)$.

Definition 3.1 A Σ-*rule* is a triple $\langle \Gamma, \delta, \pi\rangle$ where $\Gamma \cup \{\delta\} \subseteq SL(\Sigma)$ and π is a Σ-proviso. $\triangle$

When $\Gamma = \emptyset$ the conclusion δ of the rule is also known as an *axiom*. When Γ is finite the rule is said to be *finitary*.

Definition 3.2 A *deduction system* is a triple $\mathcal{D} = \langle \Sigma, \mathcal{R}_{\mathrm{d}}, \mathcal{R}_{\mathrm{p}}\rangle$ where Σ is a signature and both $\mathcal{R}_{\mathrm{p}}$ and $\mathcal{R}_{\mathrm{d}}$ are sets of finitary Σ-rules and $\mathcal{R}_{\mathrm{d}} \subseteq \mathcal{R}_{\mathrm{p}}$. $\triangle$

The elements of $\mathcal{R}_\mathrm{p}$ are called *proof rules* and those of $\mathcal{R}_\mathrm{d}$ are known as *derivation rules*.

Definition 3.3 A $\vec{x}$-*proof* within a deduction system $\mathcal{D}$ of $\varphi \in L(\Sigma, \vec{x})$ from $\Psi \subseteq L(\Sigma, \vec{x})$ is a finite sequence $\varphi_1 \dots \varphi_n$ of formulae in $L(\Sigma, \vec{x})$ such that φ_n is φ and for each $i = 1, \dots, n$:

- either $\varphi_i \in \Psi$;
- or there is a rule $\langle \{\gamma_1, \dots, \gamma_k\}, \delta, \pi \rangle \in \mathcal{R}_\mathrm{p}$ and a Σ-substitution ρ such that:
  1. $\pi(\rho) = 1$;
  2. $\varphi_i = \delta\rho$;
  3. for each $j = 1, \dots, k$, there is a $i_j \in \{1, \dots, i-1\}$ such that $\varphi_{i_j} = \gamma_j \rho$.

When there is such a $\vec{x}$-proof in $\mathcal{D}$ of φ from Ψ, we write $\Psi \vdash^{\mathcal{D}}_{\mathrm{p}\vec{x}} \varphi$. And when there is a context $\vec{x}$ such that $\Psi \vdash^{\mathcal{D}}_{\mathrm{p}\vec{x}} \varphi$ we write $\Psi \vdash^{\mathcal{D}}_{\mathrm{p}} \varphi$. $\triangle$

Definition 3.4 A $\vec{x}$-*derivation* within a deduction system $\mathcal{D}$ of $\varphi \in L(\Sigma, \vec{x})$ from $\Psi \subseteq L(\Sigma, \vec{x})$ is a finite sequence $\varphi_1 \dots \varphi_n$ of formulae in $L(\Sigma, \vec{x})$ such that φ_n is φ and for each $i = 1, \dots, n$:

- either $\varphi_i \in \Psi$;
- or $\emptyset \vdash^{\mathcal{D}}_{\mathrm{p}\vec{x}} \varphi_i$;
- or there is a rule $\langle \{\gamma_1, \dots, \gamma_k\}, \delta, \pi \rangle \in \mathcal{R}_\mathrm{d}$ and a Σ-substitution ρ such that:
  1. $\pi(\rho) = 1$;
  2. $\varphi_i = \delta\rho$;
  3. for each $j = 1, \dots, k$, there is a $i_j \in \{1, \dots, i-1\}$ such that $\varphi_{i_j} = \gamma_j \rho$.

When there is such a $\vec{x}$-derivation in $\mathcal{D}$ of φ from Ψ, we write $\Psi \vdash^{\mathcal{D}}_{\mathrm{d}\vec{x}} \varphi$. And when there is a context $\vec{x}$ such that $\Psi \vdash^{\mathcal{D}}_{\mathrm{d}\vec{x}} \varphi$ we write $\Psi \vdash^{\mathcal{D}}_{\mathrm{d}} \varphi$. $\triangle$

As usual, with respect to both proofs and derivations, we may drop the reference to the assumptions when $\Gamma = \emptyset$. Note that $\vdash^{\mathcal{D}}_{\mathrm{p}\vec{x}} \varphi$ iff $\vdash^{\mathcal{D}}_{\mathrm{d}\vec{x}} \varphi$.

Definition 3.5 A *logic system* is a tuple $\mathcal{L} = \langle \Sigma, \mathcal{M}, \mathcal{R}_\mathrm{d}, \mathcal{R}_\mathrm{p} \rangle$ such that $\mathcal{S} = \langle \Sigma, \mathcal{M} \rangle$ is an interpretation system and $\mathcal{D} = \langle \Sigma, \mathcal{R}_\mathrm{d}, \mathcal{R}_\mathrm{p} \rangle$ is a deduction system. $\triangle$

Definition 3.6 A logic system $\mathcal{L}$ is said to be *sound* iff, for $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$, any context $\vec{x}$ and every finite $\Psi \cup \{\varphi\} \subseteq L(\Sigma, \vec{x})$:

- $\Psi \vdash^{\mathcal{D}}_{\mathrm{o}\vec{x}} \varphi$ implies $\Psi \vDash^{\mathcal{S}}_{\mathrm{o}\vec{x}} \varphi$.

A logic system $\mathcal{L}$ is said to be *complete* iff, for $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$ and finite $\Psi \cup \{\varphi\} \subseteq L(\Sigma)$:

- $\Psi \models^{\mathcal{S}}_{\mathrm{o}} \varphi$ implies $\Psi \vdash^{\mathcal{D}}_{\mathrm{o}} \varphi$. △

We say that a Σ-structure M *satisfies* $\mathcal{D}$ if $\Psi \vdash^{\mathcal{D}}_{\mathrm{o}\vec{x}} \varphi$ implies $\Psi \models^{\langle \Sigma, \{M\} \rangle}_{\mathrm{o}\vec{x}} \varphi$ for every $\Psi, \varphi, \vec{x}$ and $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$.

Remark 3.7 We recall here the observation made in [4] about the strangeness of Definition 3.6. It is clear that the intended definition of soundness of a logic system $\mathcal{L}$ is, for $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$,

$$\Psi \vdash^{\mathcal{D}}_{\mathrm{o}} \varphi \text{ implies } \Psi \models^{\mathcal{S}}_{\mathrm{o}} \varphi.$$

Unfortunately, this definition is not correct in the realm of logic systems, because the (possibly) empty domains interpreting the types of Σ. In general, from $\Psi, \psi \models^{\mathcal{S}}_{\mathrm{o}} \varphi$ and $\Psi \models^{\mathcal{S}}_{\mathrm{o}} \psi$ we cannot infer $\Psi \models^{\mathcal{S}}_{\mathrm{o}} \varphi$, for $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$ (see, for instance, [2]). On the other hand, it is obvious that any deduction system $\mathcal{D}$ satisfies the following property: From $\Psi, \psi \vdash^{\mathcal{D}}_{\mathrm{o}} \varphi$ and $\Psi \vdash^{\mathcal{D}}_{\mathrm{o}} \psi$ we infer $\Psi \vdash^{\mathcal{D}}_{\mathrm{o}} \varphi$, for $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$. Therefore the standard definition of soundness must be changed, and we must live with the fact that it is possible to have

$$\Psi \vdash^{\mathcal{D}}_{\mathrm{o}} \varphi \text{ but } \Psi \not\models^{\mathcal{S}}_{\mathrm{o}} \varphi$$

even in a sound logic system $\mathcal{L}$. △

4 Local Set Theories

As mentioned in the Introduction, the logic of topoi is, in general, expressed through a sequent calculus or in natural deduction-style (see, for instance, [2, 6]). One reason for this is probably related to the problems that the definition of soundness involves for Hilbert calculi (cf. Remark 3.7 and [4]). In this section we recall the sequent calculus called *local set theory* introduced by Bell in [2], which is sound (in the usual sense) and complete for (local) topos semantics. And in Section 5 we will give a Hilbert calculus which is equivalent to local set theory, as long as $\vec{x}$-derivations are considered.

Definition 4.1 Let Σ be a signature as in Definition 1.2 but allowing terms of the form $\langle t \rangle$, which are identified with t. A Σ-*sequent* (or simply a *sequent*) is a pair $\langle \Psi, \varphi \rangle$, where $\Psi \cup \{\varphi\}$ is a finite set of formulae over Σ. △

A sequent $\langle \Psi, \varphi \rangle$ will be denoted by $(\Psi : \varphi)$ or simply $\Psi : \varphi$. If $\Psi = \emptyset$ then we will write $(: \varphi)$ or $: \varphi$. As usual,

$$\varphi, \Psi : \psi \quad \Psi, \varphi : \psi \quad \text{and} \quad \Phi, \Psi : \psi$$

will stand for $(\{\varphi\} \cup \Psi : \psi)$, $(\Psi \cup \{\varphi\} : \psi)$ and $(\Phi \cup \Psi : \psi)$, respectively. If Ψ is a finite set of formulae then Ψ^x_τ will stands for the finite set $\{\varphi^x_\tau \mid \varphi \in \Psi\}$.

Definition 4.2 *Local Set Theories (cf. [2])* A local set theory is a sequent calculus defined as follows

Tautology $\varphi : \varphi$

Unity $: x_1 = \langle\rangle$

Equality $x = y,\ \varphi^z_x : \varphi^z_y$ $\quad$ (x and y free for z in φ)

Product1 $: (\langle x_1, \ldots, x_n\rangle)_i = x_i$ $\quad$ $(1 \le i \le n)$

Product2 $x = \langle (x)_1, \ldots, (x)_n\rangle$ $\quad$ $(n \ge 1)$

Comprehension $: x \in \{x \ :\ \varphi\}$

Thinning $\dfrac{\Psi : \varphi}{\psi, \Psi : \varphi}$

Cut $\dfrac{\Psi : \varphi \qquad \varphi, \Psi : \psi}{\Psi : \psi}$ (any free variable of φ free in Ψ or ψ)

Substitution $\dfrac{\Psi : \varphi}{\Psi^x_\tau : \varphi^x_\tau}$ (τ free for x in Ψ and φ)

Extensionality $\dfrac{\Psi : x \in \sigma \Leftrightarrow x \in \tau}{\Psi : \sigma = \tau}$ (x not free in Ψ, σ, τ)

Equivalence $\dfrac{\varphi, \Psi : \psi \qquad \psi, \Psi : \varphi}{\Psi : \varphi \Leftrightarrow \psi}$ $\triangle$

In [2], the term $\langle t\rangle$ is defined to be t, for every term t. This allows us to prove the sequent $(: \forall x(x =_\theta x))$ for all θ. Recall from [2] the following: Let S be a set of sequents. Then the local set theory $\overline{S}$ generated by S is defined as follows: $(\Psi : \varphi) \in \overline{S}$ iff $\Psi \vdash_S \varphi$ iff there exists a proof of $(\Psi : \varphi)$ possibly using sequents of S as assumptions in the rules. If $S = \emptyset$ then we write $\Psi \vdash \varphi$ instead of $\Psi \vdash_S \varphi$. The following result was proved in [2].

Proposition 4.3 The following is true in any local set theory:

(1) $\dfrac{\varphi, \Psi : \psi}{\Psi : \varphi \Rightarrow \psi}$ and $\dfrac{\Psi : \varphi \Rightarrow \psi}{\varphi, \Psi : \psi}$

(2) $\dfrac{\Psi : \psi}{\Psi : \forall x\, \psi}$ provided either (i) x is not free in Ψ or (ii) x is not free in ψ.

(3) $\forall x\, \psi \vdash \psi$ provided x is free in ψ.

(4) $\dfrac{\varphi^x_\tau, \Psi : \psi}{\forall x\, \varphi, \Psi : \psi}$ provided that τ is free for x in φ, x is free in φ and any free variable of τ is free in $\forall x\, \varphi$, Ψ or ψ.

In Section 5 we will define a deduction system called *HOL* and prove that inferences in local set theories correspond to deductions in *HOL*, obtaining the theorem of adequacy of *HOL* w.r.t. topos semantics. Because the different definition of soundness we state in 3.6, we need to extend the notion of inference in local set theories (cf. Definition 4.5).

Remark 4.4 For convenience, we adopt the following notation. Let Ψ be a finite subset of $L(\Sigma)$; then $(\bigwedge \Psi)$ denotes a formula obtained from Ψ by taking the conjunction of all the formulae in Ψ in an arbitrary order and association (if $\Psi = \emptyset$ then we take $(\bigwedge \Psi)$ to be $\mathbf{t}$). It is easy to prove that, if $\Psi \cup \{\psi\}$ is a finite subset of $L(\Sigma)$ and $(\bigwedge \Psi)_1$, $(\bigwedge \Psi)_2$ are two conjunctions defined as above then $\{(\bigwedge \Psi)_1\} \vdash (\bigwedge \Psi)_2$, therefore $\{(\bigwedge \Psi)_1 \Rightarrow \psi\} \vdash (\bigwedge \Psi)_2 \Rightarrow \psi$. △

Definition 4.5 Let S be a set of sequents formed by formulae in $L(\Sigma)$ and let $\Psi \cup \{\varphi\}$ be a finite subset of $L(\Sigma, \vec{x})$ for some context $\vec{x} = x_1 \dots x_n$. Let $(\vec{x} = \vec{x})$ be a formula $(\bigwedge\{(x_1 = x_1), \dots, (x_n = x_n)\})$ obtained as in Remark 4.4. Then $\Psi \vdash_{S\vec{x}} \varphi$ will stands for $\Psi, (\vec{x} = \vec{x}) \vdash_S \varphi$. △

As usual we omit the reference to the theory S when $S = \emptyset$.

Let $\mathcal{S} = \langle \Sigma, \mathcal{M} \rangle$ be the interpretation system such that $\mathcal{M}$ is the class of all the Σ-structures $M = \langle \mathcal{E}, \cdot_M \rangle$. Using the soundness and completeness theorem of local set theory stated in [2] we obtain easily the following theorem of adequacy:

Theorem 4.6 Let $\vec{x}$ be a context, $\Psi \cup \{\varphi\}$ a finite subset of $L(\Sigma, \vec{x})$ and S a set of sequents in $L(\Sigma)$. Then $\Psi \vdash_{S\vec{x}} \varphi$ iff $\Psi \models^{\mathcal{S}_S}_{\mathrm{d}\vec{x}} \varphi$, where $\mathcal{S}_S = \langle \Sigma, \mathcal{M}_S \rangle$ and $\mathcal{M}_S$ is the subclass of $\mathcal{M}$ formed by all the models of S. In particular: $\Psi \vdash_{\vec{x}} \varphi$ iff $\Psi \models^{\mathcal{S}}_{\mathrm{d}\vec{x}} \varphi$; $\Psi \vdash_S \varphi$ iff $\Psi \models^{\mathcal{S}_S}_{\mathrm{d}} \varphi$ and $\Psi \vdash \varphi$ iff $\Psi \models^{\mathcal{S}}_{\mathrm{d}} \varphi$.

5 Hilbert-style axiomatization of Higher-order logic

In this section we will adapt the sequent calculus-style presentation of local set theory to a Hilbert-style one, defining a deduction system called *HOL*. The main results to be stated are the following:

- $\Psi \vdash^{HOL_S}_{\mathrm{d}\vec{x}} \varphi$ implies $\Psi \vdash_{S\vec{x}} \varphi$;
- $\Psi \vdash_S \varphi$ implies $\Psi \vdash^{HOL_S}_{\mathrm{d}} \varphi$,

where HOL_S is obtained from *HOL* by adding the sequents of S as axioms (under an appropriate form). In order to do this, we begin with some notation. For any schema formula δ, any type θ, any variable x of type θ and any schema term δ_1 of type θ we define the following provisos:

- $(\delta_1 \rhd x : \delta)(\rho) = 1$ iff $\delta_1\rho$ is free for x in $\delta\rho$;
- $(x \prec \delta)(\rho) = 1$ iff x occurs free in $\delta\rho$;
- $(x \not\prec \delta)(\rho) = 1$ iff x does not occur free in $\delta\rho$.

Definition 5.1 *Hilbert calculus for intuitionistic hol.*
We define the deduction system $HOL = \langle \Sigma, \mathcal{R}_{\mathrm{p}}, \mathcal{R}_{\mathrm{d}} \rangle$ as follows (here $i \in \mathbb{N}$, $k \geq 2$ and θ, θ_1, ..., θ_k are types):

- $\mathcal{R}_{\mathrm{d}}$ is the set composed by:

taut1: $\langle \emptyset, \xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1), \mathbf{u} \rangle$;
taut2: $\langle \emptyset, (\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3)), \mathbf{u} \rangle$;
taut3: $\langle \emptyset, (\xi_1 \Rightarrow \xi_2) \Rightarrow ((\xi_1 \Rightarrow \xi_3) \Rightarrow (\xi_1 \Rightarrow (\xi_2 \wedge \xi_3))), \mathbf{u} \rangle$;
taut4: $\langle \emptyset, \xi_1 \Rightarrow (\xi_2 \Rightarrow (\xi_1 \wedge \xi_2)), \mathbf{u} \rangle$;
uni: $\langle \emptyset, \forall x_1 (x_1 = \langle \rangle), \mathbf{u} \rangle$;
equa$_{i,\theta}$**:** $\langle \emptyset, (\xi_1 = \xi_2) \Rightarrow ({}^{x_i}_{\xi_1}\xi_3 \;\Rightarrow\; {}^{x_i}_{\xi_2}\xi_3), (\xi_1 \rhd x_i : \xi_3) \sqcap (\xi_2 \rhd x_i : \xi_3) \rangle$;
ref$_\theta$**:** $\langle \emptyset, \forall x_1 (x_1 = x_1), \mathbf{u} \rangle$;
proj$_{k,\theta_1,\ldots,\theta_k,i}$**:** $\langle \emptyset, \forall x_1 \cdots \forall x_k ((\langle x_1, \ldots, x_k \rangle)_i = x_i), \mathbf{u} \rangle$ for $1 \leq i \leq k$;
prod$_{k,\theta_1,\ldots,\theta_k}$**:** $\langle \emptyset, \forall x_1 (x_1 = \langle (x_1)_1, \ldots, (x_1)_k \rangle), \mathbf{u} \rangle$;
comph$_\theta$**:** $\langle \emptyset, \forall x_1 (x_1 \in \{x_1 : \xi_1\} \Leftrightarrow \xi_1), \mathbf{u} \rangle$;
subs$_{i,\theta}$**:** $\langle \emptyset, (\forall x_i \xi_2) \;\Rightarrow\; {}^{x_i}_{\xi_1}\xi_2, (\xi_1 \rhd x_i : \xi_2) \sqcap (x_i \prec \xi_2) \rangle$;
ext$_\theta$**:** $\langle \emptyset, (\forall x_1 (x_1 \in \xi_1 \Leftrightarrow x_1 \in \xi_2) \Rightarrow (\xi_1 = \xi_2), (x_1 \not\prec \xi_1) \sqcap (x_1 \not\prec \xi_2) \rangle$;
equiv: $\langle \emptyset, (\xi_1 \Rightarrow \xi_2) \Rightarrow ((\xi_2 \Rightarrow \xi_1) \Rightarrow (\xi_1 \Leftrightarrow \xi_2)), \mathbf{u} \rangle$;
MP: $\langle \{\xi_1, \xi_1 \Rightarrow \xi_2\}, \xi_2, \mathbf{u} \rangle$;

- $\mathcal{R}_{\mathrm{p}}$ is obtained by adding to $\mathcal{R}_{\mathrm{d}}$ the following rules:

  GEN$_{i,\theta}$**:** $\langle \{\xi_1 \Rightarrow \xi_2\}, \xi_1 \Rightarrow (\forall x_i \xi_2), (x_i \not\prec \xi_1) \rangle$. △

Of course, rules with subscripts are in fact "schema-rules" (in the usual sense). Thus, each $i \in \mathbb{N}$ and each type θ define a particular instance of $\mathbf{equa}_{i,\theta}$, and so on. Since, in contrast with the approach in [2], we do not define terms of the form $\langle t \rangle$ (1-tupling), we need to include the axioms $\mathbf{ref}_\theta$. From now on, we will omit the subscripts in the name of the rules.

Remark 5.2 If $\Psi \cup \{\psi\}$ is a finite subset of $L(\Sigma, \vec{x})$ and $(\bigwedge \Psi)_1$, $(\bigwedge \Psi)_2$ are two conjunctions defined as in Remark 4.4 then $\{(\bigwedge \Psi)_1\} \vdash^{HOL}_{\mathrm{d}\vec{x}} (\bigwedge \Psi)_2$. Therefore $\{(\bigwedge \Psi)_1 \Rightarrow \psi\} \vdash^{HOL}_{\mathrm{d}\vec{x}} (\bigwedge \Psi)_2 \Rightarrow \psi$. △

Given a set S of sequents we define the set of rules

$$\mathcal{R}_S = \{\langle \emptyset, (\bigwedge \Psi) \Rightarrow \varphi, \mathbf{u} \rangle \;\mid\; (\Psi : \varphi) \in S\},$$

where $(\bigwedge \Psi)$ is defined as in Remark 4.4. The system HOL_S is given by

$$\langle \Sigma, \mathcal{R}_{\mathrm{p}} \cup \mathcal{R}_S, \mathcal{R}_{\mathrm{d}} \cup \mathcal{R}_S \rangle.$$

Proposition 5.3 HOL_S satisfies the *Metatheorem of Deduction* with respect to $\vec{x}$-derivations: For every context $\vec{x}$ and finite $\Psi \cup \{\varphi, \psi\} \subseteq L(\Sigma, \vec{x})$,

$$\varphi, \Psi \vdash^{HOL_S}_{\mathrm{d}\vec{x}} \psi \;\text{ iff }\; \Psi \vdash^{HOL_S}_{\mathrm{d}\vec{x}} \varphi \Rightarrow \psi.$$

Proof: By straightforward induction on the length of a $\vec{x}$-derivation of ψ from $\Psi \cup \{\varphi\}$ we get $\Psi \vdash^{HOL_S}_{\mathrm{d}\vec{x}} \varphi \Rightarrow \psi$. The converse is immediate by **MP**. QED

The following useful properties of *HOL* can be easily proved.

Lemma 5.4 Let $\varphi, \psi, \psi' \in L(\Sigma, \vec{x})$. The following holds in *HOL*.

1. $\vdash^{HOL}_{\mathrm{d}[]} \mathbf{t}$.
2. $\{\varphi\} \vdash^{HOL}_{\mathrm{d}\vec{x}} (\mathbf{t} \Rightarrow \varphi)$.
3. $\{\varphi\} \vdash^{HOL}_{\mathrm{p}\vec{x}} (\forall x \varphi)$.
4. $\{(\varphi \Rightarrow \psi), (\psi \Rightarrow \psi')\} \vdash^{HOL}_{\mathrm{d}\vec{x}} (\varphi \Rightarrow \psi')$.
5. $\{(\varphi \Rightarrow \psi), (\varphi \Rightarrow \psi')\} \vdash^{HOL}_{\mathrm{d}\vec{x}} (\varphi \Rightarrow (\psi \wedge \psi'))$.
6. $\{\varphi \wedge \psi\} \vdash^{HOL}_{\mathrm{d}\vec{x}} \varphi$.
7. $\{\varphi \wedge \psi\} \vdash^{HOL}_{\mathrm{d}\vec{x}} \psi$.
8. $\{\varphi \Rightarrow (\psi \Rightarrow \psi')\} \vdash^{HOL}_{\mathrm{d}\vec{x}} ((\varphi \wedge \psi) \Rightarrow \psi')$.

Lemma 5.5 Let $\vec{x}$ be a context and let $\varphi \in L(\Sigma, \vec{x})$. Then $\vdash^{HOL_S}_{\mathrm{d}\vec{x}} \varphi$ implies $\vdash_{S\vec{x}} \varphi$.

Proof: By induction on the length n of a $\vec{x}$-derivation of φ from $\emptyset$ within HOL_S. If $n = 0$ then we have the following cases:
(1) φ is an instance of (**taut i**) ($i = 1, ..., 4$). The result follows from the completeness of pure local set theory and *Thinning*.
(2) φ is an instance $(\forall x)(x \in \sigma \Leftrightarrow x \in \tau) \Rightarrow (\sigma = \tau)$ of **ext**. Then x does not occur free in $\langle \sigma, \tau \rangle$. Consider the following proof from S in pure local set theory:

1. $x \in \sigma \Leftrightarrow x \in \tau : x \in \sigma \Leftrightarrow x \in \tau$ *Tautology*
2. $(\forall x (x \in \sigma \Leftrightarrow x \in \tau)) : x \in \sigma \Leftrightarrow x \in \tau$ Proposition 4.3(3), 1
3. $(\forall x (x \in \sigma \Leftrightarrow x \in \tau)) : \sigma = \tau$ *Extensionality*, 2
4. $(\vec{x} = \vec{x}), (\forall x (x \in \sigma \Leftrightarrow x \in \tau)) : \sigma = \tau$ *Thinning*, 3
5. $(\vec{x} = \vec{x}) : (\forall x (x \in \sigma \Leftrightarrow x \in \tau)) \Rightarrow \sigma = \tau$ Proposition 4.3(1), 4.

(3) φ is an instance $(\forall x \psi) \Rightarrow \psi^x_\tau$ of **subs**. Then τ is free for x in ψ and x occurs free in ψ, and we can construct the following proof from S in pure local set theory:

1. $\psi : \psi$ *Tautology*
2. $(\forall x \psi) : \psi$ Proposition 4.3(3), 1
3. $(\forall x \psi) : \psi^x_\tau$ *Substitution*, 2
4. $(\vec{x} = \vec{x}), (\forall x \psi) : \psi^x_\tau$ *Thinning*, 3
5. $(\vec{x} = \vec{x}) : (\forall x \psi) \Rightarrow \psi^x_\tau$ Proposition 4.3(1), 4.

(4) The other cases for $n = 0$ are easy.
Suppose that the result is true for every $\vec{x}$-derivation within HOL_S in $k \leq n$ steps, and let φ obtained from $\emptyset$ through a $\vec{x}$-derivation in $n+1$ steps. We have the following cases:
(a) φ is an instance of an axiom. The proof is as above.
(b) φ is obtained from ψ and $\psi \Rightarrow \varphi$ by **MP**. Thus we can construct the following proof from S in pure local set theory:

1. $(\vec{x} = \vec{x}) : \psi$ (IH)
2. $(\vec{x} = \vec{x}) : \psi \Rightarrow \varphi$ (IH)
3. $\psi, (\vec{x} = \vec{x}) : \varphi$ Proposition 4.3(1), 2
4. $(\vec{x} = \vec{x}) : \varphi$ *Cut* 3,1.

Note that the application of *Cut* is legitimated by the presence of a suitable $(\vec{x} = \vec{x})$ which captures all variable occurring free in ψ.
(c) φ is $\psi_1 \Rightarrow (\forall x \psi_2)$ obtained from $\psi_1 \Rightarrow \psi_2$ by **GEN**. Thus x does not occur free in ψ_1. We have the following cases:
CASE 1: x does not occur free in ψ_2. Then we construct the following proof from S in pure local set theory:

1. $(\vec{x} = \vec{x}) : \psi_1 \Rightarrow \psi_2$ (IH)
2. $\psi_1, (\vec{x} = \vec{x}) : \psi_2$ Proposition 4.3(1), 1
3. $\psi_1, (\vec{x} = \vec{x}) : (\forall x \psi_2)$ Proposition 4.3(2), 2
4. $(\vec{x} = \vec{x}) : \psi_1 \Rightarrow (\forall x \psi_2)$ Proposition 4.3(1), 3.

CASE 2: x occurs free in ψ_2. Then $\vec{x}$ is, let's say, $\vec{y}x$, and we can construct the following proof from S in pure local set theory:

1. $x = x, (\vec{y} = \vec{y}) : \psi_1 \Rightarrow \psi_2$ (IH)
2. $(\forall x(x = x)), (\vec{y} = \vec{y}) : \psi_1 \Rightarrow \psi_2$ Proposition 4.3(4), 1
3. $: (\forall x(x = x))$
4. $(\vec{y} = \vec{y}) : \psi_1 \Rightarrow \psi_2$ *Cut* 2,3
5. $\psi_1, (\vec{y} = \vec{y}) : \psi_2$ Proposition 4.3(1), 4
6. $\psi_1, (\vec{y} = \vec{y}) : (\forall x \psi_2)$ Proposition 4.3(2), 5
7. $(\vec{y} = \vec{y}) : \psi_1 \Rightarrow (\forall x \psi_2)$ Proposition 4.3(1), 6
8. $(\vec{x} = \vec{x}) : \psi_1 \Rightarrow (\forall x \psi_2)$ *Thinning*, 7.

This concludes the proof. QED

Proposition 5.6 Let $\vec{x}$ be a context and let $\Psi \cup \{\varphi\}$ be a finite subset of $L(\Sigma, \vec{x})$. Then $\Psi \vdash^{HOL_S}_{d\vec{x}} \varphi$ implies $\Psi \vdash_{S\vec{x}} \varphi$.

Proof: Is an immediate consequence of Propositions 5.3 and 4.3(1), and Lemma 5.5. QED

Lemma 5.7 Let $\Psi \cup \{\varphi\}$ be a finite subset of $L(\Sigma)$. Then $\Psi \vdash_S \varphi$ implies $\vdash_{\mathrm{d}}^{HOL_S} (\bigwedge \Psi) \Rightarrow \varphi$.

Proof: Induction on the length n of the proof of the sequent $(\Psi : \varphi)$ from S. If $n = 0$ then we have two possibilities:
(1) $(\Psi : \varphi)$ is an axiom of the pure local set theory. The result follows easily using the axioms of *HOL* and Lemma 5.4. For example, any instance of axiom *Equality* can be derived in *HOL* as follows:

1. $((x = y) \Rightarrow (\psi^z_x \Rightarrow \psi^z_y))$ (**equa**)
2. $(((x = y) \wedge \psi^z_x) \Rightarrow \psi^z_y)$ (Lemma 5.4(8), 1),

provided that both x, y are free for z in ψ.
(2) $(\Psi : \varphi)$ is an instance of a sequent in S. The conclusion is immediate, by the definition of $\mathcal{R}_S$.

Assume that the result is true for any proof of length $\leq n$, and consider a sequent $(\Psi : \varphi)$ which is proved from S in $n + 1$ steps. We have the following new cases:
(a) $(\Psi : \varphi)$ is of the form $(\psi, \Phi : \varphi)$, and it is obtained from $(\Phi : \varphi)$ by *Thinning*. Then there exists a context $\vec{x}$ and a $\vec{x}$-derivation within HOL_S of $((\bigwedge \Phi) \Rightarrow \varphi)$ from $\emptyset$, by induction hypothesis. From one of such $\vec{x}$-derivations we can construct the following $\vec{x}$-derivation in HOL_S:

1. $((\bigwedge \Phi) \Rightarrow \varphi)$ (IH)
2. $((\psi \wedge (\bigwedge \Phi)) \Rightarrow (\bigwedge \Phi))$ (Lemma 5.4(7), Proposition 5.3)
3. $((\psi \wedge (\bigwedge \Phi)) \Rightarrow \varphi)$ (Lemma 5.4(4), 2,1).

(b) $(\Psi : \varphi)$ is obtained from $(\Psi : \psi)$ and $(\psi, \Psi : \varphi)$ by *Cut*. There exists a context $\vec{x}$ and $\vec{x}$-derivations in HOL_S of $((\bigwedge \Psi) \Rightarrow \psi)$ and $((\psi \wedge (\bigwedge \Psi)) \Rightarrow \varphi)$, by induction hypothesis and Remark 5.2. From one of such $\vec{x}$-derivations we can construct the following $\vec{x}$-derivation in HOL_S:

1. $((\bigwedge \Psi) \Rightarrow \psi)$ (IH)
2. $((\psi \wedge (\bigwedge \Psi)) \Rightarrow \varphi)$ (IH)
3. $((\bigwedge \Psi) \Rightarrow (\bigwedge \Psi))$
4. $((\bigwedge \Psi) \Rightarrow (\psi \wedge (\bigwedge \Psi)))$ (Lemma 5.4(5) 1,3)
5. $((\bigwedge \Psi) \Rightarrow \varphi)$ (Lemma 5.4(4), 4,2).

(c) $(\Psi : \varphi)$ is of the form $(\Phi^x_\tau : \psi^x_\tau)$, and it is obtained from $(\Phi : \psi)$ by *Substitution*. Note that τ is free for x in $(\bigwedge \Phi) \Rightarrow \psi$. If x does not occur free in $(\bigwedge \Phi) \Rightarrow \psi$ then $(\Psi : \varphi)$ is $(\Phi : \psi)$ and there exists a $\vec{x}$-derivation in HOL_S of

$((\bigwedge\Psi)\Rightarrow\varphi)$, by induction hypothesis. If x occurs free in $(\bigwedge\Phi)\Rightarrow\psi$ then there exists a $\vec{x}$-derivation in HOL_S of $((\bigwedge\Phi)\Rightarrow\psi)$, by induction hypothesis. From one of such $\vec{x}$-derivations we can construct the following $\vec{y}$-derivation in HOL_S, for some context $\vec{y}$:

1. $((\bigwedge\Phi)\Rightarrow\psi)$ (IH)
2. $(\forall x((\bigwedge\Phi)\Rightarrow\psi))$ (Lemma 5.4(3), 1)
3. $(\forall x((\bigwedge\Phi)\Rightarrow\psi))\Rightarrow((\bigwedge\Phi)\Rightarrow\psi)^x_\tau$ (**subs**)
4. $(\bigwedge\Phi^x_\tau)\Rightarrow\psi^x_\tau$ (**MP** 2,3).

(d) $(\Psi:\varphi)$ is of the form $(\Psi:\sigma=\tau)$, and it is obtained from $(\Psi:x\in\sigma\Leftrightarrow x\in\tau)$ by *Extensionality*. Note that x does not occur free in $\langle(\bigwedge\Psi),\sigma,\tau\rangle$. There exists a $\vec{x}$-derivation in HOL_S of $(\bigwedge\Psi)\Rightarrow(x\in\sigma\Leftrightarrow x\in\tau)$, by induction hypothesis. From one of such $\vec{x}$-derivations we can construct the following $\vec{x}$-derivation in HOL_S:

1. $(\bigwedge\Psi)\Rightarrow(x\in\sigma\Leftrightarrow x\in\tau)$ (IH)
2. $(\bigwedge\Psi)\Rightarrow(\forall x(x\in\sigma\Leftrightarrow x\in\tau))$ (**GEN** 1)
3. $(\forall x(x\in\sigma\Leftrightarrow x\in\tau))\Rightarrow(\sigma=\tau)$ (**ext**)
4. $(\bigwedge\Psi)\Rightarrow(\sigma=\tau)$ (Lemma 5.4(4), 2,3).

(e) $(\Psi:\varphi)$ is $(\Psi:\psi_1\Leftrightarrow\psi_2)$, and it is obtained from $(\psi_1,\Psi:\psi_2)$ and $(\psi_2,\Psi:\psi_1)$ by *Equivalence*. Then there exists $\vec{x}$-derivations in HOL_S of $((\psi_1\wedge(\bigwedge\Psi))\Rightarrow\psi_2)$ and $((\psi_2\wedge(\bigwedge\Psi))\Rightarrow\psi_1)$, by induction hypothesis and Remark 5.2. From one of such $\vec{x}$-derivations we can construct the following $\vec{x}$-derivation in HOL_S:

1. $((\psi_1\wedge(\bigwedge\Psi))\Rightarrow\psi_2)$ (IH)
2. $((\psi_2\wedge(\bigwedge\Psi))\Rightarrow\psi_1)$ (IH)
3. $((\bigwedge\Psi)\Rightarrow(\psi_1\Rightarrow\psi_2))$ (1)
4. $((\bigwedge\Psi)\Rightarrow(\psi_2\Rightarrow\psi_1))$ (2)
5. $((\psi_1\Rightarrow\psi_2)\Rightarrow((\psi_2\Rightarrow\psi_1)\Rightarrow(\psi_1\Leftrightarrow\psi_2)))$ (**equiv**)
6. $((\bigwedge\Psi)\Rightarrow((\psi_2\Rightarrow\psi_1)\Rightarrow(\psi_1\Leftrightarrow\psi_2)))$ (Lemma 5.4(4), 3,5)
7. $(((\bigwedge\Psi)\Rightarrow((\psi_2\Rightarrow\psi_1)\Rightarrow(\psi_1\Leftrightarrow\psi_2)))\Rightarrow(((\bigwedge\Psi)\Rightarrow(\psi_2\Rightarrow\psi_1))\Rightarrow((\bigwedge\Psi)\Rightarrow(\psi_1\Leftrightarrow\psi_2))))$ (**taut2**)
8. $(((\bigwedge\Psi)\Rightarrow(\psi_2\Rightarrow\psi_1))\Rightarrow((\bigwedge\Psi)\Rightarrow(\psi_1\Leftrightarrow\psi_2)))$ (**MP** 6,7)
9. $((\bigwedge\Psi)\Rightarrow(\psi_1\Leftrightarrow\psi_2))$ (**MP** 4,8).

QED

Proposition 5.8 Let $\Psi \cup \{\varphi\}$ be a finite subset of $L(\Sigma)$. Then $\Psi \vdash_S \varphi$ implies $\Psi \vdash_{\mathrm{d}}^{HOL_S} \varphi$.

Proof: Immediate by Lemma 5.7 and Proposition 5.3. QED

From Propositions 5.6 and 5.8 we obtain the desired result.

Theorem 5.9 (Adequacy of *HOL*) Let $\mathcal{S}_S = \langle \Sigma, \mathcal{M}_S \rangle$ such that $\mathcal{M}_S$ is the class of all the Σ-structures satisfying S. Then:

1. HOL_S is d-sound and d-complete w.r.t. $\mathcal{S}_S$, that is, for every context $\vec{x}$ and finite $\Psi \cup \{\varphi\} \subseteq L(\Sigma, \vec{x})$:
   - $\Psi \vdash_{\mathrm{d}\vec{x}}^{HOL_S} \varphi$ implies $\Psi \vDash_{\mathrm{d}\vec{x}}^{\mathcal{S}_S} \varphi$.
   - $\Psi \vDash_{\mathrm{d}}^{\mathcal{S}_S} \varphi$ implies $\Psi \vdash_{\mathrm{d}}^{HOL_S} \varphi$.
2. HOL_S is p-sound and p-complete w.r.t. $\mathcal{S}_S$, that is, for every context $\vec{x}$ and finite $\Psi \cup \{\varphi\} \subseteq L(\Sigma, \vec{x})$:
   - $\Psi \vdash_{\mathrm{p}\vec{x}}^{HOL_S} \varphi$ implies $\Psi \vDash_{\mathrm{p}\vec{x}}^{\mathcal{S}_S} \varphi$.
   - $\Psi \vDash_{\mathrm{p}}^{\mathcal{S}_S} \varphi$ implies $\Psi \vdash_{\mathrm{p}}^{HOL_S} \varphi$.

Proof: (1) It is an immediate consequence of Theorem 4.6 and Propositions 5.6 and 5.8.
(2) Recall the notation introduced in Remark 4.4 and Definition 4.5. Let $HOL_S^{\Psi,\vec{x}}$ be the deduction system obtained from HOL_S by adding the axiom

$$\langle \emptyset, (\bigwedge \Psi) \wedge (\vec{x} = \vec{x}), \mathbf{u} \rangle,$$

and let $\mathcal{S}_S^{\Psi,\vec{x}} = \langle \Sigma, \mathcal{M}_S^{\Psi,\vec{x}} \rangle$ be the corresponding interpretation system. Then, by definition of proofs and derivations and by item (1):

$$\Psi \vdash_{\mathrm{p}\vec{x}}^{HOL_S} \varphi \text{ implies } \vdash_{\mathrm{d}\vec{x}}^{HOL_S^{\Psi,\vec{x}}} \varphi \text{ implies } [\![\varphi]\!]_{\vec{x}}^{M} = true_{\theta_{\vec{x}M}} \text{ for every } M \in \mathcal{M}_S^{\Psi,\vec{x}}.$$

But the last affirmation implies the following: For every $M \in \mathcal{M}_S$,

$$\text{if } [\![\psi]\!]_{\vec{x}}^{M} = true_{\theta_{\vec{x}M}} \text{ for every } \psi \in \Psi \text{ then } [\![\varphi]\!]_{\vec{x}}^{M} = true_{\theta_{\vec{x}M}}.$$

This means that $\Psi \vDash_{\mathrm{p}\vec{x}}^{\mathcal{S}_S} \varphi$ and then HOL_S is p-sound w.r.t. $\mathcal{S}_S$. Finally, suppose that $\Psi \vDash_{\mathrm{p}}^{\mathcal{S}_S} \varphi$. Then, for every $M \in \mathcal{M}_S$:

$$\text{if } [\![\psi]\!]_{\vec{x}}^{M} = true_{\theta_{\vec{x}M}} \text{ for every } \psi \in \Psi \text{ then } [\![\varphi]\!]_{\vec{x}}^{M} = true_{\theta_{\vec{x}M}},$$

where $\vec{x}$ is the canonical context of $\Psi \cup \{\varphi\}$. Then

$$[\![\varphi]\!]_{\vec{x}}^{M} = [\![\varphi \wedge (\vec{x} = \vec{x})]\!]_{\vec{x}}^{M} = true_{\theta_{\vec{x}M}}$$

for every $M \in \mathcal{M}_S^{\Psi,\vec{x}}$, that is, $\vDash_{\mathrm{d}}^{\mathcal{S}_S^{\Psi,\vec{x}}} (\varphi \wedge (\vec{x} = \vec{x}))$. By item (1) we infer $\vdash_{\mathrm{d}}^{HOL_S^{\Psi,\vec{x}}} (\varphi \wedge (\vec{x} = \vec{x}))$. Therefore $\Psi \vdash_{\mathrm{p}}^{HOL_S} \varphi$ and then we obtain the desired result. QED

6 Extending the language

In [4] it is shown that it is possible to extend the set $\Theta(S)$ of Definition 1.1 to a wider collection allowing functional types instead of the particular cases $P(\theta)$. That is, the language to be considered is as in Church's simple type theory (cf. [3, 5]). Of course the logic obtained is the same than *HOL*, because arbitrary exponentials can be expressed in any topos just using exponentials of the form Ω^A, finite limits and the properties of Ω (see for instance [10]).

Definition 6.1 Given a set S with distinguished element $\mathbf{1}$, we denote by $\Theta^*(S)$ the set inductively defined as follows: (i) $s \in \Theta^*(S)$ whenever $s \in S$; (ii) $(\theta_1 \times \cdots \times \theta_n) \in \Theta^*(S)$ whenever $\theta_1, \ldots, \theta_n \in \Theta^*(S)$ for integer $n \geq 2$; (iii) $(\theta \to \theta') \in \Theta^*(S)$ whenever $\theta, \theta' \in \Theta^*(S)$. △

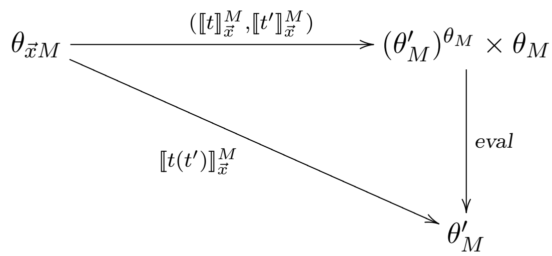
Definition 6.2 The signature Σ^* is defined analogously to Σ in Definition 1.2, replacing $\Theta(S)$ by $\Theta^*(S)$ and requiring that $\mathbf{\Omega}$ is also a distinguished element of S (therefore $\mathbf{\Omega}$ is now a primitive symbol). △

Definition 6.3 The family $ST(\Sigma^*) = \{ST(\Sigma^*)_\theta\}_{\theta \in \Theta^*(S)}$ is inductively defined as in Definition 1.3, replacing $\Theta(S)$ by $\Theta^*(S)$, $P(\theta)$ by $(\theta \to \mathbf{\Omega})$ and the clause concerning $\in_\theta$ by the following:

- if $t \in ST(\Sigma^*)_{(\theta \to \theta')}$ and $t' \in ST(\Sigma^*)_\theta$ then $(\mathbf{app}_{\theta\theta'}\langle t, t'\rangle) \in ST(\Sigma^*)_{\theta'}$. △

We write $t(t')$ instead of $(\mathbf{app}_{\theta\theta'}\langle t, t'\rangle)$. Additionally, we write $(\in_\theta \langle t_1, t_2\rangle)$ or $t_1 \in_\theta t_2$ or $t_1 \in t_2$ for $(\mathbf{app}_{\theta\mathbf{\Omega}}\langle t_2, t_1\rangle)$.

A Σ^*-structure M is a Σ-structure such that $\mathbf{\Omega}_M$ is the subobject classifier Ω, $(\theta \to \theta')_M = (\theta'_M)^{\theta_M}$ and $[\![(\mathbf{app}_{\theta\theta'}\langle t, t'\rangle)]\!]^M_{\vec{x}}$ is $eval \circ ([\![t]\!]^M_{\vec{x}}, [\![t']\!]^M_{\vec{x}})$, where $eval\colon (\theta'_M)^{\theta_M} \times \theta_M \to \theta'_M$ is the evaluation map associated to the exponential $(\theta'_M)^{\theta_M}$.



Finally, the full version of *HOL* introduced in [4] is as follows. Let $t_i \in ST(\Sigma^*)_{(\theta_i \to \mathbf{\Omega})}$ (for $i = 1, 2$). The following notation for schema terms will be useful:

- $t_1 \times t_2$ for $\{\langle x, y\rangle \;:\; (x \in t_1) \wedge (y \in t_2)\}$;
- $t_1 \subseteq t_2$ for $\forall x(x \in t_1 \Rightarrow x \in t_2)$;
- $\exists! x\varphi$ for $\exists x(\varphi \wedge \forall y(\varphi^x_y \Rightarrow (x = y)))$ where y is the first variable of the same type than x, different from x and not occurring in φ;

- $t_2^{t_1}$ for $\{z \subseteq t_1 \times t_2 \ : \ \forall x(x \in t_1 \Rightarrow \exists! y((y \in t_2) \wedge (\langle x, y\rangle \in z))\}$;
- U_θ for $\{x_1^\theta \ : \ \mathbf{t}\}$, where $\theta \in \Theta^*(S)$.

Definition 6.4 The deduction system *HOL** defined over Σ^* is obtained from *HOL* by replacing axioms $\mathbf{ext}_\theta$ by the following ones:

$$\mathbf{fun}_{\theta\theta'} : \langle \emptyset, \forall x_1(x_1 \in U_{\theta'}^{U_\theta} \Rightarrow \exists! x_2 \forall x_3 \forall x_4(\langle x_3, x_4\rangle \in x_1 \Leftrightarrow x_2(x_3) = x_4)), \mathbf{u}\rangle.$$

△

It is easy to prove that the following "extensionality" properties are derived in *HOL**:

$$\mathbf{ext}_{\theta\theta'} : \forall x_1 \forall x_2(\forall x_3(x_1(x_3) =_{\theta'} x_2(x_3)) \ \Rightarrow \ x_1 =_{(\theta\to\theta')} x_2).$$

In particular, axioms $\mathbf{ext}_\theta$ (with their provisos) are derived in *HOL**. The main result is the following.

Theorem 6.5 Let $\mathcal{S}^* = \langle \Sigma^*, \mathcal{M}^*\rangle$ be the interpretation system such that $\mathcal{M}^*$ is the class of all Σ^*-structures. Then *HOL** is sound and complete w.r.t. $\mathcal{S}^*$, that is, for $\mathrm{o} \in \{\mathrm{p}, \mathrm{d}\}$ and $\Psi \cup \{\varphi\} \subseteq L(\Sigma, \vec{x})$ finite:

1. $\Psi \vdash_{\mathrm{o}\vec{x}}^{HOL^*} \varphi$ implies $\Psi \vDash_{\mathrm{o}\vec{x}}^{\mathcal{S}^*} \varphi$.
2. $\Psi \vDash_{\mathrm{o}}^{\mathcal{S}^*} \varphi$ implies $\Psi \vdash_{\mathrm{o}}^{HOL^*} \varphi$.

Proof: Since axioms $\mathbf{ext}_\theta$ can be are derived in *HOL**, the system is rich enough to construct a canonical model (simply adapting the completeness proof of [2] or the completeness lemma in [4]). Then *HOL** is d-complete. On the other hand, axioms $\mathbf{fun}_{\theta\theta'}$ are sound in every topos. Again, the proof is adapted from [4]. It is well-known that, in any topos, using the properties of the subobject classifier Ω, finite limits and exponentials of the form Ω^A then it is possible to construct arbitrary exponentials (see, for instance, [10]). The existence of exponentials guarantees the soundness of axioms $\mathbf{fun}_{\theta\theta'}$, therefore *HOL** is d-sound. The adequacy of *HOL** for global entailment is easily obtained from the local adequacy. QED

The generalization of Theorem 6.5 to theories with additional axioms is obvious. This shows that *HOL** is another system of higher-order intuitionistic logic sound and complete for topos semantics, but defined in a richer language than those of *HOL* or local set theory. In fact, the language of *HOL** corresponds to Church's simple type theory. It should be clear that axioms $\mathbf{fun}_{\theta\theta'}$ are enough to express λ-abstraction.

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