Models for a paraconsistent set theory

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Extended abstract

We know from Russell’s paradox that the first-order axiomatization of Cantor’s naive set theory is inconsistent in classical logic. More precisely, some peculiar sets \( \{ x \mid \varphi \} \) provided by specific \( \varphi \)-instances of the comprehension scheme, i.e. \( \forall x ( x \in \{ x \mid \varphi \} \iff \varphi ) \), lead to triviality if the underlying logic is classical. The most popular of those is so called Russell’s set : \( R = \{ x \mid x \notin x \} \).

Any classical solution to the paradoxes has to get rid of such ‘contradictory sets’. It appeared besides that these virtual sets are by no means essential for the foundations of mathematics, hopefully. However, certain logicians have expressed a desire that such inconsistent objects might be handled and studied within suitable theories, namely para(in)consistent ones. After all, non-well-founded sets, as the universal set \( V = \{ x \mid x = x \} \), which are no more indispensable for the foundations of mathematics, have subsequently found interesting applications in computer science.\(^1\)

There are many examples in mathematics where the introduction of ‘imaginary/ideal’ objects, though giving some advantage to deal with them, has also forced us to give up some basic properties or principles.\(^2\) Obviously, the price to be paid here concerns the logic in which the theory is embedded, and its possible debilitating effects on classical reasoning and mathematical practice. One could therefore argue that this price seems to outweigh the advantages.\(^3\) Nevertheless, we are going to show in this talk that it is possible to construct ‘natural models’ for a paraconsistent set theory.

\(^1\) The interested reader should consult Peter Aczel’s book : “Non-well-founded sets”, CSLI Lecture Notes Number 14, Stanford, 1988 (137 pages).

\(^2\) For instance, adding to the reals an ‘imaginary’ number \( i \) such that \( i^2 = -1 \) leads to sacrifice the structure of ordered field. As well as dealing with non well-founded sets prevents us from inductive definitions. Etc.

\(^3\) Actually, it should be noted here that there exists some standard methodology producing (para)inconsistent extensions of classical theories, so that nothing classical is lost.
What we call a \textit{model} for a set theory is any $\in$-structure satisfying some fragment of the full-comprehension scheme, in a logic to be specified. All models shall be considered and described within a pre-existing classical universe of sets (e.g. a model of $ZF$, plus some reasonable large cardinal assumption if necessary).\footnote{At least, we assume that the reader is familiar with $ZF$.} Note that we will use below a small `$\in$' to denote and distinguish the membership relation in the metatheory from the big `$\in$' in the language of models for set theories.

The purpose of this extended abstract is to outline some technique common to the construction of such models in classical and non-classical logics, showing how these non-classical models can be compared with classical ones. In the talk, we shall describe in detail and apply that technique to get models of some paraconsistent set theory. So we will start here with the classical case, introducing by the way a quite unusual but very natural response to Russell’s paradox: \textit{positive set theory}.\

\textbf{The classical case}

From the semantical viewpoint, any classical $\in$-structure $\mathcal{M} := \langle M; \in_M \rangle$ can be equally thought of as a non-empty set $M$ together with a function $[\cdot]_M : M \to \mathcal{P}(M)$, where $\mathcal{P}(M)$ denotes the set of all subsets of $M$, by the way of the following definition:

$$x \in_M y \iff x \in [y]_M,$$

for any $x, y \in M$.

Thus $[y]_M$ shall be called the \textit{extension} of $y$ in $\mathcal{M}$. In this setting, $\mathcal{M}$ is \textit{extensional} exactly when $[\cdot]_M$ is injective. On the other hand, the semantical translation of Russell’s paradox, namely Cantor’s theorem, tells us that $[\cdot]_M$ can never be surjective. More precisely, if $[M]_M$ denotes the range of $[\cdot]_M$, then some $\in$-definable subsets of $\mathcal{M}$, e.g. $\{x \in M \mid x \notin_M x\}$, never belong to $[M]_M$.

Anyway, it is a fact that very interesting models can be constructed by providing $M$ with some suitable structure and defining $[\cdot]_M$, in relation to that structure, onto a set of distinguished subsets of $M$.

A well-known and very simple example is given by the set of natural numbers $\mathbb{N}$, with its arithmetical structure, where, for each $n \in \mathbb{N}$, $[n]_\mathbb{N}$ is defined to be the set of those natural numbers which appear as exponent in the binary expansion of $n$. Explicitly, $[0]_\mathbb{N} = \emptyset$, $[1]_\mathbb{N} = \{0\}$, $\ldots$, $[11]_\mathbb{N} = \{0, 1, 3\}$, etc. Thus, $[\cdot]_\mathbb{N}$ is one-one and $[\mathbb{N}]_\mathbb{N}$ is the set of all \textit{finite} subsets of $\mathbb{N}$. In this way, it can be readily proved that $\langle \mathbb{N}; \in_\mathbb{N} \rangle$ is a model of Zermelo-Fraenkel set theory without infinity.
In $ZF$, the informal argument which shows that Russell’s paradox is blocked is that the well-founded part of the universe already fulfils the axioms of $ZF$.\footnote{Some authors include the axiom of foundation (regularity) in $ZF$. Notice that this axiom is not an instance of the comprehension scheme (nor is the axiom of extensionality).} Indeed, in this cumulative hierarchy, Russell’s set would be nothing but the universal class $V$, but this latter cannot be a set, otherwise $V \in V$. Incidentally, it is worth observing that the universe of the model $N$ described above is actually well-founded, seeing that $m \in_n n$ implies $m < n$ by definition. As a matter of fact, the universal class is clearly not a set \textit{in the sense of $N$} because $[n]_\nu$ is finite for any $n \in \mathbb{N}$ (that is precisely why the axiom of infinity fails to be satisfied in $N$).

There is another route to get around Russell’s paradox in classical logic, which is based on a radically different idea: just restrict comprehension to those \textit{positive} formulae in which classical negation does not occur ‘explicitly’.\footnote{We are deliberately not very explicit on that point. For precise references on ‘positive comprehension’, the reader should consult [For;Hin] or [Hin;Lib], where a brief historical account of the subject is given.} Thus the universal class $V$ should be a set, and this is a singular departure from the ‘limitation of size’ doctrine of $ZF$ and related set theories.

It was shown that very interesting models for so called \textit{positive set theories} can be defined as above by means of a topological space $M$ and a homeomorphism $[\cdot]_\nu$ from $M$ onto the set of closed subsets of $M$, endowed with the Vietoris topology. These topological models, called \textit{hyperuniverses}, can be constructed by \textit{projective limit}, in a quite similar way as Scott’s models for the lambda calculus.\footnote{Hyperuniverses are described in detail in [For;Hon].} The properties of these models give rise to a very interesting \textquoteleft topological\textquoteright{} set theory, which might be called $PF$ (for Positive Foundations) and which was axiomatized and deeply investigated by Olivier Esser in [Ess1], where he showed for instance that $PF$ interprets $ZF$.\footnote{Note that Esser called his theory ‘$GPK^\infty$’. Needless to say why we think $PF$ would have been luckier.} On this point, it should be remarked that a pertinent formulation of the axiom of infinity is actually required in $PF$ in order to recover it in $ZF$ (that is why a large cardinal assumption is necessary for the construction of hyperuniverses fulfilling this axiom). Incidentally, Esser also showed in [Ess2] that $PF$, with that pertinent axiom of infinity, proves the negation of the axiom of choice, whatever the formulation of this latter one adopts. As a similar result holds for Quine’s set theory $NF$, this seems to be a characteristic feature of a set theory with an universal set and infinity.\footnote{A comprehensive bibliography on set theories with a universal set can be found at the following address : http://math.boisestate.edu/~holmes/setbiblio.html.}
In any positive set theory, beside the universal class \( V \), the ‘complement’ of Russell’s class, that is \( R^c = \{ x \mid x \in x \} \), should be a set as well. So the next step is to deal with Russell’s set itself.

In terms of boolean operations, it is rather clear that this further step has to tamper with our conception of ‘complement’, for if \( R \) was a set in some model \( \mathcal{M} \), we would have to choose between \( R \in [R]_{\mathcal{M}} \cap [R^c]_{\mathcal{M}} \) and \( R \notin [R]_{\mathcal{M}} \cup [R^c]_{\mathcal{M}} \). In other words, from the logical viewpoint, \( \in \) and \( /\in \) might be conceived as weak negation of each other, and the underlying logic of the model might be paraconsistent or paracomplete.

We shall only focus on the paraconsistent case here, showing how such a weak negation can be incorporated into positive comprehension.

**The paraconsistent case**

One might intuitively think of a paraconsistent set \( S \) as an ordered pair of covering parts of the universe: the first part collecting those objects which are supposed to belong to \( S \), the second gathering those which are supposed not to belong to \( S \), and where it is now agreed that these two parts might have a non-empty intersection.

Such a ‘membership ambiguity’ can be easily concocted in the following way. Consider an universe \( \mathcal{U} \) which consists of a collection \( U \) of objects together with a topology on it, which might materialize some notion of ‘indiscernibility’ on \( U \) or something like that. Then, for \( x \in U \) and \( S \subseteq U \), we define

\[
\begin{align*}
{x \in_u S} & \iff x \in \overline{S} \\
{x \notin_u S} & \iff x \in U \setminus \overline{S} \\
\end{align*}
\]

(where \( \overline{\cdot} \) is the closure operator on \( U \)).

Note that \( x \in S \Rightarrow x \in_u S \), as well as \( x \notin S \Rightarrow x \notin_u S \). Also, \( x \in_u U \setminus S \iff x \notin_u S \). But it is now possible that both \( x \in_u S \) and \( x \notin_u S \), for some \( x \in U \) and \( S \subseteq U \), and therefore some kind of relative inconsistency may be observed. It is also clear that the consistent subsets of \( U \) are thus exactly the clopen subsets in \( \mathcal{U} \).

Of course, one cannot define a model for a (paraconsistent) set theory in this way, simply because the binary relations \( \in_u \) and \( \notin_u \) are subsets of \( U \times \mathcal{P}(U) \), not of \( U \times U \). However, the underlying idea is tied up with so called ‘topological models’ we are going to present.

For any set \( M \), let \( \mathcal{P}_p(M) \) denote the set of all ordered pairs \((A, B)\) of subsets of \( M \) such that \( A \cup B = M \).\(^{10}\) Then, any paraconsistent \( \in \)-structure can be conceived as a non-empty set \( M \) together with a function \([\cdot]_M : M \rightarrow \mathcal{P}_p(M)\).

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\(^{10}\) In the paracomplete case, this latter condition should be replaced by \( A \cap B = \emptyset \).
Since $\mathcal{P}(M) \subset \mathcal{P}(M) \times \mathcal{P}(M)$, it is convenient here to set $\langle \cdot \rangle_M = (\lceil \cdot \rceil_M, \lfloor \cdot \rfloor_M)$ and thus define, for any $x, y \in M$,

\[
\left\{ \begin{array}{ll}
x \in_M y & \iff x \in \langle y \rangle_M \\
x \notin_M y & \iff x \notin \langle y \rangle_M
\end{array} \right.
\]

Then we shall call $\langle y \rangle_M$ the extension of $y$ in $\mathcal{M}$, and $\lfloor y \rfloor_M$ its anti-extension. Naturally, $\mathcal{M}$ is said to be extensional if $\lceil \cdot \rceil_M$ is injective. Here again, Cantor’s theorem keeps $\lceil \cdot \rceil_M$ from being surjective.

Nevertheless, it is possible to construct by projective limit a suitable topological space $M$ together with a bijective function $\lceil \cdot \rceil_M$ from $M$ onto the set of all ordered pairs of covering closed subsets of $M$. Thus, in such a structure $\mathcal{M}$, any covering pair of closed subsets of the universe $M$ really becomes identified with an $\mathcal{M}$-set. To express a straightforward consequence of this, let us adopt the following definition:

for any $x, y \in M$, $x$ is said to be less inconsistent than $y$ if

\[
\left\{ \begin{array}{ll}
\lceil x \rceil_M \subseteq \lceil y \rceil_M \\
\lfloor x \rfloor_M \subseteq \lfloor y \rfloor_M
\end{array} \right.
\]

It follows that for each covering pair $(A, B)$ of subsets of $M$, there exists a minimal inconsistent $\mathcal{M}$-set $y$ such that $A \subseteq \lceil y \rceil_M$ and $B \subseteq \lfloor y \rfloor_M$.

Indeed, it is nothing but the (unique) $\mathcal{M}$-set $y$ such that $\langle y \rangle_M = (A, B)$.

Let us just mention here that the $\mathcal{M}$-sets define a ‘paraconsistent boolean algebra’ (in the sense of [daCos;Bue]), in which the classical part coincides with the boolean algebra of clopen subsets of $M$. Furthermore, it can be shown that the class of hereditarily classical, well-founded $\mathcal{M}$-sets yields a model of $ZF$...

In the talk, we shall sketch the construction of such structures and show that these are extensional models of a ‘maximal’ fragment of the comprehension scheme in a ‘natural’ three-valued logic.\textsuperscript{11} Incidentally, we will point out another way of formalizing set theory, namely by using set abstracts in the language, as in [Bra;Rou]. To compare this latter with our approach, we will show the incompatibility of extensionality and set abstracts with equality in the language. Finally, we shall also introduce and discuss the paraconsistent counterpart of the theory $PF$.

\textbf{References}


\textsuperscript{11} Details can be found in [Hin] & [Lib].

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(available at http://homepages.ulb.ac.be/~oesser)


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