Representation by tuples of Łukasiewicz–Moisil algebras of order $n+1$ and applications

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Abstract

We intend to present a generalization of the techniques developed by Fidel, and independently by Vakarelov, widely known as *twist–structure semantics*, to the more general context of n–valued logics.

We focus here on the well–known Łukasiewicz–Moisil algebras of order $n+1$. Departing from a twist–structure representation for De Morgan algebras, we construct an structure with more axes ($2n+2$ axes actually) that shows to represent the Moisil operators more efficiently. With this representation, the theory of homomorphisms, filters and free algebras is simplified significantly. Finally, we complete this study with the presentation of a logic sound and complete with these new models.

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1 Introduction

In order to study non–classical logics, for many years, a wide range of algebraic structures have been considered and studied. The study of such algebras presents many similarities and analogies in both concepts and demonstrations. This may lead to think that there are not methods and principles more general than those. However, a different line of study have also been considered: it is possible to study the structure and meaning of algebraic models of certain non–classical logics by means of tuples. The method of tuples, also known as the method of twist structures, have shown to be simpler and illuminating. This technique can be displayed, for example, studying the Nelson logic in [3] and [9].

In this work, we apply the technique of tuples to the well–known *Łukasiewicz–Moisil algebras of order* $n+1$. In these algebras, they are introduced operators, widely known as *Moisil possibility operators*, which could be considered as modals. However, some of their properties could prevent

us of thinking them as such. In order to clarify this conflict, we shall introduce the propositional calculi $\mathcal{M}_{n+1}$ as well as semantic models for them, in the sense of Kripke, exhibiting the corresponding soundness and completeness theorems. This will show that Moisil possibility operators can be thought as modals and that their unusual properties are due to the fact that they are valid in their models.
The main difference between the treatment here and the one of [3] is that our algebraic models will be formed by $(2n+2)$-tuples. That is, we shall consider the product of $n+1$ lattices with their $n+1$ dual lattices. With this representation, the theory of homomorphisms, filters and free algebras will be simplified significantly.

2 Preliminaries

Early last century J. Łukasiewicz introduced a class of propositional calculi by means of matrices in the following way: if x and y denote numbers in the real interval $[0,1]$, then the implication $\rightarrow$ (known as Łukasiewicz implication) is defined by: (A) $x \rightarrow y = \min(1, 1-x+y)$, the negation $\sim$ is defined by: (B) $\sim x = 1-x$, and the set of designated values is (C) $D = \{1\}$. With $\omega-$ŁPC we shall denote the propositional calculus which has $\langle [0,1], \rightarrow, \sim, \{1\} \rangle$ as characteristic matrix. Let $L_{n+1} = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$ and $J = \{1, 2, \ldots, n\}$. If we consider the chain L_{n+1}, instead of the segment $[0,1]$, along with the operators $\rightarrow, \sim$ and the set D as above, we have the matrix of the $(n+1)-$valued Łukasiewicz propositional calculus (for short $(n+1)-$ŁPC).
Recall that it was Moisil (in 1941) who first intended to present an algebraic semantic for $(n+1)-$ŁPC. In order to do so, he introduced what today are widely known as Łukasiewicz-Moisil algebras of order $(n+1)$ (or $(n+1)$-valued Łukasiewicz-Moisil algebras) (see [1]), for n being a positive integer, as follows:

Definition 2.1 *A* Łukasiewicz-Moisil algebra of order $(n+1)$ *is an algebra* $\langle A, \vee, \wedge, \sim, (\phi_i)_{i\in J}, 0, 1\rangle$ *of type* $(2,2,1,(1)_{i\in J},0)$ *where* $J = \{1, 2, \ldots, n\}$ *and such that* $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ *is a De Morgan algebra and* $(\phi_i)_{i\in J}$ *are unary operations on* A *such that: for all* $i, j \in J$,

(C1) $\phi_i(x \vee y) = \phi_i x \vee \phi_i y$,

(C2) $\phi_i x \vee \sim \phi_i x = 1$,

(C3) $\phi_i \phi_j x = \phi_j x$,

(C4) $\phi_i \sim x = \sim \phi_{n-i} x$,

(C5) $i \leq j$ *implies* $\phi_i x \leq \phi_j x$, *and*

(C6) $\phi_i x = \phi_i y$, *for all* $i \in J$, *implies* $x = y$.

The success of Moisil was partial since Rose verified later that, for $n \geq 4$, $(n+1)$-valued Łukasiewicz-Moisil algebras are not the algebraic counterpart of the $(n+1)$-ŁPC calculus.

On the other hand, these algebras have been extensively studied by different authors standing out among them R. Cignoli who studied them in his Ph.D. thesis under the name of *Moisil algebras* (see[2]).

The *standard Łukasiewicz-Moisil algebra of order $(n+1)$* is $\mathcal{L}_{n+1} = \langle L_{n+1}, \vee, \wedge, \sim, (\sigma_i)_{i\in J}, 0, 1\rangle$ where: L_{n+1} and $\sim$ are as above, $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$ and

$$\sigma_i\left(\frac{j}{n}\right) = \begin{cases} 0 & \text{if } j+i < n+1 \\ \\ 1 & \text{if } j+i \geq n+1 \end{cases}, \text{ for every } j \in \{0\} \cup J, i \in J. \tag{1}$$

The following result is well-known.

Theorem 2.1 *([2]) Every Łukasiewicz-Moisil algebra of order $(n+1)$ is isomorphic to a subdirect product of algebras $\mathcal{L}_{n+1}$.*

Remark 2.1 *Every Łukasiewicz-Moisil algebra is a Heyting algebra where the pseudo-complement relative to $\rightarrow$ can be defined in terms of $\wedge$, $\vee$, $\sim$, ϕ_i. Besides, the ϕ_i's can be defined in terms of $\rightarrow$, $\wedge$, $\vee$ and $\sim$, but this is not possible any more for $n \geq 5$ ([2]).*

Let A be a De Morgan algebra and let $\mathcal{E}(A)$ the family of all prime filters of A. We denote by Φ the well–known Birula–Rasiowa mapping defined on $\mathcal{E}(A)$ as

$$\Phi(P) = A\backslash \sim P, \text{ where } \sim P = \{\sim p : p \in P\}.$$

Recall that in [5] it was showed that any De Morgan algebra can be embedded in the product $L\times L^\star$ where L is a bounded distributive lattice and $L^\star$ is its dual lattice. Indeed, let $\mathcal{F} = \{P_i\}_{i\in I}$ be any family of prime filters of A such that

$$\mathcal{F} \cup \Phi(\mathcal{F}) = \mathcal{E}(A)$$

Consider, now, the bounded distributive lattice $L = \prod_{i\in I} L_i$ where L_i is the two–element distributive lattice **2** for all $i \in I$. Then, A can be embedded in $L \times L^\star$ by means of $f_A : A \rightarrow L \times L^\star$

$$f_A(x) = ((x_i^1)_{i\in I}, (x_i^2)_{i\in I}) \text{ where } x_i^1 = 1 \text{ iff } x \in P_i \text{ and } x_i^2 = 0 \text{ iff } x \in \Phi(P_i) \text{ for all } i \in I. \tag{2}$$

It can be proved that $f_A(A) \subseteq L\times L^\star$ is a De Morgan algebra where, if $(x^1, x^2), (y^1, y^2) \in f_A(A)$ then $\sim (x^1, x^2) = (x^2, x^1)$, $(x^1, x^2) \vee (y^1, y^2) = (x^1 \vee y^1, x^2 \wedge y^2)$, $(x^1, x^2) \wedge (y^1, y^2) = (x^1 \wedge y^1, x^2 \vee y^2)$, $0 = (0, 1)$, and $1 = (1, 0)$.

If $\pi_1 : L \times L^\star \rightarrow L$ and $\pi_2 : L \times L^\star \rightarrow L^\star$ are the standard projections then $\pi_i(f_A(A))$ is a sub-lattice of $L \times L^\star$, $i = 1, 2$ and $\pi_1(f_A(A)) = \pi_2(f_A(A)$.

It is worth mentioning that the above representation for a De Morgan algebra A is not unique: it strongly depends on the family of prime filters $\mathcal{F}$ that we choose. On the other hand, there always exists such a family: it is enough to consider $\mathcal{E}(A)$.

Let $\mathbf{T}(A)$ be the subalgebra $f_A(A)$ of the product $L \times L^\star$ where L is the lattice that is constructed by considering $\mathcal{E}(A)$ as the family $\mathcal{F}$. It is clear that f_A is a De Morgan isomorphism between A and $f(A)$. Then, this representation of De Morgan algebras can be recast in categorical terms. Let $\mathcal{DM}$ be the category whose objects are all De Morgan algebras and its morphisms are all the homomorphisms between De Morgan algebras. On the other hand, let $\mathcal{LL}^\star$ be the category whose objects are De Morgan subalgebras of products $L \times L^\star$ where $L \simeq \prod_{i \in I} \mathbf{2}$ for any arbitrary set of indexes.

Then, we can extend the operator $\mathbf{T}$ to a functor $\mathbf{T} : \mathcal{DM} \to \mathcal{LL}^\star$ by $\mathbf{T}(A) = f_A(A)$ and $\mathbf{T}h = f_B \circ h \circ f_A^{-1}$. It is not difficult to see that $\mathbf{T}$ is full, faithful and dense. Then

Theorem 2.2 *The categories $\mathcal{DM}$ and $\mathcal{LL}^\star$ are equivalent.*

3 $(n+1)$-valued Moisil twist structure

Next, we are shall introduce our *models* in terms of $(2n+2)$-tuples, namely, $(n+1)$*-valued Moisil twist structure.*

Definition 3.1 *Let L be a bounded distributive lattice (the base lattice). Let L_i, $1 \leq i \leq n$, be sub-lattices of L, let $L^\star$ and $L_i^\star$ be the dual lattices (with respect to the order) of L and $L_i^\star$ respectively. Consider the product of $(2n+2)$ lattices: $L \times L_1 \times \cdots \times L_n \times L_n^\star \times \cdots \times L_1^\star \times L^\star = L \times \bigotimes_{i=1}^{n} L_i \times \bigotimes_{i=0}^{n+1} L_{n-i}^\star \times L^\star$. We shall call $(n+1)$-valued Moisil twist structure to any sub-lattice N of the product $L \times \bigotimes_{i=1}^{n} L_i \times \bigotimes_{i=0}^{n+1} L_{n-i}^\star \times L^\star$ such that for any $x = (x_0, x_1, \ldots, x_n, x_n^\star, \ldots, x_1^\star, x_0^\star) = ((x_i)_0^n, (x_{n-i}^\star)_0^n)$, $y = (y_0, y_1, \ldots, y_n, y_n^\star, \ldots, y_1^\star, y_0^\star) = ((y_i)_0^n, (y_{n-i}^\star)_0^n)$ in N it is verified:*

(Tw0) *if $x_0 \leq y_0$ then $x_i \leq y_i$, for $i \in J$,*

(Tw1) *$1 \leq x_i \vee x_i^\star$ and $x_i \wedge x_i^\star \leq 0$, for $i \in J$,*

(Tw2) *if $i \leq j$ then $x_i \leq x_j$ and $x_i^\star \geq x_j^\star$, for $i, j \in J$,*

(Tw3) *$x_0 \leq x_n$ and $x_0^\star \geq x_n^\star$,*

(Tw4) if $x_i = y_i$ for all $i \in J$ then $x = y$, and

(Tw5) if $(x_0, x_1, \ldots, x_n, x_n^\star, \ldots, x_1^\star, x_0^\star) \in N$, then $(x_0^\star, x_n^\star, \ldots, x_1^\star, x_1, \ldots, x_n, x_0) \in N$ and $(x_i, \underbrace{x_i, \ldots, x_i}_{n\ places}, \underbrace{x_i^\star, \ldots, x_i^\star}_{n\ places}, x_0^\star) \in N$, for $1 \leq i \leq n$.

Remark 3.1 *(i) Intuitively, x_0 represents a positive value, $x_0^\star$ represents a negative value (independent from the positive one); $x_1, \ldots, x_n$ represent the graded positive possibility of some proposition and $x_1^\star, \ldots, x_n^\star$ the respective negative values. The pairs $(x_i, x_i^\star)$ for $i \in J$ stand for the classical possibility values as it can be seen in the following proposition.*

(ii) (Tw4) is the well–known Moisil's determination principle and it is equivalent to (i.e. it could be replaced by)

$$(Tw4)' \quad x_0 \wedge x_i^\star \wedge y_{i+1} \leq y_0 \text{ and } y_0^\star \leq x_0^\star \vee x_i \vee y_{i+1}^\star.$$

This last axiom is due to Cignoli [2] and will be very useful as we shall see.

Proposition 3.1 *If* $L \times \bigotimes_{i=1}^{n} L_i \times \bigotimes_{i=0}^{n+1} L_{n-i}^\star \times L^\star$ *is a $(n+1)$-valued Moisil twist structure, then $L_i \times L_i^\star$ is a Boolean algebra, $i \in J$*

Proof. Immediate from Definition 3.1. □

Example 3.1 *Let's see the chain L_5 as a 5-valued Moisil twist structures. We take L_3 as the base lattice and $L_i = L_2 = \{0, 1\}$ for $i = 1, 2, 3, 4$. Then*

$$0 = (0, 0, 0, 0, 0, 1, 1, 1, 1, 1)$$

$$\frac{1}{4} = (\frac{1}{2}, 0, 0, 0, 1, 0, 1, 1, 1, 1)$$

$$\frac{2}{4} = (1, 0, 0, 1, 1, 0, 0, 1, 1, 1)$$

$$\frac{3}{4} = (1, 0, 1, 1, 1, 0, 0, 0, 1, \frac{1}{2})$$

$$1 = (1, 1, 1, 1, 1, 0, 0, 0, 0, 0)$$

In general, we can represent $\mathcal{L}_{n+1}$ by a Moisil twist structure taking $L = \mathcal{L}_{n-1}$ and $L_i = \{0,1\}$ for $i \in J$. Besides, if n is even natural number L_{n+1} has a center, that is, an element c such that $\sim c = c$ and its coordinates are

$$\frac{n/2}{n} = (\frac{n/2}{n-2}, \underbrace{0,\ldots,0}_{n/2 \ places}, \underbrace{1,\ldots,1}_{n/2 \ places}, \underbrace{0,\ldots,0}_{n/2 \ places}, \underbrace{1,\ldots,1}_{n/2 \ places}, 1)$$

The standard algebra $\mathcal{L}_{n+1}$ presented as a twist structure will be noted $\mathcal{T}_{n+1}$. Now we are going to define operations on a given $(n+1)$-valued Moisil twist structure.

Definition 3.2 *Let $x = ((x_i)_0^n, (x^\star_{n-i})_0^n), y = ((y_i)_0^n, (y^\star_{n-i})_0^n) \in N$. We define on N the following operations.*

(i) $\sim x =_{def} (x^\star_0, x^\star_n, \ldots, x^\star_1, x_1, \ldots, x_n, x_0)$,

(ii) $x \vee y =_{def} ((x_i \vee y_i)_0^n, (x^\star_{n-i} \wedge y^\star_{n-i})_0^n)$,

(iii) $x \wedge y =_{def} ((x_i \wedge y_i)_0^n, (x^\star_{n-i} \vee y^\star_{n-i})_0^n)$,

(iv) $\phi_i x =_{def} (x_i, \underbrace{x_i, \ldots, x_i}_{n \ places}, \underbrace{x^\star_i, \ldots, x^\star_i}_{n \ places}, x^\star_0)$, *for $i \in J$,*

(v) $0 =_{def} (\underbrace{0,\ldots,0}_{n+1 \ places}, \underbrace{1,\ldots,1}_{n+1 \ places})$,

(vi) $1 =_{def} (\underbrace{1,\ldots,1}_{n+1 \ places}, \underbrace{0,\ldots,0}_{n+1 \ places})$.

Proposition 3.2 *$\langle N, \vee, \wedge, \sim, 1\rangle$ is a De Morgan algebra.*

Proof. Immediate. □

The following property will be useful in the sequel.

Proposition 3.3 *If $(x_0, x_1, \ldots, x_n, x^\star_n, \ldots, x^\star_1, x^\star_0) \in N$ then $x_1 \leq x_0$ and $x^\star_1 \geq x^\star_0$.*

Proof. By Definition 3.1 (Tw5) and (Tw3). □

From definitions 3.1 and 3.2 and Proposition 3.2, the next result is quite obvious.

Theorem 3.1 *If N is $(n+1)$-valued Moisil twist structure then $\langle N, \vee, \wedge, \sim, (\phi_i)_{i\in J}, 0, 1\rangle$ is a Łukasiewicz-Moisil algebra of order $(n+1)$.*

The converse is also valid.

Theorem 3.2 *For every $(n+1)$-valued Łukasiewicz-Moisil algebra A there exists a $(n+1)$-valued Moisil twist structure N such that A and $\langle N, \vee, \wedge, \sim, (\phi_i)_{i\in J}, 0, 1\rangle$ are isomorphic.*

Proof. Let $\langle A, \vee, \wedge, \sim, (\phi_i)_{i\in J}, 0, 1\rangle$ a $(n+1)$-valued Łukasiewicz-Moisil algebra. Following what it was done in [5], let $\mathcal{F} = \{P_j\}_{j\in I}$ any family of prime filters of A such that $\mathcal{F} \cup \Phi(\mathcal{F}) = \mathcal{E}(A)$ and let $L = \prod_{j\in I} R_j$ where R_j is the two–element distributive lattice $\mathbf{2}$ for all $i \in I$.
Let $f : A \hookrightarrow L \times L^\star$ as in (2) and π_1, π_2 the respective projections. We know that $\pi_1(f(A)) = \pi_2(f(A))$. Now, we have to choose the lattices L_i for $1 \leq i \leq n$, and they have to be sub–lattices of L. It is well–known that, if $K_i = \{x \in A : \phi_i(x) = x\}$ for $1 \leq i \leq n$ then $K_1 = \cdots = K_n = B(A)$ where $B(A)$ is the set of all Boolean elements of A. So, let $L_i = \pi_1(f(B(A)))$, for $1 \leq i \leq n$. It is not difficult to see that $L_i^\star = \pi_2(f(\sim B(A)))$, $1 \leq i \leq n$.
Since p is a lattice homomorphism, every L_i is a sub–lattice of L and all $x_i \in L_i$ has a complement that belongs to $L_i^\star$. In order to prove that A can be immersed in $L \times \bigotimes_{i=1}^{n} L_i \times \bigotimes_{i=0}^{n+1} L_{n-i}^\star \times L^\star$ consider the function $\bar{f} : A \to L \times \bigotimes_{i=1}^{n} L_i \times \bigotimes_{i=0}^{n+1} L_{n-i}^\star \times L^\star$ defined as follows

$$\bar{f}(x) = (\pi_1(f(x)), (\pi_1(f(\phi_i(x))))_{i=1}^{n}, (\pi_2(f(\sim \phi_{n-i}(x))))_{i=1}^{n}, \pi_2(f(x)))$$

Then, $\bar{f}(A) \subseteq L \times \bigotimes_{i=1}^{n} L_i \times \bigotimes_{i=0}^{n+1} L_{n-i}^\star \times L^\star$ is a $n+1$-valued Moisil twist structure. For example, let's see that (Tw2) holds. Let $\vec{x} \in \bar{f}(A)$, then there is $x \in A$ such that $\bar{f}(x) = \vec{x}$. Since A is a $(n+1)$-valued Łukasiewicz-Moisil algebra, $\phi_i(x) \leq \phi_j(x)$, and since f and π_1 are lattice homomorphism we have that $\pi_1(f(\phi_i(x))) \leq \pi_1(f(\phi_j(x)))$.
On the other hand, by Theorem 3.1, $\bar{f}(A)$ is Łukasiewicz-Moisil algebra of order $(n+1)$. Besides, it is not difficult to see that $\bar{f}$ is $(n+1)$-valued Łukasiewicz-Moisil homomorphism and that $\bar{f}$ is injective since f is injective. Therefore, $\bar{f}(A)$ and A are isomorphic algebras. $\square$

Remark 3.2 *Any $(n+1)$-valued Moisil twist structure is, in particular, a Łukasiewicz-Moisil algebra of order $(n+1)$. Therefore, the notion of homomorphism, congruence, filter, etc., is the usual in the context of universal algebra.*

4 Prime filters, homomorphisms and representation

In order to prove soundness and completeness of the calculus $\mathcal{M}_{n+1}$, which will be presented in Section 6, with respect to their semantic models we need to know in detail the structure of the prime filters of any $(n+1)$-valued Moisil twist structure.

Recall that in [5] it was proved that every De Morgan algebra D can be seen as a subset $\emptyset \neq M \subseteq L \times L^\star$ where L and $L^\star$ are a lattice and its dual respectively such that; (DM1) $(0,1)$ and $(1,0)$ are in M, (DM2) $(x_1,x_2) \in M$ implies $(x_2,x_1) \in M$, and (DM3) $(x_1,x_2),(y_1,y_2) \in M$ implies $(x_1 \wedge y_1, x_2 \vee y_2) \in M$. Besides, in M, the operations $\sim$, $\vee$ and $\wedge$ are defined as follows: $\sim (x_1,x_2) =_{def} (x_2,x_1)$, $(x_1,x_2) \vee (y_1,y_2) =_{def} (x_1 \vee y_1, x_2 \wedge y_2)$, and $(x_1,x_2) \wedge (y_1,y_2) =_{def} (x_1 \wedge y_1, x_2 \vee y_2)$.

At this point, it is worth correcting an omission in [5] concerning the form of prime filters in a De Morgan algebra presented by a twist–product.

Proposition 4.1 *Let M be the De Morgan algebra $L \times L^\star$ as indicated above. If P is a prime filter of M then $P = P_0 \times I_1^\star$ where $P_0 = \pi_1(P)$, $\sim P = \{(x_1,x_0) : (x_0,x_1) \in P\}$ and $I_1 = \{\sim P\}$. Besides, P_0 is a prime filter of L and I_1 is a prime ideal of $L^\star$.*

Proof. Since π_1 is an epimorphism, we have that P_0 and I_1 are a prime filter and a prime ideal of L, respectively.
It is clear that $P \subseteq P_0 \times I_1^\star$. Let's see that $P_0 \times I_1^\star \subseteq P$. Let $x \in P_0 \times I_1^\star$. Then, $x = (x_0,x_1)$ where $x_0 \in P_0$, $x_1 \in I_1$, and there exists $y \in P$ such that $y = (x_0,y_1) \leq y' = (x_0,0) \in P$. Analogously, there exists $z \in P$ such that $z = (z_0,x_1) \leq z' = (1,x_1) \in P$. Since P is a filter, we have that $y' \wedge z' = x = (x_0,x_1) \in P$. □

The reciprocal is stated in the following two propositions.

Proposition 4.2 *Let P be a proper prime filter of $M = L \times L^\star$. Then, $P = P_0 \times L^\star$ or $P = L \times I_1^\star$ where P_0 and $I_1^\star$ are as in the above proposition.*

Proof. We know that $P = P_0 \times I_1^\star = (P_0 \times L^\star) \cap (L \times I_1^\star)$. Since P is a prime filter and $P_0 \times L^\star$, $L \times I_1^\star$ are filters of M, we have that $P = P_0 \times L^\star$ or $P = L \times I_1^\star$. □
and,

Proposition 4.3 *Let P be a proper prime filter of $M = L \times L^\star$ and suppose that $P = P_0 \times I_1^\star$. Then, $P_0 = L$ or $I_1^\star = L^\star$.*

Proof. Let P be a proper filter of $M = L \times L^\star$ and suppose that $P_0 \subsetneq L$ and $I_1^\star \subsetneq L^\star$. Then, $P_0 \times I_1^\star \subset P_0 \times L^\star$ and $P_0 \times I_1^\star \subset L \times I_1^\star$. So, there are

$$(x_0,x_1') \in P_0 \times L^\star \text{ and } (x_0,x_1') \notin P_0 \times I_1^\star,$$
$$(x_0',x_1) \in L \times I_1^\star \text{ and } (x_0,x_1') \notin P_0 \times I_1^\star.$$

Then, $(x_0,x_1) \in P_0 \times L^\star$ and $(x_0,x_1) \in L \times I_1^\star$; and so $(x_0,x_1) \in (P_0 \times L^\star) \cap (L \times I_1^\star) = P_0 \times I_1^\star$. On the other hand, $(x_0,1) \leq (x_0,x_1')$, so $(x_0,1) \notin P_0 \times I_1^\star$. Analogously, $(0,x_1) \notin P_0 \times I_1^\star$. But $(x_0,1) \vee (0,x_1) = (x_0 \vee 0, x_1 \wedge 1) = (x_0,x_1) \in P_0 \times I_1^\star$ which is a contradiction. □

In what follows, we shall rediscover many results by Cignoli ([2]) in the context of our $(n+1)$-valued Moisil twist structures. In particular, we shall see that every prime filter of a given $n+1$–valued Moisil twist structure belongs to one, and only one, chain of filters with at most $n-1$ elements.

Let $N \subseteq L \times \bigotimes_{i=1}^{n} L_i \times \bigotimes_{i=0}^{n+1} L^\star_{n-i} \times L^\star$ be a $n+1$-valued Moisil twist structure. Let $N^R = \{(x, x^\star) : ((x_i)_0^n, (x^\star_{n-i})_0^n) \in N\} \subseteq L \times L^\star$. It is clear that N^R is a De Morgan algebra presented as a twist product. Let $\pi_j : N \to L_j$, $0 \leq j \leq n$ the projection morphisms from N into L_j. Without loss of generality, we may assume that the $\pi_j' s$ are surjective, i.e., $\pi_j(N) = L_j$.

From what we have just discussed if P is a prime filter of N^R then

$$(1)\, (P_0 \times L^\star)|_{N^R} \ \text{ or } \ (2)\, (L \times I_0^\star)|_{N^R}$$

where $P_0 = \pi_1(P)$ and $I_0 = \pi_1(\sim P)$.

Let P be a prime filter of N^R of the form (1), i.e., $P = (P_0 \times L^\star)_{N^R}$ then we define

Definition 4.1 *Let $P_0' =_{def} P_0|_{L_{n-1}}$. For every i, $1 \leq i \leq n-1$*

$$P_0^i =_{def} \{x_0 \in L : x = ((x_i)_0^n, (x^\star_{n-i})_0^n) \in N^R,\ x_i \in P_0'\}.$$

Then, it is immediate that

Proposition 4.4 *P_0' is an ultrafilter of the Boolean algebra L_n.*

Besides,

Proposition 4.5 *For $1 \leq i \leq n$, it holds:*

(i) P_0^i is a prime filter of L,

(ii) if $x_0 \notin P_0^i$ then $x_0^\star \in P_0^i$.

Proof. (i) Let's see that P_0^i is a prime filter. Clearly $1 \in P_0^i$. Suppose that $x_0 \in P_0^i$ and there is $y_0 \in L$, $x_0 \leq y_0$. Then, there is $y = ((y_i)_0^n, (y^\star_{n-i})_0^n) \in N^R$ and, by (Tw0), $x_i \leq y_i$. So, $y_i \in P_0'$ and therefore $y_0 \in P_0^i$.
On the other hand, let $x_0, y_0 \in L$ such that $x_0 \vee y_0 \in P_0^i$. Since the projection $\pi_0 : M \to L$ is surjective, we have that there exist $x, y \in N^R$ such that $x \in ((x_i)_0^n, (x^\star_{n-i})_0^n)$ and

$y \in ((y_i)_0^n, (y_{n-i}^\star)_0^n)$. But $x \vee y = ((x_i \vee y_i)_0^n, (x_{n-i}^\star \wedge y_{n-i}^\star)_0^n) \in L$ and, by hypothesis, $x_0 \vee y_0 \in P_0^i$ and so $x_i \vee y_i \in P_0'$. By Proposition 4.4, $x_i \in P_0'$ or $y_i \in P_0'$ and so $x_0 \in P_0^i$ or $y_0 \in P_0^i$.

(ii) Finally, suppose that there is $x_0 \in L \setminus P_0^i$. Then, there is $x = ((x_i)_0^n, (x_{n-i}^\star)_0^n) \in N^R$ such that $x_i \notin P_0'$. By Proposition 4.4, $x_i^\star \in P_0'$ and so $x_0^\star \in P_0^i$. $\square$

Next, we shall see that the filters P_0^i's form a chain.

Proposition 4.6 $P_0^i \subseteq P_0^{i+1}$, *for* $1 \leq i < n$.

Proof. Let $x_0 \in P_0^i$. Then, there is $x = ((x_i)_0^n, (x_{n-i}^\star)_0^n) \in N^R$ such that $x_0 \in L$ and $x_i \in P_0'$. Since $x_i \leq x_{i+1}$ we have that $x_{i+1} \in P_0'$; and then $x_0 \in P_0^{i+1}$. $\square$

Then, the family $\{(P_0^i \times L^\star)\}_{i=1}^n$ is a chain of prime filters of N^R. We can locate the prime filter P_0 in the chain $\{P_0^i\}_{i=1}^n$.

Proposition 4.7 $P_0^1 \subseteq P_0 \subseteq P_0^n$.

Proof. Let $x_0 \in P_0^1$. Then, there is $x = ((x_i)_0^n, (x_{n-i}^\star)_0^n) \in M$ such that $x_0 \in L$ and $x_1 \in P_0'$. By 3.3, $x_1 \leq x_0 \leq x_n$ we have that $x_0 \in P_0'$. Besides, the elements $x = ((x_0)_0^n, (x_0^\star)_0^n) \in N^R$. Analogously we prove that $P_0 \subseteq P_0^n$. $\square$

As an immediate consequence we have:

Corollary 4.1 $(P_0^1 \times L^\star) \subseteq (P_0 \times L^\star) \subseteq (P_0^n \times L^\star)$.

The next result will be fundamental when proving the completeness of the login $\mathcal{M}_n$ with respect to our models.

Proposition 4.8 *Let* $x_0 \in L$. *The following conditions are equivalent:*

(i) $x_0 \in P_0^i$,

(ii) for every r, $i \leq r \leq n$, $x_0 \in P_0^r$,

(iii) there is r, $i \leq r \leq n$, $x_0 \in P_0^r$.

Proof. It follows essentially from the fact that the family $\{P_0^i\}_{i=1}^n$ form a chain, choosing $r = i$ when necessary. $\square$

Remark 4.1 *The corresponding results for the filters* $\{(P_0^i \times L^\star)\}_{i=1}^n$ *of* N^R *follows immediately. We have an analogous situation with the corresponding results for the filters* $\{(P_0^i \times L_1 \times \cdots \times L_n \times L_n^\star \times \cdots \times L_1^\star \times L^\star)\}_{i=1}^n$ *of* N^R

Next proposition justifies Cignoli's choice of the equations defining Łukasiewicz-Moisil algebras of order $n+1$.

Proposition 4.9 *Let P_0 be a prime filter of L. Then, for every* $1 \leq i < n$

$$P_0 \subseteq P_0^i \quad or \quad P_0^{i+1} \subseteq P_0$$

Proof. Suppose that $P_0 \not\subseteq P_0^i$ and $P_0^{i+1} \not\subseteq P_0$. Then, there exist $x_0 \in P_0$ and $y_0 \in P_0^{i+1}$ such that $x_0 \notin P_0^i$ and $y_0 \notin P_0$. By Proposition 4.5 (ii), $x^\star \in P_0^i$ and so $x_i^\star \in P_0' \subseteq P_0$. On the other hand, $y_0 \in P_0^{i+1}$ and then $y_{i+1} \in P_0' \subseteq P_0$. Therefore, $x_0 \wedge x_i^\star \wedge y_{i+1} \in P_0$ and by (Tw4)$'$, $y_0 \in P_0$ which is a contradiction. □

Theorem 4.1 *Let P_0 be a prime filter of L. Then, there exist a unique i, $1 \leq i \leq n$ such that*

$$P_0 = P_0^i$$

Proof. Let $i_0 = max\{i : P_0^i \subseteq P_0\}$. By Proposition 4.7, we know that $\{i : P_0^i \subseteq P_0\} \neq \emptyset$. If $i_0 = n$ then, by Proposition 4.7 and the definition of i_0, $P_0^{i_0} = P_0$. On the other hand, if $i_0 \leq n-1$ then $P_0^{i_0} \subseteq P_0$ and $P_0^{i_0+1} \not\subseteq P_0$. By Proposition 4.9, $P_0 \subseteq P_0^{i_0}$. □

Again, it is easy to state the same result for filters of N.

Lemma 4.1 *If* $P_0 \subseteq Q_0$ *then $P_0^i \subseteq Q_0^i$, for all $1 \leq i \leq n$.*

Proof. Immediate. □

Proposition 4.10 *Every filter P_0 of L belongs to one and only one chain of filters $\{P_0^i\}_{i=1}^n$.*

Proof. Let P_0 be a prime filter of L. We know by Theorem 4.1, that P_0 belongs to the chain of filters $\{P_0^i\}_{i=1}^n$, i.e., there is i_0 such that $P_0 = P_0^{i_0}$. On the other hand, suppose that there exist a filter Q_0 of L such that P_0 belongs to the chain $\{Q_0^j\}_{j=1}^n$, i.e., there is j_0 such that $P_0 = Q_0^{j_0}$. Then, $P_0 = Q_0^{j_0} \subseteq Q_0^{j_0+1}$ and, by propositions 4.6 and 4.9, we have that $P_0 \subseteq Q_0$ or $Q_0 \subseteq P_0$. If $P_0 \subseteq Q_0$ then, for every $1 \leq k \leq n$, $P_0^k \subseteq Q_0^k$, by Proposition 4.1.
Let $k \in \{1, \ldots, n\}$ and let $x_0 \in P_0^k$, then $x_0 \in P_0^i$ for all $i = 1, \ldots, n$ (Proposition 4.8). Then, $x_0 \in P_0^{i_0} = P_0 = Q_0^{j_0}$ and by Proposition 4.8, $x_0 \in Q_0^k$. Therefore, both chains of filters coincide. We proceed analogously if $Q_0 \subseteq P_0$. □

All these results related to prime filters of L can be easily be formulated and proved for filters of M. We leave this task to the patient reader. Besides, these same results can be obtained if we assume that the filter P of M is of the form $L \times I_0^\star$.
Next theorem is consequence of all the stated above and it will be used in the sections of semantical models. This formulation is due to Cignoli:

Theorem 4.2 *(Cignoli) Every prime filter of $n+1$-valued Moisil twist structures belongs to one, and only one, chain of prime filters with al most n elements. The set of all prime filters ordered by inclusion is the cardinal sum of chains with at most n elements.*

Lemma 4.2 *Let $P = P_0 \times L^\star$ be a prime filter of N^R. If $P_0 = P_0^i$ then $\varphi(P) = P_0^{(n+1-i)} \times L^\star$.*

Proof. $x \in \varphi(P) \Leftrightarrow x \notin \sim P \Leftrightarrow \sim x \in P = P_0^i \times L^\star \Leftrightarrow x^\star_{n+1-i} \notin P_0' \Leftrightarrow x_0^\star \notin P_0^{n+1-i} \Leftrightarrow x_0 \in P_0^{n+1-i} \Leftrightarrow x \in P_0^{n+1-i} \times L^\star$. □

As a consequence we have the following corollary.

Corollary 4.2 $(n+1)$*-valued Moisil twist structures verify the Kleene's property.*

Proof. It is enough to see that for every prime filter P of L it holds: $P \subseteq \varphi(P)$ or $\varphi(P) \subseteq P$. But this is consequence of the above lema and the fact that both filters belong to the same chain. □

Regarding simple and semisimple Łukasiewicz-Moisil algebras of order $(n+1)$ the treatment in our context is quite simple. This is due to the fact that the study of the homomorphisms of $(n+1)$-valued Moisil twist structures is reduced to the study of lattice homomorphisms.
Indeed, we have already seen that the standard algebras $\mathcal{L}_{n+1}$ can be presented by $(2n+2)$-tuples using the lattice chain with $n-1$ elements and the lattice $L_2 = \{0,1\}$ for the subbases. The only homomorphic images of these lattices are chains with less elements and the same L_2 respectively. Then, the only homomorphic images of $\mathcal{L}_{n+1}$ are the subalgebras of $\mathcal{L}_{n+1}$. Therefore, by definition, the algebras $\mathcal{L}_{n+1}$ are simple.
On the other hand, any other simple algebra have $\mathcal{L}_{n+1}$ as homomorphic image since any chain of prime filters (of the kind we have seen above) in the lattice have a maximal filter P_n. Then, these algebras are semisimple and it holds the next representation theorem due to Moisil.

Theorem 4.3 *(Moisil) Every $n+1$-valued Moisil twist structures is isomorphic to a subdirect product of algebras $\mathcal{T}_{n+1}$.*

5 Construction of the free $n+1$-valued Moisil twist structure

In this section we shall construct the free Łukasiewicz-Moisil algebras of order $(n+1)$ using its representation as a twist product. The simplicity of this construction will show, once again, the power of twist structures.

Let $\langle G\rangle_{\mathcal{T}_n}$ be the $n+1$-valued Moisil twist structure generated by a subset

$$G \subseteq L \times \bigotimes_{i=1}^{n} L_i \times \bigotimes_{i=0}^{n+1} L^{\star}_{n-i} \times L^{\star},$$

satisfying (Tw0), (Tw1), (Tw2), (Tw3) and (Tw4).
That is, $\langle G\rangle_{\mathcal{T}_n}$ is the least $n+1$-valued Moisil twist structure that contains G. On the other hand, let $\langle G\rangle_{\mathcal{L}}$ be the lattice generated by G in the same product. Consider now the following sets.

$$\sim G =_{def} \{(g_0^{\star}, g_n^{\star}, \ldots, g_1^{\star}, g_1, \ldots, g_n, g_0) : (g_0, g_1, \ldots, g_n, g_n^{\star}, \ldots, g_1^{\star}, g_0^{\star}) \in G\},$$

$$\phi_i G =_{def} \{(g_i, g_i, \ldots, g_i, g_i^{\star}, \ldots, g_i^{\star}, g_i^{\star}) : (g_0, g_1, \ldots, g_n, g_n^{\star}, \ldots, g_1^{\star}, g_0^{\star}) \in G\},$$

$$\sim \phi_i G =_{def} \{(g_i^{\star}, g_i^{\star}, \ldots, g_i^{\star}, g_i, \ldots, g_i, g_i) : (g_0, g_1, \ldots, g_n, g_n^{\star}, \ldots, g_1^{\star}, g_0^{\star}) \in G\},$$

for $i = 1, \ldots, n$. Then, it is possible to prove the next proposition.

Proposition 5.1 $\langle G \cup \bigcup_{i=1}^{n} \phi_i G \cup \bigcup_{i=1}^{n} \sim \phi_i G \cup \sim G\rangle_{\mathcal{L}} = \langle G\rangle_{\mathcal{T}_n}$.

Proof. (I) $\langle G\rangle_{\mathcal{T}_n}$ is a lattice which contains $G \cup \bigcup_{i=1}^{n} \phi_i G \cup \bigcup_{i=1}^{n} \sim \phi_i G \cup \sim G$, therefore $\langle G \cup \bigcup_{i=1}^{n} \phi_i G \cup \bigcup_{i=1}^{n} \sim \phi_i G \cup \sim G\rangle_{\mathcal{L}} \subseteq \langle G\rangle_{\mathcal{T}_n}$.
(II) $L_G = \langle G \cup \bigcup_{i=1}^{n} \phi_i G \cup \bigcup_{i=1}^{n} \sim \phi_i G \cup \sim G\rangle_{\mathcal{L}}$ is a $n+1$-valued Moisil twist structure. Indeed, this is proved using induction on the length of the expressions of L_G and taking into account that these expressions are built from elements (disjunctions of conjunctions of elements) of $G \cup \bigcup_{i=1}^{n} \phi_i G \cup \bigcup_{i=1}^{n} \sim \phi_i G \cup \sim G$. On the other hand, the elements of $G \cup \bigcup_{i=1}^{n} \phi_i G \cup \bigcup_{i=1}^{n} \sim \phi_i G \cup \sim G$ satisfy (Tw0), (Tw1), (Tw2), (Tw3), (Tw4) and (Tw5); and taking into account the De Morgan Laws and the properties of the ϕ_i's we have that (Tw0), (Tw1), (Tw2), (Tw3), (Tw4) and (Tw5) hold for every expression. Therefore, we have $\langle G\rangle_{\mathcal{T}_n} \subseteq L_G$. $\square$

For every $i = 1, \ldots, n$, let $L_i = \langle \phi_i G\rangle_{\mathcal{L}}$. Then, it is clear that $\langle \sim \phi_i G\rangle_{\mathcal{L}} = L_i^{\star}$ and both, L_i and $L_i^{\star}$, are sublattices of $L = \langle G\rangle_{\mathcal{L}}$.
Let H be a lattice such that $\pi_0 \langle G\rangle_{\mathcal{T}_n} \subseteq H$. Consider the following sets

$$G'_0 = \pi_0(G),$$

$$G'_i = \pi_i(\phi_i G), \text{ for } 1 \leq i \leq n,$$

$$G'_{n+i} = \pi_i(\sim \phi_{n-i+1}G), \text{ for } 1 \leq i \leq n,$$

$$G'_{2n+1} = \pi_0(\sim G).$$

Then, the following result constitute a link between the free $n+1$-valued Moisil twist structure and its base lattice.

Proposition 5.2 *If* $H = \langle \bigcup_{i=0}^{2n+1} G'_i \rangle_{\mathcal{L}}$ *then* $H = \pi_0 \langle G \rangle_{\mathcal{T}_n}$.

Proof. Let's see that $H \subseteq \pi_0\langle G\rangle_{\mathcal{T}_n}$. It is enough to see that $\bigcup_{i=0}^{2n+1} G'_i \subseteq \pi_0\langle G\rangle_{\mathcal{T}_n}$. Suppose that $x \in \bigcup_{i=0}^{2n+1} G'_i$. If $x \in G'_0$, then $x \in \pi_0(G)$. Since $G \subseteq \langle G\rangle_{\mathcal{T}_n}$, we have that $\pi_0 G \subseteq \pi_0\langle G\rangle_{\mathcal{T}_n}$. Therefore, $x \in \pi_0\langle G\rangle_{\mathcal{T}_n}$. If $x \in G'_i = \pi_i(\phi_i G) \subseteq \pi_0(\phi_i G) \subseteq \pi_0(G)$, so $x \in \pi_0\langle G\rangle_{\mathcal{T}_n}$.
The other cases are analogous. $\square$

Now we are going to exhibit a way to build free $n+1$-valued Moisil twist structures by means of free distribuitve bounded lattices.
Let m be a denumerable cardinal number and let $\mathbf{Free}_L(K)$ the free bounded distributive lattice generated by a set $K = \{g_i\}$ of size $(2n+2) \times m$ if m is finite or ω otherwise. Consider the following sets for $1 \leq i \leq n$.

$$K_0 = \{g_j \in K : j \equiv 0 \,(\mathrm{mod}\;\; 2n+2)\},$$

$$K_i = \{g_j \in K : j \equiv i \,(\mathrm{mod}\;\; 2n+2)\},$$

$$K_{n+i} = \{g_j \in K : j \equiv (n+i) \,(\mathrm{mod}\;\; 2n+2)\},$$

$$K_{2n+1} = \{g_j \in K : j \equiv (2n+1) \,(\mathrm{mod}\;\; 2n+2)\}.$$

It is clear that $K_k \cap K_l = \emptyset$ if $k \neq l$ and $K = \bigcup_{k=0}^{2n+1} K_k$. Besides, $|K_0| = \cdots = |K_{2n+1}| = m$. Let

$$G = \{g = (g_{(2n+2)r_g}, g_{(2n+2)r_g+1}, \ldots, g_{(2n+2)r_g+(2n+1)}) \in \bigotimes_{j=0}^{2n+1} K_j : \text{ for some } r_g \in \mathbb{N}, 0 \leq r < m\},$$

and let G'_i as above for $1 \leq i \leq 2n+2$.

Remark 5.1 *$|G| = m$ and for every $g_i \in K$ there exists one, and only one, $g \in G$ such that g_i is a coordinate of g.*

Consider now the product

$$\langle K\rangle_{\mathcal{L}} \times \bigotimes_{i=1}^{2n} \langle K_i\rangle_{\mathcal{L}} \times \langle K\rangle_{\mathcal{L}}^{\star},$$

and let $\langle G\rangle_{\mathcal{T}_n}$ the $n+1$-valued Moisil twist structure generated by G in this product.

Theorem 5.1 *If $\langle K\rangle_{\mathcal{L}}$ is the free distributive bounded lattice generated by K, then $\langle G\rangle_{\mathcal{T}_n}$ is the free $n+1$-valued Moisil twist structure generated by G.*

Proof. Let $f : G \to M$, where $M \subseteq \bar{L} \times \bigotimes_{i=1}^{n} \bar{L}_i \times \bigotimes_{i=0}^{n+1} \bar{L}_{n-i}^{\star} \times \bar{L}^{\star}$ is an arbitrary $n+1$- valued Moisil twist structure. Then f induces a function $\bar{f}$ from K in $\bar{L}$ in the following way:

$$\bar{f}(g_i) = \pi_j(f(g))$$

where g is the unique element of G such that g_i is one of its coordinates and j is such that $0 \leq j < 2n+1$ and $i \equiv j \pmod{2n+2}$.
By Remark 5.1, we know that $\bar{f}$ is well defined. Then, there exists a unique homomorphism $\bar{h} : \langle K\rangle_{\mathcal{L}} \to \bar{L}$ that extends $\bar{f}$. Let h be the function defined as follows: if $x = (x_0, \ldots, x_{2n+1}) \in \langle G\rangle_{\mathcal{T}_n}$ then

$$h(x) = (\bar{h}(x_1), \ldots, \bar{h}(x_{2n+1}))$$

It is clear that h is well defined and that it is a homomorphism. Let's see that h extends f on G. Let $g \in G$,

$$\begin{array}{rcl} h(g) & = & h((g_{(2n+2)r_g}, g_{(2n+2)r_g+1}, \ldots, g_{(2n+2)r_g+(2n+1)})) \\ & = & (\bar{h}(g_{(2n+2)r_g}), \bar{h}(g_{(2n+2)r_g+1}), \ldots, \bar{h}(g_{(2n+2)r_g+(2n+1)})) \\ & = & (\bar{f}(g_{(2n+2)r_g}), \bar{f}(g_{(2n+2)r_g+1}), \ldots, \bar{f}(g_{(2n+2)r_g+(2n+1)})) \\ & = & (\pi_0(f(g)), \pi_1(f(g)), \ldots, \pi_{2n+1}(f(g))) \\ & = & f(g) \end{array}$$

Finally, we need to verify that $h(\langle G\rangle_{\mathcal{T}_n}) \subseteq M$. Consider the following set

$$S = \{x \in \langle G\rangle_{\mathcal{T}_n} : h(x) \in M\}$$

Then, (1) $G \subseteq S$ since h extends f; and (2) S is a sub-algebra of $\langle G\rangle_{\mathcal{T}_n}$ since h is a homomorphism. From (1) and (2), we have that $S = \langle G\rangle_{\mathcal{T}_n}$ and so $h(\langle G\rangle_{\mathcal{T}_n}) \subseteq M$ and then $h : \langle G\rangle_{\mathcal{T}_n} \to M$. That is to say that $\langle G\rangle_{\mathcal{T}_n}$ is the free $n+1$-valued Moisil twist structure with m generators. $\square$

Also, it can be proved that:

Proposition 5.3 *If $\langle K\rangle_{\mathcal{L}}$ is free, then $\langle K\rangle_{\mathcal{L}}$ and $\langle G\rangle_{\mathcal{T}_n}$ are isomorphic lattices.*

The above construction was made with the sole intention of showing that, due to the De Morgan laws, the complete distributivity of the ϕ_i's and other properties ($\phi_i\phi_j = \phi_j$), the expressions in the correspondent logic are precisely those that are obtained applying the connectives $\vee$, $\wedge$ to the propositional variables, their negations, the modal connectives and their negations to these same variables.

6 The propositional calculus $\mathcal{M}_{n+1}$

In this section, we introduce the calculus $\mathcal{M}_{n+1}$, based on the *positive intuitionistic propositional calculus*, that will be demonstrated to be sound and complete with respect to $(n+1)$-valued Moisil twist structures. This calculus contains the operators ϕ_i whose intended meaning is that of the graduation of the possibility of a given proposition.
The formulas of this calculus are built up in the usual way from the countable set $Var = \{p_i\}_{i\in I}$ of propositional variables by means of the connectives $\rightarrow$ (implication), $\vee$ (disjunction), $\wedge$ (conjunction), $\sim$ (negation) and ϕ_i, for $i \in J$ (modal operators). With **Fm** we denote the term algebra in this language, i.e., $\mathbf{Fm} = \langle Fm, \rightarrow, \vee, \wedge, \sim, \{\phi_i\}_{i\in I}\rangle$.
The connective $\leftrightarrow$ (equivalence) is defined in the usual way

$$p \leftrightarrow q \equiv_{def} (p \rightarrow q) \wedge (q \rightarrow p)$$

We shall eliminate parenthesis following the usual convention that connectives $\sim, \phi_1, \ldots, \phi_n$ binds more strongly than either $\wedge$ or $\vee$ and each of them, in turn, binds more strongly that either $\rightarrow$ or $\leftrightarrow$.
The following are the axiom schemas of $\mathcal{M}_{n+1}$ (where $\alpha, \beta, \gamma \in Fm$ denote formulas):

(A1) $\alpha \rightarrow (\beta \rightarrow \alpha)$,

(A2) $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$,

(A3) $\alpha \rightarrow \alpha \vee \beta$,

(A4) $\alpha \rightarrow \beta \vee \alpha$,

(A5) $(\alpha \rightarrow \beta) \rightarrow ((\gamma \rightarrow \beta) \rightarrow (\alpha \vee \gamma \rightarrow \beta))$,

(A6) $\alpha \wedge \beta \rightarrow \alpha$,

(A7) $\alpha \wedge \beta \rightarrow \beta$,

(A8) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta \wedge \gamma))$,

(A9) $\alpha \leftrightarrow \sim\sim \alpha$, (A10i) $\phi_i(\alpha \vee \beta) \to \phi_i\alpha \vee \phi_i\beta$, $i \in J$,

(A11i) $\phi_i\alpha \vee \sim \phi_i\alpha$, $i \in J$,

(A12i) $\phi_i\phi_j\alpha \leftrightarrow \phi_j\alpha$, $i \in J$,

(A13i) $\phi_i \sim \alpha \leftrightarrow \sim \phi_n\alpha$, $i \in J$,

(A14i) $\phi_i\alpha \to \phi_{i+1}\alpha$, $i \in J$,

(A15i) $\alpha \to \phi_n\alpha$, $i \in J$,

(A16i) $\alpha \wedge \sim \phi_i\alpha \wedge \phi_{i+1}\beta \to \beta$, $i \in J$,

The rules of $\mathcal{M}_{n+1}$ are the following:

$$\text{(MP)}\frac{\alpha,\ \alpha \to \beta}{\beta} \qquad \text{(R2)}\frac{\alpha \to \beta}{\sim \beta \to \sim \alpha} \qquad \text{(R3i)}\frac{\alpha \to \beta}{\phi_i\alpha \to \phi_i\beta}, i \in J$$

As it is usual, we shall say that $\alpha \to \beta$ or $\alpha \leftrightarrow \beta$ are provable in $\mathcal{M}_{n+1}$, denoted $\vdash_{n+1} \alpha \to \beta$ or $\vdash_{n+1} \alpha \leftrightarrow \beta$, if they belong to the least set of formulas that contain the axioms listed above and that is closed by the rules (MP), (R2) and (R3i).

Proposition 6.1 *The following formulas are provable in* $\mathcal{M}_{n+1}$

(i) $\vdash_{n+1} \sim (\alpha \vee \beta) \leftrightarrow \sim \alpha \wedge \sim \beta$,

(ii) $\vdash_{n+1} \phi_i\alpha \vee \phi_i\beta \to \phi_i(\alpha \vee \beta)$.

We say that $\alpha \in Fm$ is valid (in the $(n+1)$-valued Moisil twist structures), noted $\models \alpha$, iff for every $(n+1)$-valued Moisil twist structures N and every $h \in Hom(Fm, N)$ we have that $h(\alpha) = 1$. By the usual methods it can be proved the soundness and completeness of this logic with respect to $(n+1)$-valued Moisil twist structures. Indeed, first, we define the equivalence relation $\equiv$ on the set of formulas as $\alpha \equiv \beta$ iff $\vdash_{n+1} \alpha \leftrightarrow \beta$. Then, it can be proved that this relation is compatible with all the connectives of the logic $\mathcal{M}_{n+1}$. Therefore, it is possible to define the $(n+1)$-valued Moisil twist structures L^{n+1}, the Lindenbaum algebra of $\mathcal{M}_{n+1}$, considering $(2n+2)-$tuples in the following way:

$$(|\alpha|, |\phi_1\alpha|, \ldots, |\phi_n\alpha|, |\sim \phi_n\alpha|, \ldots, |\sim \phi_1\alpha|, |\sim \alpha|)$$

where $|\alpha|$ is the equivalence class of the formula α. So,

Theorem 6.1 *Let* $\alpha \in Fm$. *The following conditions are equivalent:*

(i) $\vdash_{n+1} \alpha$,

(ii) $\models \alpha$.

It is worth mentioning that in the proof of Theorem 6.1, it is only necessary to use properties of distributive lattices.
On the other hand, it can be showed that L^{n+1} is the free $(n+1)$-valued Moisil twist structures with $\aleph_0$ generators which is the cardinal of the set of propositional variables. Besides, $\mathcal{T}_{n+1}$ is the characteristic matrix of the calculus $\mathcal{M}_{n+1}$, i.e., $\vdash_{n+1} \alpha \to \beta$ iff $|\alpha| \leq |\beta|$.

As a consequence of Theorem 4.3 we have that the chain $\mathcal{T}_{n+1}$ is a characteristic matrix of $\mathcal{M}_{n+1}$ and therefore:

Corollary 6.1 $\mathcal{M}_{n+1}$ *is decidable.*

7 Relational Semantic Models

In this section, we shall introduce relational semantic models for the logic $\mathcal{M}_m$ *à la* Kripke. This will show that some conditions are fulfilled in such a way that the operators ϕ_i can be seen as modal.

These models are different from the ones introduced in [4].

Definition 7.1 *A Moisil* $n+1$*-structure is a poset* $\langle K, \leq \rangle$ *where* $K = \bigcup_{i=1}^{m} K_i$ *with* $m \geq 1$ *and* K_i *(*$1 \leq i \leq m$*) is a chain with at most* n *elements. Each element of any chain will be called a chain link and* K *will be said a set of possible worlds.*

To sum up, a Moisil $n+1$–structure is the cardinal sum of chains with at most n links.
Now, we shall introduce the concept of semantic models for the $\mathcal{M}_{n+1}$.

Definition 7.2 *A Moisil model of order* $n+1$ *is a system* $\langle K, \leq, a \rangle$ *where* $\langle K, \leq \rangle$ *is one chain of one Moisil* $n+1$*-structure. We shall assume that* $K = \{e_k\}_{k=1}^{r}$ *for some* $1 < r \leq n$ *and that* $k_i \leq k_j$ *iff* $i \leq j$*. An assignment is a map* $a = \langle a_k^+, a_k^- \rangle$ *for* $1 \leq k \leq r$ *formed by a pair of applications from the product* $K \times Var$ *into the set* $\{0, 1\}$ *where* Var *is the set of propositional variables.*
There is a unique extension $v = \langle v_k^+, v_k^- \rangle$ *of* a *such that* v *is a homomorphism from* $K \times Fm$ *to* $\{0, 1\}$*. We shall say that* v *is a valuation. This extension can be constructed inductively in the following way:*

(V1) $v_k^+(p) = a_k^+(p)$ *and* $v_k^-(p) = a_k^-(p)$ *where* p *is a propositional variable.*

(V2) $v_k^+(p \vee q) = 1$ *iff* $v_k^+(p) = 1$ *or* $v_k^+(q) = 1$. *Otherwise,* $v_k^+(p \vee q) = 0$.
$v_k^-(p \vee q) = 0$ *iff* $v_k^-(p) = 0$ *and* $v_k^-(q) = 0$. *Otherwise,* $v_k^-(p \vee q) = 1$.

(V3) $v_k^+(p \wedge q) = 1$ *iff* $v_k^+(p) = 1$ *and* $v_k^+(q) = 1$. *Otherwise,* $v_k^+(p \wedge q) = 0$.
$v_k^-(p \wedge q) = 0$ *iff* $v_k^-(p) = 0$ *or* $v_k^-(q) = 0$. *Otherwise,* $v_k^-(p \wedge q) = 1$.

(V4) $v_k^+(\sim p) = 1$ *iff* $v_k^+(p) = 0$. *Otherwise,* $v_k^+(\sim p) = 0$.
$v_k^-(\sim p) = 0$ *iff* $v_k^-(p) = 1$. *Otherwise,* $v_k^-(\sim p) = 1$.

(V5) $v_k^+(\phi_i p) = 1$ *iff* $v_r^+(p) = 1$ *for all* $r \in \{i, \ldots, n\}$ *iff* $v_r^+(p) = 1$ *for some* $r \in \{1, \ldots, i\}$. *Otherwise,* $v_k^+(\phi_i p) = 0$.
$v_k^-(\phi_i p) = 0$ *iff* $v_r^-(p) = 0$ *for all* $r \in \{i, \ldots, n\}$ *iff* $v_r^-(p) = 0$ *for some* $r \in \{1, \ldots, i\}$. *Otherwise,* $v_k^-(\phi_i p) = 1$.

(V6) $v_k^+(\sim \phi_i p) = 1$ *iff* $v_k^+(\phi_i p) = 0$. *Otherwise,* $v_k^+(\sim \phi_i p) = 0$.
$v_k^-(\sim \phi_i p) = 0$ *iff* $v_k^-(\phi_i p) = 1$. *Otherwise,* $v_k^+(\sim \phi_i p) = 1$.

(V7) $v_k^+(p \to q) = 1$ *iff* $v_k^+(p) = 1$ *implies* $v_k^+(q) = 1$. *Otherwise,* $v_k^+(p \to q) = 0$.
$v_k^+(p \to q) = 0$ *iff* $v_k^+(p) = 0$ *implies* $v_k^+(q) = 0$. *Otherwise,* $v_k^+(p \to q) = 1$.

Recall that in (V6) $\sim$ is the Boolean complement.

Definition 7.3 *We shall say that the formula* $\alpha \to \beta$ *is valid, noted* $\models_n \alpha \to \beta$ *if and only if for every Moisil* $n+1$*-structure; every valuation* $v = (v_k^+, v_k^-)_{k=1}^n$*; and every* k,

$$(v_k^+(\alpha), v_k^-(\alpha)) = (1, 0) \text{ implies } (v_k^+(\beta), v_k^-(\beta)) = (1, 0)$$

Taking into account (V5) and Kripke's interpretation of modal operators we can interpret ϕ_1 as a necessity operator, ϕ_n as a possibility operator and ϕ_i for $1 < i < n$ as some intermediate graduation between necessity and possibility.

Lemma 7.1 *Let* $\alpha \in Fm$. *If there is a formal proof for* α *in* $\mathcal{M}_{n+1}$*, i.e.* $\vdash_{n+1} \alpha$*, then* $\models_{n+1} \alpha$.

Proof. It is not difficult to see that all axioms of $\mathcal{M}_n$ are valid and that the rules preserve validity in every Moisil $n+1$-structure. As an example, we shall see that (A16i) is valid. Suppose that $v_k^+(\alpha \wedge \sim \phi_i \alpha \wedge \phi_{i+1}\beta) = 1$ and $v_k^+(\beta) = 0$ in some chain of some Moisil $n+1$-structure. Then, by (V4), $v_k^+(\alpha) = 1$, $v_k^+(\phi_i \alpha) = 0$ and $v_k^+(\phi_{i+1}\beta) = 1$. If $k \leq i$, from $v_k^+(\alpha) = 1$, by (V5), we have that $v_k^+(\phi_i \alpha) = 1$ which is a contradiction. Then $k \not\leq i$, i.e., $k \geq i+1$, but then $v_k^+(\phi_{i+1}\beta) = 1$ contradicts $v_k^+(\beta) = 0$, by (V5). Analogously, we work with v^-. □

Reciprocally, every formula that is valid in every Moisil $n+1$-structure is also valid in the Moisil $n+1$-structure formed by the prime filters of the Lindenbaum algebra L^{n+1}; and therefore it is a theorem of $\mathcal{M}_{n+1}$.

In order to present a completeness theorem we are going to construct a canonical Moisil $n+1$-structure for the logic $\mathcal{M}_n$. This canonical model will verify the following fundamental property:

$$\vdash_{n+1} \alpha \to \beta \quad \text{iff} \quad \alpha \to \beta \text{ is valid in the canonical model.}$$

Let $\mathcal{P}_{L^{n+1}}$ the set of all prime filters of the of L^{n+1}. We already know that this set is a cardinal sum of chains with at most n elements. It can be seen without any difficulty that $\mathcal{P}_{L^{n+1}}$ is a Moisil $n+1$-structure.
Consider now the following assignment from one chain of $\mathcal{P}_{L^{n+1}}$:

$$a = (a_k^+(p), a_k^-(p)) = (1, 0) \text{ iff } (|p|, |\sim p|) \in k.$$

The classic argument here is the following: if α is not a theorem of $\mathcal{M}_{n+1}$ then $|\alpha| \neq 1$, then there is a prime filter that contains 1 but do not contain $|\alpha|$. This prime filter belongs to every canonical model obtained from L^{n+1}, where α is not valid. Therefore, we have:

Lemma 7.2 *Let $\alpha \in For$. If $\models_{n+1} \alpha$ then $\vdash_{n+1} \alpha$.*

8 Conclusions

We have apply successfully the technique of tuples to represent Łukasiewicz-Moisil algebras of order $n+1$ and in this way provide a very suitable semantic model to the $n+1$-valued logic $\mathcal{M}_{n+1}$.
We show that, when we are dealing with many valued logics, more than two axes in the product structure can be very suitable for describing the logic as well as the algebraic structure naturally associated to it. In many cases, these models have shown to be simpler, needing simpler technical work since the study is reduced to the axes which usually are well-known algebraic structures. Besides, it is semantically clearer when we are trying to determine the logical values of propositions.
We think that these ideas can be explored in order to provide new algebraic semantic models for a wide number of non–classical logics.

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