Paraconsistent Logics and Paraconsistency: Technical and Philosophical Developments*

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> "I predict a time when there will be mathematical investigations of calculi containing contradictions, and people will actually be proud of having emancipated themselves from contradictions."
>
> L. Wittgenstein

> "As a lightning clears the air of unpalatable vapors, so an incisive paradox frees the human intelligence from the lethargic influence of latent and unsuspected assumptions. Paradox is the slayer of Prejudice."
>
> J. J. Sylvester

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*The philosophical counterpart of this paper is not included in this first draft. Criticisms are welcome.

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1 Introduction

Saying in a few words, paraconsistent logics (PL) are the logics that can be the logics of inconsistent but non-trivial theories. A deductive theory is paraconsistent if its underlying logic is paraconsistent. A theory is inconsistent if there is a formula (a grammatically well formed expression of its language) such that the formula and its negation are both theorems of the theory; otherwise, the theory is called consistent. A theory is trivial if all formulas of its language are theorems. Roughly speaking, in a trivial theory 'everything' (expressed in its language) can be proved. If the underlying logic of a theory is classical logic, or even any of the standard logical systems like intuitionistic logic, inconsistency entails triviality, and conversely. So, how can we speak of inconsistent but non trivial theories? Of course by exchanging the underlying logic to one which may admit inconsistency without making the system trivial. Paraconsistent logics do the job.

Let us remark that our use of terms like 'consistency', 'inconsistency', 'contradictory' and similar ones is syntactical, what is in accord with the original metamathematical terminology of Hilbert and his school. On the other hand, in order to treat such terms from the semantical point of view, in the field of paraconsistency, one must be able to build, for instance, a paraconsistent set theory beforehand. This is possible, as we shall see, although most semantics for paraconsistent logics are classical, i.e., constructed inside classical set theories. So, the best, to begin with, is to employ the above terms syntactically.

1.1 The origins

The origins of paraconsistent logics go back to the first systematic studies dealing with the possibility of rejecting or of restricting the law (or principle) of contradiction, which (in one of its possible formulations) says that a formula and its negation cannot be both true. The law of contradiction is one of the basic laws of traditional, or classical (Aristotelian) logic. This principle is important for –since inconsistency entails triviality– an inconsistent set of premises yields any well-formed statement as a consequence. The result is that the set of consequences of an inconsistent theory or set of premises will explode into triviality and the theory is rendered useless.

Another way of expressing this fact is by saying that under classical logic the closure of any inconsistent set of sentences includes every sentence. It is this which lies behind Popper's famous declaration that the acceptance of inconsistency "... would mean the complete breakdown of science" and that an inconsistent system is ultimately uninformative.[1] Anyway, inconsistencies appear in various levels of discussion on science and philosophy. For instance, Peirce's world of 'signs' (in which we inhabit), is an inconsistent and incomplete world. Bohr's theory for the atom is one of the known examples in science of an inconsistent theory; the old quantum theory of black-body radiation, Newtonian cosmology, the (early) theory of infinitesimals in calculus, the Dirac δ-function, Stokes analysis of pendulum motion, Michelson's 'single-ray' analysis of the Michelson-Morley interferometer arrangement, among others, could also be recalled as cases when inconsistencies appear in science. Taken this for granted, it seems quite obvious that we should not eliminate a priori inconsistent theories, but rather to investigate them. In this context, paraconsistent logics acquire an important and fundamental role within science itself, so as in its philosophy. As we shall see below, due to the wide field of applications which nowadays have been found for these logics, they have also an important role in applied science as well.

The forerunners of paraconsistent logics are Jean Łukasiewicz and Nicolai I.

[1] For further details and references, see [95, Chap. 5].

Vasiliev. Independently of each other, both suggested in 1910 and 1911 that 'non-Aristotelian' logics could be achieved by rejecting the law of contradiction.[2] Although Łukasiewicz did not construct any system of paraconsistent logic, his ideas on the principle of contradiction in Aristotle influenced his student S. Jaśkowski in the construction of 'discussive' (or 'discursive') logic in 1948. We shall comment on Jaśkowski's systems below. In 1911, 1912 and 1913, inspired in the works of Lobachewski on non-Euclidian geometry, initially called 'imaginary geometry', Vasiliev envisaged an *imaginary logic*, a non-aristotelian logic where the principle of contradiction was not valid in general. According to Arruda, Vasiliev did not believe that there exist contradictions in the real world, but only in a possible world created by the human mind. Thus he hypothesized imaginary worlds where the aristotelian principles could not be valid, although Vasiliev has not not developed his ideas in full.[3]

The very first logician to construct a formal system of paraconsistent logic was Stanislaw Jaśkowski in 1948. His motivations came from his interests in systematizing systems which contain contradictions, such as dialectics, so as the study of theories where contradictions are caused by vagueness. He was also interested in the study of empirical theories whose postulates include contradictory assumptions (see section 4). Notwithstanding this wide field of possible applications, Jaśkowski's discussive logic was restricted to the propositional level. In 1958, the first author of this paper, independently of Jaśkowski, began the general study of contradictory systems [63]. Ever since then, da Costa has developed several systems related to paraconsistency (for instance, 'paraclassical logic' –see section 7.3), showing how to deal with inconsistencies from different perspectives, apparently becoming the first logician to really develop strong logical systems involving contradictions which could be useful for substantive parts of mathematics and the empirical and human sciences. By the way, it should be remarked that the adjective 'paraconsistent' (something like 'at the side of consistency') was suggested by F. Miró-Quesada in 1976, in a letter to da Costa.

Already in the sixties, the interest in logics dealing with inconsistencies began in other parts of the world as well, as in Poland, Australia, the United States, Italy, Argentina, Belgium, Ecuador, Peru, mainly by its relationships with da Costa's logics and with relevant and dialectical logic. Of course in this paper we cannot make reference neither to all these tendencies nor to make justice to all the involved authors. For historical details, we suggest the reading of the above mentioned papers. Anyway, it should be remarked at least two facts which contribute to emphasize the relevance of these developments. The first is that in 1990, Mathematical Reviews added a new entry, 03B53, termed 'Paraconsistent Logic'. From 2000 on, the title was exchanged to 'Logics admitting inconsistency (paraconsistent logics, discussive logics, etc.)', so involving a wider subject. The second fact is that ever since 1996, there have been orgainzed the World Congresses on Paraconsistency.[4] Nowadays 'paraconsistency' can be regarded as a field of knowledge. But perhaps the most surprising fact concerning paraconsistent logic is related to its applications. As we shall comment in between the text, there are applications not only into the foundation and philosophical analysis of scientific contexts, but even in technology. Here we have no space for details, but the references contain the original sources.

[2]For further historical details on PL, see [15], [16], [17], [46], [100], [121], [209], [138], [107].

[3]Arruda systematized some points on Vasiliev's imaginary logic, given raise to three systems of paraconsistent logic [14]. Nowadays, the logical works of Vasiliev have been studied in Russia, specially by V. A. Bazhanov.

[4]See `www.cle.unicamp.br/wcp3` for the page of the third congress.

1.2 On the nature of logic

If one wishes to understand the meaning and nature of logic, it is important to take into account that logic is today a field of knowledge of the same nature of mathematics. The results achieved in logic can be compared to those of the empirical sciences and even to mathematics in deepness and originality (let us just mention Gödel's theorems, the results in recursion theory and in the theory of models). So, as in mathematics, we can divide its field into two domains: pure logic and applied logic. 'Pure' logic, as pure mathematics, can in principle be developed *in abstracto*, independently of possible applications. So, we can study paraconsistent logic or intuitionistic logic by themselves, basically with the aim of exploring their abstract mathematical properties. From this point of view, in developing a logical system, the logician can proceed as Hilbert suggested, when he said that "[t]he mathematician [or the logician] will have to take account not only of those theories that come near to reality, but also, as in geometry, of all logically possible theories" [142]. To sum up, from the pure viewpoint, logic studies certain abstract structures, such as formal languages, models, and Turing machines.

Then, following Hilbert's suggestion, we might develop abstract (pure) systems where some principle of classical logic is violated, for instance the principle which entails that from contradictory premises any formula can be derived, in symbols, $\alpha \wedge \neg\alpha \vdash \beta$ (the corresponding law, $(\alpha \wedge \neg\alpha) \rightarrow \beta$, is Duns Scotus Law, valid not only in classical logic, but in almost all the known logical systems, like intuitionistic logic).[5] This is the way taken by Vasiliev in the construction of his imaginary logics.

But in developing logic we can also proceed from the *applied* point of view, looking at some domain of knowledge where our intuition feels that some logic (in particular a paraconsistent one) could be useful for describing certain abstract structures which might reflect the way certain deductive inferences are made within such a domain. One of the best known examples is provided by Birkhoff and von Neumann's approach to quantum logic, in saying that quantum mechanics would demand a logic distinct from the classical one, giving raise to a wide field of investigation, termed 'quantum logic' (see [117]).

Either defending positions like that one, or even by simple misunderstanding, sometimes we find someone saying that non-classical logics need to be developed *because* classical logic is wrong and that it must be replaced by a suitable one (in accordance with some philosophical criterion) [139, p. 1]. This would be the case, for example, of intuitionistic Brouwer-Heyting logic, if we consider it as a culmination of Brouwer's original philosophy of mathematics. Brouwer's stance implies that, in a certain sense, classical mathematics has basic shortcomings and that a constructive mathematics should take its place; the underlying logic of this constructive mathematics being a new one, different from the classical. In particular, it might be guessed that, in domains involving inconsistency (if really there are any), some other logic is to be used instead of classical logic. Nowadays, there are also philosophers, some of all who strongly believe that in these fields classical logic should be replaced by another logic (most of them think that the good logic would be a relevant one).

But this opinion regarding the rejection of classical logic does not fit ours. We think that classical logic is a fantastic subject which has and will continue to have strong interest and applications. The only problem is that in some specific domains, other logics, in particular paraconsistent logics, may be more adequate for expressing certain philosophical or even technical reasons so as to make explicit some of the underlying structures which (apparently) fit more adequately what is being assumed in these fields, since classical logic (apparently) can't do that in full. This does

[5] Some authors attribute this principle, also known as 'the principle of explosion', to Pseudo Scotus.

not show that classical logic is wrong, but that its area of application should be restricted. At least, the use of non-classical logics in systematizing certain domains helps us to better understand important aspects of these domains.

An important question is that of the nature of negation, which has been better understood with the rise of PL (another question is the significance of Russell's set –see section 3). Furthermore, it should be recalled that PL (in our approach) keeps classical logic valid in its particular domain of application, as we shall see below. Really, in this sense PL can be viewed not only as a 'heterodox' logic (or 'rival' logic [139], that is, as a logic which deviate from classical logic in what respects some of its principles), but also as a *supplement* to classical logic, for it coincides, in certain paraconsistent logics, with this one if we take into consideration just what are called 'well-behaved propositions' (roughly, those propositions that obey the principle of contradiction). In short, and we hope this can be put definitively, at least according to our point of view, we don't intend to pray according to PL rules. PL may be useful in some domains, as shown below, but we shall continue to use classical logic, or other logics, when we find they are convenient or necessary.

So, we are inclined to agree at least partially with a tendency derived from Gonseth [129, Chap. 8]), by sustaining that (applied) logic has an empirical counterpart. Nevertheless, we have some observations to make on Gonseth's view. First, taking into consideration our distinction between pure and applied logic, it is not necessary to eliminate the a priori traces of logic, as he apparently do (according to him, "logic is the science of the object whatever"). This does not imply that we are endorsing the position that there is just one logic and that it is independent of whatever field of knowledge. According to our opinion, logic *can* be studied independently of any application, as a pure mathematical system, hence being a priori in certain sense. So, even an applied logical system possesses an a priori dimension, in addition to its a posteriori one. For instance, we could begin by studying a logical system (like some quantum logic), motivated by the empirical science, by verifying whether this domain can be axiomatized, afterwards proving a completeness theorem for the resulting system, and so on. From another point of view, logic deals with the underlying structures of inference in particular domains or theories, and in this sense a particular field (like the quantum world, to pursue the example) may suggest that a different logic (that is, other than classical logic) could be useful to cope with certain features which cannot be dealt with by means of classical logic. As an instance, if we accept the view (advanced by E. Schrödinger, M. Born and others) that quantum objects are *non-individuals*, having no individuality in the sense that one is always indistinguishable of any other of a similar species, then it seems that in looking at the quantum world as constituted by entities of this kind, classical logic (with its Leibniz's Principle of the Identity of Indiscernibles) and classical mathematics (founded on the very notion of set, that is, collections of *distinguishable* objects), should be revised to deal with entities which can be regarded as 'individuals' from one point of view (precisely when the apparatuses are prepared to work with particles), and are not in others (when waves are taken into consideration; concerning these points, see [126]). So, different 'perspectives' of a domain of science may demand for distinct logical tools, which put us on a philosophical point of view very different from the classical.

However, let us insist, the possibility of using non-standard systems does not necessarily entail that classical logic is totally wrong, or that domains like quantum theory *need* at the moment another logic. Physicists and other scientists probably will continue to use classical (informal) logic in the near future. But we should realize that other forms of logic may help us in the better understanding of certain features of these domains, which are not easily treated by classical means, as the concepts of complementarity and of non-individuality in the quantum domain show (ibid.), [97], [98].

We think that there is not just one 'true logic', for distinct logical (so as mathematical and perhaps even physical) systems can be useful to approach different aspects of knowledge. If we push this view a little bit, although we shall not develop this philosophical point here, we could say that our philosophical position may be called pluralist (but not relativist). In what concerns PL, we make the claim, with Granger [135], that paraconsistent logic can and should be employed in some developments, but only as a preliminary tool; in future researches, classical logic could finally be a substitute for it, as the underlying logic of those developments. Our position does not excludes Granger's.

In synthesis, there are in principle various 'pure' logics whose potential applications depend not only on a priori and philosophical reasons, but, above all, on the nature of the applications one has in mind. This is true also with respect to paraconsistent logic.

This paper is organized as follows. In the next section, we present da Costa's $\mathcal{C}$-logics. Then we turn to paraconsistent set theories, by showing in particular how to deal with things like Russell's set. Jaśkowski's discussive logic, so as its application to the concept of partial truth, is then introduced. Annotated logic is developed here from one of its possible ways of approaching. Since these logics have been applied so extensively nowadays, these applications are here only touched upon, but references are given. The limitations of space enable us neither to develop nor to mention all the related paraconsistent systems which have been presented to the literature in recent times. We hope that all the 'paraconsistent people' in all around the world, whose list of names would occupy several pages, will excuse us for neither mentioning them nor their works. It would be impossible to do here justice to all developments of PL and the corresponding applications; so, we limit ourselves to expound only the ideas and results with which our work is more related.

2 $\mathcal{C}$-logics

In this section we shall study a class of logics termed $\mathcal{C}$-logics, which show that it is possible to elaborate strong paraconsistent logical systems. In particular, within some of these systems, it is possible to build set theories and paraconsistent mathematics such that they contain standard mathematics. So, we can say that certain paraconsistent logico-mathematical systems enlarge the scope of traditional mathematics; so it is possible to construct strong inconsistent systems without the immediate danger of trivialization. Really, there is no difficulty of reproducing within these systems the usual theories of logic and of mathematics.

We begin with the propositional paraconsistent logic and, little by little, we show how it is possible to get paraconsistent set theories and paraconsistent mathematics.

The way we shall approach the subject will be that of pure mathematics, in the same way as group theory or projective geometry of several dimensions are developed. The significance of all of this for the 'real world', that is, the possible applications of the resulting systems, shall be discussed throughout the paper.

2.1 The propositional calculi $\mathcal{C}_n$

Our initial goal is to develop propositional calculi that can be the base of inconsistent but non trivial theories.

We recall that a theory T, whose underlying logic is L and whose language is $\mathcal{L}$ is *inconsistent* if there is a formula α such that both α and $\neg\alpha$ (the negation of α) are theorems of T; otherwise, T is *consistent*. T is *trivial* if all formulas of $\mathcal{L}$ are theorems of T; otherwise, T is *non-trivial*. L is *paraconsistent* if it can be the underlying logic of inconsistent but non-trivial theories.

An expression of the form $\alpha \wedge \neg\alpha$, where $\wedge$ is the symbol for the conjunction, is called a *contradiction*. In general, if a theory is inconsistent, its logic enables us to derive, for whatever formula α, a contradiction $\alpha \wedge \neg\alpha$ from α and $\neg\alpha$; on the other hand, in most logics, from $\alpha \wedge \neg\alpha$ we can deduce both α and $\neg\alpha$. So, it is usual to call the trivial theories contradictory, which means that in such theories there are contradictions as theorems (or, equivalently, contradictory theorems).

We shall begin by presenting the propositional calculus $\mathcal{C}_1$. It seems natural that it should contain the usual connectives: $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction), and $\neg$ (negation); equivalence ($\leftrightarrow$) is defined as usual (see Definition 2.1.1 below). Furthermore, it seems also natural that $\mathcal{C}_1$ should be composed of most all of the valid schemes and rules of classical propositional calculus, obeying the following conditions:[6]

I. In $\mathcal{C}_1$ it should be not generally valid the principle of contradiction (or of non-contradiction).

II. From two contradictory propositions, that is, one being the negation of the another, it should not be possible to deduce any proposition whatever.

The language $\mathcal{L}$ of $\mathcal{C}_1$ contains the following primitive symbols: (i) propositional variables: a denumerable (infinite) set of propositional variables (formulas which are not analysed at the propositional level); (ii) connectives: $\rightarrow$, $\wedge$, $\vee$ and $\neg$; (iii) parentheses.

Formulas are defined as follows: (i) any propositional variable is a formula; (ii) if α and β are formulas, then $(\alpha \rightarrow \beta)$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$ and $\neg\alpha$ are formulas; (iii) the only formulas are those obtained from the preceding (i) and (ii).

To facilitate the reading, we shall adopt some conventions: (a) the symbol $\rightarrow$ is stronger than the others; (b) $\wedge$ and $\vee$ are stronger than $\neg$; (c) the external parentheses can be dispensed with.

Definition 2.1.1 $\alpha \leftrightarrow \beta =_{\mathsf{def}} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$

Now we list the postulates of $\mathcal{C}_1$ (axiom schemes, axioms, and primitive inference rules). To begin with, we shall assume the postulates of positive intuitionistic propositional logic:

1. $\alpha \rightarrow (\beta \rightarrow \alpha)$
2. $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma))$
3. $\alpha \wedge \beta \rightarrow \alpha$
4. $\alpha \wedge \beta \rightarrow \beta$
5. $\alpha \rightarrow (\beta \rightarrow \alpha \wedge \beta)$
6. $\alpha \rightarrow (\alpha \vee \beta)$
7. $\beta \rightarrow (\alpha \vee \beta)$
8. $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \vee \beta \rightarrow \gamma))$
9. $\alpha,\ \alpha \rightarrow \beta \,/\, \beta$

[6]Until the section 5, validity means syntactical validity: a formula is valid in a calculus if it has a proof in such calculus or if it is a theorem of this calculus.

Let us consider negation. We could think in adding to the above postulates the characteristic postulate of negation taken from the minimal calculus, namely,

$$(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha), \tag{1}$$

but this is not convenient, for from the above postulates and from (1) we can deduce the schema

$$\neg(\alpha \wedge \neg\alpha),$$

which stands for the Principle of Contradiction, which by (I) above should not be valid in $\mathcal{C}_1$. Furthermore, in the minimal calculus we can prove that from a contradiction the negation of any proposition can be derived, which is not convenient. Really, in the minimal calculus we have (using (1))

$$\alpha, \neg\alpha, \beta \vdash \alpha \quad \text{and} \quad \alpha, \neg\alpha, \beta \vdash \neg\alpha,$$

hence

$$\alpha, \neg\alpha \vdash \beta,$$

so

$$\vdash \alpha \to (\neg\alpha \to \beta),$$

by applying twice the deduction theorem, which is a consequence of the postulates for implication of the minimal calculus.

However, it seems interesting to accept (1), once we have, for β such that

$$\neg(\beta \wedge \neg\beta),$$

that is, the schema

$$\neg(\beta \wedge \neg\beta) \to ((\alpha \to \beta) \to ((\alpha \neg\beta) \to \neg\alpha)).$$

Since (in principle) we think convenient that every proposition be either *true* or *false*, yet we accept the possibility of existing propositions which are true so well as their negations, it seems convenient to include in addition the following schema

$$\alpha \vee \neg\alpha.$$

$\neg(\alpha \wedge \neg\alpha)$ means that α satisfies the law of contradiction, that is, α is *well-behaved*. If this is not the case, that is, if $\alpha \wedge \neg\alpha$ holds, then α is *ill-behaved*. Then, we introduce the following definition:

Definition 2.1.2 $\alpha^o =_{\mathsf{def}} \neg(\alpha \wedge \neg\alpha)$.

Let us consider the schema

$$\neg\neg\alpha \to \alpha. \tag{2}$$

We can reason heuristically as follows: if α is well-behaved, we can suppose that it obeys classical logic and so (2) holds; if α is ill-behaved, then both α and $\neg\alpha$ are true and, by the postulates of the implication it follows that whatever proposition entails α, in particular $\neg\neg\alpha$ entails α. Thus, (2) seems to be acceptable.

Finally, taking into account what was said above, we shall adopt further the following postulates, which entail that those formulas built with well-behaved formulae are also well-behaved, that is,

$$\alpha^o \wedge \beta^o \to (\alpha \wedge \beta)^o, \quad \alpha^o \wedge \beta^o \to (\alpha \vee \beta)^o, \quad \alpha^o \wedge \beta^o \to (\alpha \to \beta)^o.$$

These new postulates can be written as follows:

$$\alpha^o \wedge \beta^o \to (\alpha \wedge \beta)^o \wedge (\alpha \vee \beta)^o \wedge (\alpha \to \beta)^o.$$

Then, we can deduce

$$\alpha^o \wedge \beta^o \to (\alpha \leftrightarrow \beta)^o.$$

There is no necessity of assuming that

$$\alpha^o \to (\neg\alpha)^o,$$

for this can be proven from the postulates above. So, we have the following list of postulates for $\mathcal{C}_1$:

($\to_1$) $\alpha \to (\beta \to \alpha)$

($\to_2$) $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$

($\to_3$) $\alpha,\ \alpha \to \beta \,/\, \beta$

($\wedge_1$) $\alpha \wedge \beta \to \alpha$

($\wedge_2$) $\alpha \wedge \beta \to \beta$

($\wedge_3$) $\alpha \to (\beta \to \alpha \wedge \beta)$

($\vee_1$) $\alpha \to (\alpha \vee \beta)$

($\vee_2$) $\beta \to (\alpha \vee \beta)$

($\vee_3$) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$

($\neg_1$) $\beta^o \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$

($\neg_2$) $\alpha^o \wedge \beta^o \to (\alpha \wedge \beta)^o \wedge (\alpha \vee \beta)^o \wedge (\alpha \to \beta)^o$

($\neg_3$) $\alpha \vee \neg\alpha$

($\neg_4$) $\neg\neg\alpha \to \alpha$.

In the sequel we shall show that $\mathcal{C}_1$ has a bivalent semantics. But now let us study the main properties of $\mathcal{C}_1$.

The concept of (formal) deduction of a formula from a set of formulas, that is, using the standard notation, $\Gamma \vdash \alpha$, is defined as usual; in this case, we say that α is a syntactical consequence of the formulas in Γ. From now on, capital Greek letters stand for collections of formulae, while small Greek letters stand for formulas. We have in $\mathcal{C}_1$:

Theorem 2.1.1

(a) $\{\alpha\} \vdash \alpha$,

(b) $\Gamma \vdash \alpha$ entails $\Gamma \cup \Delta \vdash \alpha$,

(c) if $\Gamma \vdash \gamma$ for any $\gamma \in \Delta$ and $\Delta \vdash \alpha$, then $\Gamma \vdash \alpha$.

Proof: Immediate, from the standard definition of the syntactical consequence ($\vdash$).
▮

Definition 2.1.3 *α is a theorem of $\mathcal{C}_1$ iff $\vdash \alpha$.*

As usual, $\vdash \alpha$ means $\emptyset \vdash \alpha$. The symbols $\Rightarrow$ and $\Leftrightarrow$ are metalinguistic abbreviations of implication and bi-implication respectively.

Theorem 2.1.2 *In $\mathcal{C}_1$:*

(a) (Deduction Theorem) $\Gamma \cup \{\alpha\} \vdash \beta \Rightarrow \Gamma \vdash \alpha \rightarrow \beta$

(b) (Modus Ponens) $\{\alpha, \alpha \rightarrow \beta\} \vdash \beta$

(c) $\{\alpha, \beta\} \vdash \alpha \wedge \beta$, $\{\alpha, \beta\} \vdash \alpha$, $\{\alpha, \beta\} \vdash \beta$

(d) $\{\alpha\} \vdash \alpha \vee \beta$, $\{\beta\} \vdash \alpha \vee \beta$

(e) (Proof by Cases) $(\Gamma \cup \{\alpha\} \vdash \gamma$ and $\Gamma \cup \{\beta\}) \vdash \gamma \Rightarrow \Gamma \cup \{\alpha \vee \beta\} \vdash \gamma$

(f) (Paraconsistent Reduction at Absurdum) $(\Gamma \vdash \beta^o, \Gamma \cup \{\alpha\} \vdash \beta$ and $\Gamma \cup \{\alpha\}) \vdash \neg\beta \Rightarrow \Gamma \vdash \neg\alpha$

(g) (Elimination of the double negation) $\{\neg\neg\alpha\} \vdash \alpha$.

Proof: As in classical logic. The deduction of the paraconsistent reductio ad absurdum principle is made with the help of the rules for implication and of the schema $(\neg_1)$. ∎

The next theorem is important for it shows that the conditions (I) and (II) which $\mathcal{C}_1$ should obey are of course satisfied.

Theorem 2.1.3 *In $\mathcal{C}_1$, the following schemas do not hold:*

1. $\neg\alpha \rightarrow (\alpha \rightarrow \beta)$
2. $\neg\alpha \rightarrow (\alpha \rightarrow \neg\beta)$
3. $\alpha \rightarrow (\neg\alpha \rightarrow \beta)$
4. $\alpha \rightarrow (\neg\alpha \rightarrow \neg\beta)$
5. $\alpha \wedge \neg\alpha \rightarrow \beta$
6. $\alpha \wedge \neg\alpha \rightarrow \neg\beta$
7. $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \beta) \rightarrow \neg\alpha)$

9. $\alpha \rightarrow \neg\neg\alpha$
10. $(\alpha \leftrightarrow \neg\alpha) \rightarrow \beta$
11. $(\alpha \leftrightarrow \neg\alpha) \rightarrow \neg\beta$
12. $\neg(\alpha \wedge \neg\alpha)$
13. $\alpha \wedge \neg\alpha \rightarrow \neg(\alpha \wedge \neg\alpha)$

Proof: By using the tables below, with 1 and 2 as designated values:

α	β	$\alpha \rightarrow \beta$	$\alpha \wedge \beta$	$\alpha \vee \beta$
1	1	1	1	1
2	1	1	1	1
3	1	1	3	1
1	2	1	1	1
2	2	1	1	1
3	2	1	3	1
1	3	3	3	1
2	3	3	3	1
3	3	1	3	3

α	$\neg\alpha$
1	3
2	1
3	1

∎

Then, in $\mathcal{C}_1$ the following principles, which play an important role in certain paradoxes, leading to trivialization, are not valid:

$$\neg\alpha \to (\alpha \to \beta),\ \neg\alpha \to (\alpha \to \neg\beta),\ \alpha \wedge \neg\alpha \to \beta,\ \alpha \wedge \neg\alpha \to \neg\beta.$$

Theorem 2.1.4 *In $\mathcal{C}_1$, the following version of Reductio at Absurdum holds:*

$$(\Gamma \vdash \beta^o, \Gamma \cup \{\alpha\} \vdash \beta, \Gamma \cup \{\alpha\} \vdash \neg\beta) \Rightarrow \Gamma \vdash \neg\alpha.$$

Proof: By the deduction theorem and $(\neg_1)$. ▮

Corolary 2.1.1 *In $\mathcal{C}_1$, the following rules hold:*

$$(\Gamma \cup \{\alpha\} \vdash \beta^o, \Gamma \cup \{\alpha\} \vdash \beta, \Gamma \cup \{\alpha\} \vdash \neg\beta) \Rightarrow \Gamma \vdash \neg\alpha,$$

$$(\Gamma \cup \{\neg\alpha\} \vdash \beta^o, \Gamma \cup \{\neg\alpha\} \vdash \beta, \Gamma \cup \{\neg\alpha\} \vdash \neg\beta) \Rightarrow \Gamma \vdash \neg\alpha.$$

Proof: From the precedent theorem and from then fact that in $\mathcal{C}_1$ we have that $\vdash \alpha \vee \neg\alpha$ and $\vdash \neg\neg\alpha \to \alpha$. ▮

Theorem 2.1.5 *If we add to the axioms of $\mathcal{C}_1$ the principle of contradiction $\neg(\alpha \wedge \neg\alpha)$ as an additional postulate, we get the classical propositional calculus (CPC).*

Proof: Really, in doing so we get $\vdash (\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$, and, therefore, the postulate system for the CPC presented by Kleene [154, p. 82]. ▮

From now on, to simplify the notation, in general we shall eliminate the symbols { and } in the writing of deductions.

Theorem 2.1.6 *In $\mathcal{C}_1$, we have:*

1. $\alpha \to \alpha$
2. $\alpha \leftrightarrow \alpha$
3. $\alpha \to \beta, \beta \to \gamma \vdash \alpha \to \gamma$
4. $\alpha \to (\beta \to \gamma) \vdash \beta \to (\alpha \to \gamma)$
5. $\alpha \to (\beta \to \gamma) \vdash \alpha \wedge \beta \to \gamma$
6. $\alpha \wedge \beta \to \gamma \vdash \alpha \to (\beta \to \gamma)$
7. $\alpha \to \beta \vdash (\beta \to \gamma) \to (\alpha \to \gamma)$
8. $\alpha \to \beta \vdash (\gamma \to \alpha) \to (\gamma \to \beta)$
9. $\alpha \to \beta \vdash \alpha \wedge \gamma \to \beta \wedge \gamma$
10. $\alpha \to \beta \vdash \gamma \wedge \alpha \to \gamma \wedge \beta$
11. $\alpha \to \beta \vdash \alpha \vee \gamma \to \beta \vee \gamma$
12. $\alpha \to \beta \vdash \gamma \vee \alpha \to \gamma \vee \beta$
13. $\alpha \leftrightarrow \beta \vdash \beta \leftrightarrow \alpha$
14. $\alpha \leftrightarrow \beta, \beta \leftrightarrow \gamma, \gamma \vdash \alpha \leftrightarrow \gamma$
15. $\vdash (\alpha \leftrightarrow \beta) \leftrightarrow (\beta \leftrightarrow \alpha)$
16. $\vdash (\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$

17. $\vdash (\alpha \to (\alpha \to (\alpha \to \beta))) \to (\alpha \to (\alpha \to \beta))$
18. $\vdash \underbrace{(\alpha \to \ldots (\alpha \to \beta) \ldots)}_{n+1\, times} \to \underbrace{(\alpha \to \ldots (\alpha \to \beta) \ldots)}_{n\, times}$

Proof: Immediate consequences of the way $\mathcal{C}_1$ was constructed. ▮

Theorem 2.1.7 *Propositional positive intuitionistic logic is contained in* $\mathcal{C}_1$.

Proof: In effect, such a logic is characterized by the postulates $(\neg_1)$ to $(\vee_3)$. ▮

Theorem 2.1.8 *In* $\mathcal{C}_1$*, we have:*

1. $\beta^o, \alpha \to \beta \vdash \neg\beta \to \neg\alpha$
2. $\beta^o, \alpha \to \neg\beta \vdash \beta \to \neg\alpha$
3. $\beta^o, \neg\alpha \to \beta \vdash \neg\beta \to \alpha$
4. $\beta^o, \neg\alpha \to \neg\beta \vdash \beta \to \alpha$
5. $(\alpha \to \neg\alpha) \to \neg\alpha$
6. $(\neg\alpha \to \alpha) \to \alpha$

Proof: We shall prove item 3. We have: $\beta^o, \neg\alpha \to \beta, \neg\alpha \vdash \beta^o$; $\beta^o, \neg\alpha \to \beta, \neg\beta, \neg\alpha \vdash \neg\beta$; $\beta^o, \neg\alpha \to \beta, \neg\beta, \neg\alpha \vdash \beta$. Hence, $\beta^o, \neg\alpha \to \beta, \neg\beta \vdash \neg\neg\alpha$, therefore $\beta^o, \neg\alpha \to \beta, \neg\beta \vdash \alpha$. So, $\beta^o, \neg\alpha \to \beta \vdash \neg\beta \to \alpha$. ▮

Remark Since $\mathcal{C}_1$ has several of the usual relevant properties of classical propositional calculus, we could prove various other schemes, as it is easy to see.

Theorem 2.1.9 $\mathcal{C}_1$ *is a sub-calculus of classical propositional calculus.*

Proof: All the postulates of $\mathcal{C}_1$ are valid in the classical propositional calculus. Furthermore, Theorem (2.1.3) shows that there are schemes which are valid in this last calculus which are not valid in $\mathcal{C}_1$. ▮

Theorem 2.1.10 *In* $\mathcal{C}_1$ *the following scheme are not valid:*

$$(\alpha \wedge \beta) \wedge \neg\alpha \to \beta \;\; \textit{and} \;\; \alpha \vee \beta \to (\neg\alpha \to \beta).$$

Proof: If these schemes were valid in $\mathcal{C}_1$, then with the help of the deduction theorem, modus ponens and the schema $\alpha \to \alpha \vee \beta$, we could derive in $\mathcal{C}_1$ the schema $\alpha \to (\neg\alpha \to \beta)$. ▮

The last theorem could also be proven by using the above tables, by means of which several other scheme, which are valid in classical logic, are not theorems of $\mathcal{C}_1$.

Corolary 2.1.2 *The following rules are not valid in* $\mathcal{C}_1$ *(disjunctive syllogisms):*

$$\frac{\alpha \vee \beta, \neg\alpha}{\beta} \;\; \textit{and} \;\; \frac{\neg\alpha \vee \beta, \alpha}{\beta}.$$

Theorem 2.1.11 *If* $\alpha_1, \ldots, \alpha_n$ *are the prime components of a formula* α*, then a necessary and sufficient condition for* α *to be provable in the classical propositional calculus is that* $\alpha_1^o, \ldots, \alpha_n^o \vdash \alpha$ *in* $\mathcal{C}_1$.

Proof: If $\alpha_1^o, \ldots, \alpha_n^o \vdash \alpha$ in $\mathcal{C}_1$, then $\vdash \alpha$ in the classical propositional calculus, since β^o is an abbreviation of $\neg(\beta \wedge \neg\beta)$. Now, if $\vdash \alpha$ in the classical propositional calculus (supposed axiomatized as in [154]), then there exists a proof P of α in such calculus, in which there appear only those formulas whose prime components (propositional variables) are among $\alpha_1, \ldots, \alpha_n$. So, if k is one of the formulas in P, due to postulate $(\neg_2)$, we have that $\alpha_1^o, \ldots, \alpha_n^o \vdash k^o$ in $\mathcal{C}_1$ and, furthermore, in $\mathcal{C}_1$ we have $\alpha_1^o, \ldots, \alpha_n^o \vdash (\gamma \to \delta) \to ((\gamma \to \neg\delta) \to \neg\gamma)$, with the usual restrictions. But, since whatever postulate of the classical propositional calculus of [154] is valid in $\mathcal{C}_1$, with the exception of $(\theta \to \pi) \to ((\theta \to \neg\pi) \to \neg\theta)$, we see that P can be transformed into a deduction, in $\mathcal{C}_1$, of α from $\alpha_1^o, \ldots, \alpha_n^o$. ▮

Theorem 2.1.12 *If $\alpha_1, \ldots, \alpha_n$ are the prime components of the formulas Γ and of a formula α, then a necessary and sufficient condition for $\Gamma \vdash \alpha$ in the classical propositional calculus is that $\Gamma, \alpha_1^o, \ldots, \alpha_n^o \vdash \alpha$ in $\mathcal{C}_1$.*

Proof: Analogous to the previous one. ▮

Theorem 2.1.13 (A. I. Arruda) *In $\mathcal{C}_1$, we have $\vdash \alpha^{oo}$.*

Proof: $\vdash \alpha^{oo}$ means $\neg(\alpha^o \wedge \neg\alpha^o)$, that is, $\neg(\alpha^o \wedge \neg\neg(\alpha \wedge \neg\alpha))$. But $\alpha^o \wedge \neg\neg(\alpha \wedge \neg\alpha) \vdash \alpha^o$ and $\alpha^o \wedge \neg\neg(\alpha \wedge \neg\alpha) \vdash \alpha \wedge \neg\alpha$, hence $\vdash \neg(\alpha^o \wedge \neg\neg(\alpha \wedge \neg\alpha))$, that is, $\vdash \alpha^{oo}$. ▮

Theorem 2.1.14 *In $\mathcal{C}_1$, we have $\vdash \alpha^o \to (\neg\alpha)^o$.*

Proof: We have: $\alpha^o, \neg\alpha \wedge \neg\neg\alpha \vdash \alpha$ and $\alpha^o, \neg\alpha \wedge \neg\neg\alpha \vdash \neg\alpha$, so as $\alpha^o, \neg\alpha \wedge \neg\neg\alpha \vdash \alpha^o$, hence $\alpha^o \vdash \neg(\neg\alpha \wedge \neg\neg\alpha)$, so $\alpha^o \vdash (\neg\alpha)^o$. Then, $\vdash \alpha^o \to (\neg\alpha)^o$. ▮

Definition 2.1.4 (Strong Negation) $\neg^\star\alpha =_{\mathsf{def}} \neg\alpha \wedge \alpha^o$

Theorem 2.1.15 *In $\mathcal{C}_1$, $\vdash (\alpha \to \beta) \to ((\alpha \to \neg^\star\beta) \to \neg^\star\alpha)$*

Proof: We have

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \alpha^o \vdash \neg^\star\alpha, \quad \text{and} \tag{1}$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \neg\alpha^o \vdash \alpha \wedge \neg\alpha. \quad \text{But}$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \alpha \wedge \neg\alpha \vdash \beta,$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \alpha \wedge \neg\alpha \vdash \neg\beta \ \text{ and}$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta \vdash \alpha^o. \quad \text{Therefore,}$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \neg\alpha^o \vdash \alpha,$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \neg\alpha^o \vdash \neg\alpha,$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \neg\alpha^o \vdash \alpha^o, \quad \text{and}$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \neg\alpha^o \vdash \neg^\star\alpha. \tag{2}$$

¿From (1) and (2), using proof by cases and the excluded middle, we have

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta \vdash \neg^\star\alpha, \text{ hence}$$

$$\vdash (\alpha \to \beta) \to ((\alpha \to \neg^\star\beta) \to \neg^\star\alpha). ▮$$

Theorem 2.1.16 $\vdash \alpha \to (\neg^\star\alpha \to \beta)$

Proof: We have:

$\alpha,\ \neg\alpha \wedge \alpha^o,\ \neg\beta \vdash \alpha,$

$\alpha,\ \neg\alpha \wedge \alpha^o,\ \neg\beta \vdash \neg\alpha,$

$\alpha,\ \neg\alpha \wedge \alpha^o,\ \neg\beta \vdash \alpha^o$, then

$\alpha,\ \neg\alpha \wedge \alpha^o \vdash \neg\neg\beta,$

$\alpha,\ \neg\alpha \wedge \alpha^o \vdash \beta$ and $\alpha,\ \neg^\star\alpha \vdash \beta$. So,

$\vdash \alpha \to (\neg^\star\alpha \to \beta)$. ▮

Theorem 2.1.17 $\vdash \alpha \vee \neg^\star\alpha$

Proof: We have:

$\vdash (\alpha \vee \neg^\star\alpha) \leftrightarrow \alpha \vee (\neg\alpha \wedge \alpha^o),$

$\vdash (\alpha \vee \neg^\star\alpha) \leftrightarrow (\alpha \vee \neg\alpha) \wedge (\alpha \vee \alpha^o)$, and

$\vdash (\alpha \vee \neg^\star\alpha) \leftrightarrow (\alpha \vee \alpha^o).$

But i) $\alpha^o \vdash \alpha \vee \alpha^o$ and ii) $\neg\alpha^o \vdash \alpha \wedge \neg\alpha \vdash \alpha \vdash \alpha \vee \alpha^o$. So, from i) and ii) there follows $\vdash \alpha \vee \alpha^o$, hence $\vdash \alpha \vee \neg^\star\alpha$. ▮

Theorem 2.1.18 *Then connectives* $\to$, $\wedge$, $\vee$ *and* $\neg^\star$*, in* $\mathcal{C}_1$*, satisfy all schemes and rules of classical propositional calculus.*

Proof: In $\mathcal{C}_1$ the following schemes are valid:

$(\alpha \to \beta) \to ((\alpha \to \neg^\star\beta) \to \neg^\star\alpha),$

$\alpha \to (\neg^\star\alpha \to \beta)$ and

$\alpha \vee \neg^\star\alpha.$

If we add to these schemes the postulates $(\to_1)$ to $(\vee_3)$ above, we get the postulates for the classical propositional calculus (see [154]). ▮

Theorem 2.1.19 *In* $\mathcal{C}_1$*, Peirce's Law is valid, that is,*

$$\vdash ((\alpha \to \beta) \to \alpha) \to \alpha.$$

Proof: In the classical propositional calculus, Peirce's Law is a thesis. So, by the last theorem, it is provable in $\mathcal{C}_1$. ▮

Theorem 2.1.20 *In* $\mathcal{C}_1$*, we have:* $\vdash \alpha \vee (\alpha \to \beta)$.

Proof: $\vdash \alpha \vee (\alpha \to \beta)$ is valid in the classical propositional calculus. ▮

The above results show that classical propositional calculus is, in a certain sense, contained in $\mathcal{C}_1$, although $\mathcal{C}_1$ is a sub-calculus of it.

Definition 2.1.5 *A schema or formula* α trivializes *a calculus* $\mathcal{C}$ *when, by adjoining* α *to* $\mathcal{C}$*, the new calculus is trivial, that is, all its formulas are theorems.*

Theorem 2.1.21 *Any formula of the form* $\alpha \wedge \neg^\star\alpha$ *trivializes* $\mathcal{C}_1$*.*

Proof: By the above results, we see that $\neg^\star$ is a classical negation, so $\alpha \wedge \neg^\star\alpha$ entails any formula, that is, $\vdash \alpha \wedge \neg^\star\alpha \to \beta$. ▮

Theorem 2.1.22 *The schema $\alpha \leftrightarrow (\alpha \to \beta)$, where β is any formula, trivializes $\mathcal{C}_1$.*

Proof: In $\mathcal{C}_1$, we have (i) $\vdash \alpha \vee (\alpha \to \beta)$. Hence, from α and $\alpha \leftrightarrow (\alpha \to \beta)$, we deduce β. From $\alpha \to \beta$ and from $\alpha \leftrightarrow (\alpha \to \beta)$, we deduce β again. So, from (i), $\vdash \beta$. ∎

Since in $\mathcal{C}_1 \vdash \alpha \vee \underbrace{(\alpha \to (\alpha \to \ldots (\alpha \to \beta)\ldots)}_{n \text{ occurences of } \alpha}$, then the schema

$$\alpha \leftrightarrow \alpha \vee (\alpha \to (\alpha \to \ldots (\alpha \to \beta)\ldots),$$

where β is any formula, also trivializes $\mathcal{C}_1$.

Theorem 2.1.23 *In $\mathcal{C}_1$,*

$$\nvdash (\alpha \leftrightarrow (\alpha \vee \alpha)) \leftrightarrow (\neg\alpha \leftrightarrow (\neg(\alpha \vee \alpha)).$$

Proof: Using the table given at Theorem 2.1.3, giving to α to assume the value 2. ∎

Theorem 2.1.24 *In $\mathcal{C}_1$, the following schema and rule are not valid:*

$$(\alpha \leftrightarrow \beta) \to ((\neg\alpha \leftrightarrow \neg\beta), \ \frac{\alpha \leftrightarrow \beta}{\neg\alpha \leftrightarrow \neg\beta}.$$

Proof: Immediate consequence of the last theorem. ∎

2.2 The hierarchy $\mathcal{C}_n$, $0 \leq n \leq \omega$

The calculus $\mathcal{C}_1$ is not the only which satisfy the conditions I and II formulated above (see page 6). Among other possible solutions, we shall indicate, in what follows, a hierarchy of calculi which satisfy such conditions, excepting the first one, which will be taken to be the classical propositional calculus. The hierarchy is the following:

$$\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n, \ldots, \mathcal{C}_\omega, \qquad (3)$$

where $\mathcal{C}_0$ is the classical propositional calculus and the remaining ones are defined below.

To begin with, let us introduce the following definition:

Definition 2.2.1

(i) $\alpha^{(1)}$ stands for α^o

(ii) $\alpha^{(n)}$ stands for $\alpha^{n-1} \wedge (\alpha^{(n-1)})^o$, $2 \leq n \leq \omega$.

Then, the calculus $\mathcal{C}_n$ $(0 < n < \omega)$ is individualized by the postulates $(\to_1$ to $(\vee_3)$ above, plus $\alpha \vee \neg\alpha$ and $\neg\neg\alpha \to \alpha$ and the following ones:

(n_1) $\beta^{(n)} \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$

(n_2) $\alpha^{(n)} \wedge \beta^{(n)} \to ((\alpha \to \beta)^{(n)} \wedge (\alpha \wedge \beta)^{(n)} \wedge (\alpha \vee \beta)^{(n)})$.

$\mathcal{C}_\omega$ has as postulates $(\to_1)$ to $(\vee_3)$ plus $\alpha \vee \neg\alpha$ and $\neg\neg\alpha \to \alpha$.

We can see that, in $\mathcal{C}_n, 1 \leq n \leq \omega$, we can substitute the schema $\beta^{(n)} \wedge \beta \wedge \neg\beta \to \gamma$ for the postulate (n_1). It is also easy to verify that the above results, proven for $\mathcal{C}_1$, with suitable adaptation, can be also proven for $\mathcal{C}_n, n = 2, 3, \ldots$.

Theorem 2.2.1 *Each one of the calculi in the hierarchy (3) is strictly stronger than those which follow it.*

Proof: (A. I. Arruda) The tables of the proof of Theorem 2.1.3 which we call here T_1, show that $\mathcal{C}_0$ properly contains $\mathcal{C}_1$. With the aim of proving that $\mathcal{C}_i$ properly contains $\mathcal{C}_{i+1}$, $i = 1, 2, \ldots$, we define the tables $T_2, T_3, \ldots$ as follows, with the exception of the tables for negation, which will be defined separately. T_2 is obtained from T_1 (as in the proof of Theorem 2.1.3 by the addition of a new value, 4, which will be the only not designated; we obtain T_n from T_{n-1} by adding a new value $n+2$, which will be the only not designated etc.. The tables T_n are then obtained from T_{n-1}, $n = 2, 3, \ldots$, as follows (the rules below refer to the new arrangements of values, which result from the addition of a new value, and no changes are made, in the new tables, of those already obtained in the tables of order $n-1$): (1) *conjunction*: if the components have different values, the conjunction will have the greatest value among the values of its conjuncts; if the values are equal, this will be the value of the conjunction. (2) *disjunction*: if the values of the components are distinct, the disjunction will have the smaller value among the values of the components; if they are equal, this value will be the value of the disjunction. (3) *implication*: if the values of the components are distinct, the implication will have the value of the consequent; if they are equal, the implication will have the value 1. As for negation, in T_n we have the following table:

α	1	2	3	$\cdots$	n	n+1	n+2
$\neg\alpha$	n+2	1	2	$\cdots$	n-1	n	1

$n = 1, 2, \ldots, n, n+1$ being the designated values and $n+2$ the only not designated, the tables T_n show that $\mathcal{C}_{n-1}$ strictly contains $\mathcal{C}_n$; for instance, the postulate $\beta^{(n-1)} \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$ and the schema $\alpha^{(n-1)} \to (\neg\alpha)^{(n-1)}$ are not valid in $\mathcal{C}_n$, which may be verified with some work ($n > 1$). Furthermore, $\mathcal{C}_\omega$ is strictly weaker than all the other calculi of the hierarchy. ∎

2.3 Theories

In this section, we shall make reference to the calculus $\mathcal{C}_1$, although the exposition can be extended to all the calculi $\mathcal{C}_n, 0 \leq n \leq \omega$.

Definition 2.3.1 $\overline{\Delta} =_{\mathsf{def}} \{\alpha : \Delta \vdash \alpha\}$

Theorem 2.3.1

(i) $\Delta \subset \overline{\Delta}$

(ii) $\Delta \subset \Gamma \Rightarrow \overline{\Delta} \subset \overline{\Gamma}$

(iii) $\overline{\overline{\Delta}} \subset \overline{\Delta}$

Proof: Immediate. ∎

Definition 2.3.2 Δ *is a theory if and only if* $\overline{\Delta} = \Delta$.

Definition 2.3.3 *Let* Υ *be the set of all formulas of* $\mathcal{C}_1$. *Then* Δ *is inconsistent if there exists* α *such that* $\Delta \vdash \alpha$ *and* $\Delta \vdash \neg\alpha$, *otherwise* Δ *is consistent.* Γ *is trivial if* $\overline{\Gamma} = \Upsilon$; *otherwise,* Γ *is non-trivial. An inconsistent set is also called* contradictory. *A theory* Δ *is* paraconsistent *if it is inconsistent but non-trivial.*

Definition 2.3.4 *Expressions of either the form* $\alpha \wedge \neg\alpha$ *or* $\neg\alpha \wedge \alpha$ *are called contradictions.*

It is easy to see that Δ is inconsistent if and only if $\Delta \vdash \alpha \wedge \neg\alpha$ for some formula α.

Theorem 2.3.2 *$\mathcal{C}_1$ can be the underlying calculus of paraconsistent theories.*

Proof: It suffices to consider the theory $\overline{\{\alpha \wedge \neg\alpha\}}$, where α is a propositional variable and to apply the tables of Theorem 2.1.3. ▮

Definition 2.3.5 *A formula α trivializes a calculus $\mathcal{C}$ if, by adjoining α to $\mathcal{C}$ as a new axiom, the new resulting system is trivial. In this case, $\mathcal{C}$ is said to be finitely trivializable.*

For instance, the intuitionistic or classical implicative propositional calculi and the classical positive propositional calculus are not finitely trivializable, while the classical predicate calculus is.

Theorem 2.3.3 *A formula of the form $\alpha \wedge \neg\alpha \wedge \alpha^{(n)}$ trivializes $\mathcal{C}_n, 1 \leq n < \omega$.*

Proof: Immediate. ▮

Theorem 2.3.4 *$\mathcal{C}_\omega$ is not finitely trivializable.*

Proof: Let us consider the following matrix M, where 1 is designated:

α	β	$\alpha \to \beta$	$\alpha \wedge \beta$	$\alpha \vee \beta$
0	0	1	0	0
1	0	0	0	1
0	1	1	0	1
1	1	1	1	1

If α does not begin by the symbol $\neg$, then:

1) If the value of α is 1, then the value of $\neg^n\alpha$ is 1, where $\neg^n$ stands for $\neg\neg\ldots\neg$, ($\neg$ repeated n times, $n \geq 1$).

2) If the value of α is 0, then the value of $\neg^{2k}\alpha$ is 0, and the value of $\neg^{2k+1}\alpha$ is 1, for all $k = 0, 1, 2, \ldots$.

M is sound for $\mathcal{C}_\omega$, as it is easy to see. By induction on the length of the formulas, we can show that no formula assumes the value 0. So, there is no formula γ such that $\gamma \vdash \alpha$ (or $\vdash \gamma \to \alpha$), for any formula α of $\mathcal{C}_\omega$. ▮

The following results can also be proven without difficulty:

Theorem 2.3.5 *The calculi $\mathcal{C}_n$, $0 < n < \omega$, with a finite number of propositional variables, are trivializable by formulas of the form $\alpha \wedge \neg\alpha \wedge \alpha^{(n)}$.*

Theorem 2.3.6 *$\mathcal{C}_\omega$, with a finite number of propositional variables, cannot be finitely trivializable.*

It should be remarked that if we base a theory on $\mathcal{C}_n$, there is more risk of trivialization than if we base it on $\mathcal{C}_{n+1}$. The 'maximum of security' in avoiding trivialization we could achieve by using $\mathcal{C}_\omega$, but as far as we go on the hierarchy (3), we get weaker and weaker calculi.

Theorem 2.3.7 (A. I. Arruda) *In $\mathcal{C}_n$, $1 \leq n \leq \omega$, there are no reduction of negations, that is, expressions like $\neg^p\alpha \leftrightarrow \neg^q\alpha$, for $p \neq q$ ($p, q = 0, 1, 2, \ldots$), are not valid in these calculi.*

Proof: It suffices to use the following truth tables: the values of the tables are the integer numbers ≥ 1 and the only not designated value is 1. If d stands for designated value and $v(\alpha)$ is the value of α, then the tables of $\to$, $\wedge$, $\vee$ and $\neg$ are as follows:

$\to$: $v(\alpha \to \beta) = 1$ iff $v(\alpha) = d$ and $v(\beta) = 1$; $v(\alpha \to \beta) = 2$ otherwise.

$\wedge$: $v(\alpha \wedge \beta) = 2$ iff $v(\alpha) = d$ and $v(\beta) = d$, and $v(\alpha \wedge \beta) = 1$ otherwise.

$\vee$: $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$, and $v(\alpha \vee \beta) = 2$ otherwise.

$\neg$: $v(\neg\alpha) = 2$ iff $v(\alpha) = 1$ and $v(\neg\alpha) = 1$ otherwise. ∎

Corolary 2.3.1 (A. I. Arruda) *The calculi $\mathcal{C}_n$, $1 \leq n \leq \omega$ do not have finite characteristic truth tables, that is, finite tables such that a necessary and sufficient conditions for α to be a theorem is that it assumes only designated values in these tables.*

Proof: From the hypothesis of the corollary, it is easy to see that if the calculi had characteristic finite tables, they would enable reduction of negations. ∎

2.4 Quantification

Corresponding to the hierarchy (3), we construct the corresponding first order predicate calculi. These new calculi will be denoted

$$\mathcal{C}_0^\star, \mathcal{C}_1^\star, \mathcal{C}_2^\star, \ldots, \mathcal{C}_n^\star, \ldots, \mathcal{C}_\omega^\star. \tag{4}$$

To begin with, we shall construct the calculus $\mathcal{C}_1^\star$. We suppose given a first order language $\mathcal{L}^\star$, which contains the usual symbols, as connectives and quantifiers, individual variables (a denumerable family), predicate symbols (each one of a certain arity) and auxiliary symbols. We could suppose that $\mathcal{L}^\star$ encompasses functional symbols, in particular individual constants, but we shall not make such a supposition here, except when explicitly mentioned. The notions of formula, free and bound variables in a formula, sentence (formula without free variables) etc. are standard. The notations and metalogical conventions extend those made for the propositional calculi, and are obvious.

$\mathcal{C}_0^\star$ is the classical first-order predicate calculus. The postulates of $\mathcal{C}_1^\star$ are those of $\mathcal{C}_1$ (conveniently adapted) plus the following:

(I) $\frac{\alpha \to \beta(x)}{\alpha \to \forall x \beta(x)}$

(II) $\forall x \alpha(x) \to \alpha(y)$

(III) $\alpha(x) \to \exists x \alpha(x)$

(IV) $\frac{\alpha(x) \to \beta}{\exists x \alpha(x) \to \beta}$

(V) $\forall x(\alpha(x))^o \to (\forall x \alpha(x))^o$

(VI) $\forall x((\alpha(x))^o \to (\exists x \alpha(x))^o$

where the variables x and y and the formulas α and β are subjected to the usual restrictions.

Furthermore, we shall adopt the following additional postulate, where A and B are congruent formulas, that is, one can be obtained from the another by exchanging bound variables or by suppressing vacuous quantifications (without confusion of variables):

(VII) $A \leftrightarrow B$

A reason to accept I - IV cames from the fact that they hold in classical logic and are acceptable intuitively; as for V and VI, they are assumed due to reasons similar to those which motivate the postulates of $\mathcal{C}_1$; finally, VII is necessary for it seems licit to say that two congruent formulas are equivalent, and this fact cannot be deduced from the remaining postulates. Then we have the following theorem, whose proof is done as in classical logic.

Theorem 2.4.1 *All derived rules of theorem 2.1.2 hold in $\mathcal{C}_1^\star$. Furthermore, we have:*

$\alpha(x) \vdash \forall x\alpha(x)$ $\qquad$ $\forall x\alpha(x) \vdash \alpha(y)$

$\alpha(t) \vdash \exists x\alpha(x)$ $\qquad$ If $\Gamma \cup \{\alpha(x)\} \vdash \beta$, then $\Gamma \cup \{\exists x\alpha(x)\} \vdash \beta$,

with the usual restrictions (*cf.* [154]).

It should be remarked that in $\mathcal{C}_1^\star$, as in the classical calculus $\mathcal{C}_0^\star$, the above rules impose restrictions to the Deduction Theorem (see [154]). The proof of the following theorems are also obtained without difficulty.

Theorem 2.4.2 *In $\mathcal{C}_1^\star$:*

$\vdash \forall x(\alpha(x) \to \beta(x)) \to (\forall\alpha(x) \to \forall x\beta(x))$

$\vdash \forall x(\alpha(x) \to \beta(x)) \to (\exists\alpha(x) \to \exists x\beta(x))$

Theorem 2.4.3 *If α and β are congruent formulas, then in $\mathcal{C}_1^\star$ we have: $\vdash \alpha \leftrightarrow \beta$.*

Theorem 2.4.4 *If the components of the formulas and Γ and of α which are quantificationally prime are $\gamma_1, \ldots, \gamma_n$, then if $\Gamma \vdash \alpha$ in the classical predicate calculus, then $\{\gamma_1^o, \ldots, \gamma_n^o\} \cup \Gamma \vdash \alpha$ in $\mathcal{C}_1^\star$ and conversely.*

Corolary 2.4.1 *Being $\alpha_1, \ldots, \alpha_n$ the component of α which are quantificationally prime, then a necessary and sufficient condition for $\vdash \alpha$ in $\mathcal{C}_0^\star$ is that $\{\alpha_1^o, \ldots, \alpha_n^o\} \vdash \alpha$ in $\mathcal{C}_1^\star$.*

Theorem 2.4.5 *$\mathcal{C}_0^\star$ is included in $\mathcal{C}_1^\star$.*

Proof: When we deal with formulas whose prime components are well-behaved, Theorem 2.4.4 shows that for these formulas the laws and rules of $\mathcal{C}_0^\star$ hold. But, from Theorem 2.1.18, $\mathcal{C}_1$ is, in a certain sense, included in $\mathcal{C}_1^\star$, and so is $\mathcal{C}_0^\star$. ▮

Corolary 2.4.2 *$\mathcal{C}_1^\star$ is undecidable.*

Proof: $\mathcal{C}_0^\star$ is included in $\mathcal{C}_1^\star$; since $\mathcal{C}_0^\star$ is undecidable (Church's theorem), so is $\mathcal{C}_1^\star$. ▮

Theorem 2.4.6 *The following schemes are valid in $\mathcal{C}_1^\star$(with the usual restrictions –*[154, p. 162]*), where γ is a formula in which x does not appear free:*

1. $\forall x\,\gamma \leftrightarrow \gamma$
2. $\exists x\,\gamma \leftrightarrow \gamma$
3. $\forall x\forall y\alpha(x, y) \leftrightarrow \forall y\forall x\alpha(x, y)$
4. $\exists x\exists y\alpha(x, y) \leftrightarrow \exists y\exists x\alpha(x, y)$
5. $\forall x\forall y\alpha(x, y) \to \forall x\alpha(x, x)$

6. $\exists x\alpha(x,x) \to \exists x\exists y\alpha(x,y)$
7. $\forall x\alpha(x) \to \exists x\alpha(x)$
8. $\exists x\forall y\alpha(x,y) \to \forall y\exists x\alpha(x,y)$
9. $\forall x(\alpha(x) \wedge \beta(x)) \leftrightarrow \forall x\alpha(x) \wedge \forall x\beta(x)$
10. $\exists x(\alpha(x) \vee \beta(x)) \leftrightarrow \exists x\alpha(x) \vee \exists x\beta(x)$
11. $\alpha \wedge \forall x\beta(x) \leftrightarrow \forall x(\alpha \wedge \beta(x))$
12. $\alpha \vee \exists x\beta(x) \leftrightarrow \exists x(\alpha \vee \beta(x))$
13. $\alpha \wedge \exists x\beta(x) \leftrightarrow \exists x(\alpha \wedge \beta(x))$
14. $\alpha \vee \forall x\beta(x) \to \forall x(\alpha \vee \beta(x))$
15. $\exists x(\alpha(x) \wedge \beta(x)) \to \exists x\alpha(x) \wedge \exists x\beta(x)$
16. $\forall x\alpha(x) \vee \forall x\beta(x) \to \forall x(\alpha(x) \vee \beta(x))$

Proof: As in the classical calculus for these and also for some other schemes which hold in $\mathcal{C}_1^\star$. ▮

Theorem 2.4.7 *In* $\mathcal{C}_1^\star$,

$\forall x(\alpha(x))^o \vdash \exists x\alpha(x) \leftrightarrow \neg\forall\ \neg\alpha(x)$

$\forall x(\alpha(x))^o \vdash \forall x\alpha(x) \leftrightarrow \neg\exists x\neg\alpha(x)$

$\forall x(\alpha(x))^o \vdash \neg\forall x\alpha(x) \leftrightarrow \exists x\neg\alpha(x)$

$\forall x(\alpha(x))^o \vdash \neg\exists x\alpha(x) \leftrightarrow \forall x\neg\alpha(x)$

Now we shall define the *k-transform* of a formula α, where k is a numeral among $\underline{1}, \underline{2}, \ldots$, that is, constant symbols in correspondence with the natural numbers $1, 2, \ldots$ [154, pp. 177ff]. By hypothesis, these symbols do not belong to the language of $\mathcal{C}_0^\star$ and of $\mathcal{C}_1^\star$.

Case 1: If α has no free variables, its k-transform is obtained as follows: each part of α of the form $\forall x\beta(x)$ or of the form $\exists x\beta(x)$ is substituted respectively by $\beta(\underline{1}) \wedge \beta(\underline{2}) \wedge \ldots \wedge \beta(\underline{n})$ or by $\beta(\underline{1}) \vee \beta(\underline{2}) \vee \ldots \vee \beta(\underline{n})$. Then such k-transform has no (individual) variables.
Case 2: If α has free variables, that is, if α is $\alpha(x_1, \ldots, x_m)$, then its k-transforms are obtained as follows: (a) we exchange the variables $x_1, \ldots, x_m$ by permutations of $1, 2, \ldots, k$ with repetitions of order k; (b) then, we take the k-transforms of the resulting formulas of (a).

Theorem 2.4.8 *If* $\Gamma \vdash \alpha$ *in* $\mathcal{C}_1^\star$*, then any* k*-transform of* α *is deducible, in* $\mathcal{C}_1$*, from the* k*-transforms of the formulas of* Γ.

Proof: By induction on the length of a deduction of α from Γ in the calculus $\mathcal{C}_1^\star$, as in the classical case, that is, as in $\mathcal{C}_0^\star$. ▮

Corolary 2.4.3 *If* $\vdash \alpha$ *in* $\mathcal{C}_1^\star$*, then the* k*-transforms of* α *are provable in* $\mathcal{C}_1$.

Corolary 2.4.4 *If the formula* α *has only predicate symbols of arity zero, that is, with zero associated variables and* $\vdash \alpha$ *in* $\mathcal{C}_1^\star$*, then* $\vdash \alpha$ *in* $\mathcal{C}_1$.

Proof: It suffices to note that in this case the only k-transform of α is α itself. ▮

Remark Corollary 2.4.4 is important for it emphasizes that those propositional schemes that do not hold in $\mathcal{C}_1$ continue not holding in $\mathcal{C}_1^\star$. In other words, if we adjoin to $\mathcal{C}_1$ the postulates and specific formation rules of $\mathcal{C}_1^\star$, no new result is obtained relatively to the *pure* formulas of $\mathcal{C}_1$. Furthermore, Theorem 2.4.8 can be extended to sub-systems of the classical propositional calculus (and of intuitionistic propositional calculus) and their corresponding quantification theories. This is the case, for instance, of classical positive logic (and intuitionistic positive logic), so as of the propositional intuitionistic calculus and the minimal intuitionistic calculus. Theorems 2.4.8 applies to the classical calculi $\mathcal{C}_0$ and $\mathcal{C}_0^\star$ and constitutes the so-called Hilbert-Bernays Theorem on k-transforms.

Theorem 2.4.9 *In $\mathcal{C}_1^\star$, the following schemes are not valid:*

1. $\neg\exists x\neg\alpha(x) \leftrightarrow \forall x\alpha(x)$
2. $\neg\forall x\neg\alpha(x) \leftrightarrow \exists x\alpha(x)$
3. $\neg\exists x\alpha(x) \leftrightarrow \forall x\neg\alpha(x)$
4. $\exists x\neg\alpha(x) \leftrightarrow \neg\forall x\alpha(x)$

Proof: It is easy to see that the 2-transforms of these schemes are not provable in $\mathcal{C}_1^\star$ (it suffices to take the 2-transforms and to use the tables of Theorem 2.1.3). ▮

Theorem 2.4.10 *$\mathcal{C}_1^\star$ is strictly weaker than $\mathcal{C}_0^\star$, and $\mathcal{C}_0^\star$ can be obtained from $\mathcal{C}_1^\star$ by adding to its postulates following scheme: $\neg(\alpha \wedge \neg\alpha)$.*

Proof: Really, $\mathcal{C}_0$ cames from $\mathcal{C}_1$ if we add such a schema to this last calculus as a postulate. ▮

The remaining calculi $\mathcal{C}_n^\star$, $2 \leq n < \omega$ of the hierarchy (4) are obtained by adjoining to $\mathcal{C}_n$, with obvious adaptations, the postulates I-IV and VII above, plus the following ones:

(V_n) $\forall\ (\alpha(x))^{(n)} \rightarrow (\forall x\alpha(x))^{(n)}$

(VI_n) $\forall\ (\alpha(x))^{(n)} \rightarrow (\exists x\alpha(x))^{(n)}$

$\mathcal{C}_\omega^\star$, is obtained from $\mathcal{C}_n$ by adjoining to its postulates the schemes IV and VII.

It is clear that the calculi $\mathcal{C}_n^\star$, $2 \leq n < \omega$ have properties similar to those of $\mathcal{C}_1^\star$. In particular, the following results hold.

Theorem 2.4.11 *If the quantificationally prime components of the formulas of Γ and of the formula α are $\alpha_1, \ldots, \alpha_m$, then if $\Gamma \vdash \alpha$ in $\mathcal{C}_0^\star$, then $\alpha_1^{(n)}, \ldots \alpha_m^{(n)}, \Gamma \vdash \alpha$ in $\mathcal{C}_n^\star$, $1 \leq n < \omega$, and conversely.*

Theorem 2.4.12 *$\mathcal{C}_n^\star$, $0 \leq n < \omega$ is undecidable.*

Theorem 2.4.13 *(Essenin-Volpin) $\mathcal{C}_\omega^\star$ is undecidable.*

Theorem 2.4.14 *If $\Gamma \vdash \alpha$ in $\mathcal{C}_n^\star$, $0 \leq n \leq \omega$, then the k-transforms of α are deducible, in $\mathcal{C}_n$, from the k-transforms of the formulas of Γ.*

Theorem 2.4.15 *If $\vdash \alpha$ in $\mathcal{C}_n^\star$, $0 \leq n \leq \omega$, then the k-transforms of α are provable in $\mathcal{C}_n$.*

Theorem 2.4.16 *If formula α belongs to $\mathcal{C}_n$, $0 \leq n < \omega$, and $\vdash \alpha$ in $\mathcal{C}_n^\star$, then $\vdash \alpha$ in $\mathcal{C}_n$.*

Theorem 2.4.17 *Every calculus of the hierarchy* $\mathcal{C}_n^\star$, $1 \leq n \leq \omega$ *is a proper subsystem of* $\mathcal{C}_0^\star$.

We can prove for the hierarchy (4) analogous relations as those established for the corresponding propositional calculi. For instance, we have:

Theorem 2.4.18 *Every calculus of the hierarchy* $\mathcal{C}_n^\star$, $0 \leq n \leq \omega$ *is strictly stronger than those which follows it.*

Theorem 2.4.19 *Every calculus* $\mathcal{C}_n^\star$, $0 \leq n < \omega$ *is finitely trivializable.*

Theorem 2.4.20 $\mathcal{C}_\omega^\star$ *is not finitely trivializable.*

2.5 Equality

¿From the hierarchy (4), one constructs the following hierarchy of predicate calculi with equality,

$$\mathcal{C}_0^=, \mathcal{C}_1^=, \mathcal{C}_2^=, \ldots, \mathcal{C}_n^=, \ldots, \mathcal{C}_\omega^=, \tag{5}$$

by adding to their languages the binary predicate symbol of equality, =, with suitable modifications in the concept of formula, and by adding the following two postulates:

(I') $\forall x(x = x)$

(II') $x = y \to (\alpha(x) \to \alpha(y))$

with the usual restrictions as in the classical calculus. Here, $\mathcal{C}_0^=$ stands for the classical first order predicate calculus with equality. We begin by studying $\mathcal{C}_1^=$.

Theorem 2.5.1 *We have in* $\mathcal{C}_1^=$*:*

1. $\vdash x = x$
2. $\vdash x = y \to y = x$
3. $\vdash x = y \wedge y = z \to x = z$

Theorem 2.5.2 *We have in* $\mathcal{C}_1^=$*:*

1. $\vdash x = y \leftrightarrow (\alpha(x) \to \alpha(y))$
2. $\forall t(\alpha(t))^o \vdash \alpha(x) \wedge \neg\alpha(y) \to x \neq y$, where $x \neq y$ stands for $\neg(x = y)$.

Proof: The first schema is proven as in $\mathcal{C}_0^=$, As for the second, firstly we note that $\forall t(\alpha(t))^o \vdash (\alpha(y))^o$ and $x = y \to (\alpha(x) \to \alpha(y))$. So,

$\forall t(\alpha(t))^o, \alpha(x) \wedge \neg\alpha(y), x = y \vdash (\alpha(y))^o$

$\forall t(\alpha(t))^o, \alpha(x) \wedge \neg\alpha(y), x = y \vdash \neg\alpha(y)$

$\forall t(\alpha(t))^o, \alpha(x) \wedge \neg\alpha(y), x = y \vdash \alpha(y)$.

Consequently,

$\forall t(\alpha(t))^o, \alpha(x) \wedge \neg\alpha(y) \vdash \neg(x = y)$

$\forall t(\alpha(t))^o \vdash \alpha(x) \wedge \neg\alpha(y) \to x \neq y$. ∎

Theorem 2.5.3 *In* $\mathcal{C}_1^=$*:*

$\forall y(\alpha(y) \leftrightarrow \exists x(x = y \wedge \alpha(x))$

$\forall y(\alpha(y) \leftrightarrow \forall x(x = y \rightarrow \alpha(x))$

$\forall y \exists x(x = y)$

Theorem 2.5.4 *If we add to the postulates of* $\mathcal{C}_1^=$ *the scheme* $\neg(\alpha \wedge \neg\alpha)$, *we obtain* $\mathcal{C}_0^=$.

Theorem 2.5.5 *If* $\Gamma \vdash \alpha$ *and* $\alpha_1, \ldots, \alpha_m$ *are the quantificationally prime components of the formulae of* Γ *and of* α, *then* $\Gamma \vdash \alpha$ *in* $\mathcal{C}_0^=$ *iff* $\alpha_1^o, \ldots, \alpha_m^o, \Gamma \vdash \alpha$ *in* $\mathcal{C}_1^=$.

Theorem 2.5.6 $\mathcal{C}_\omega^=$ *is undecidable.*

Proof: $\mathcal{C}_1^=$ contains $\mathcal{C}_0^=$ as $\mathcal{C}_1^\star$ contains $\mathcal{C}_0^\star$. So, $\mathcal{C}_1^=$ is undecidable. ▮

Really, all the valid schemes of $\mathcal{C}_0^=$ which do not explicitly contain the symbol $\neg$ are valid in $\mathcal{C}_1^=$.

Theorem 2.5.7 *If* $\Gamma \vdash \alpha$ *in* $\mathcal{C}_0^=$ *and if in the formulas of* $\Gamma \cup \{\alpha\}$ *we exchange* $\neg^\star$ *for* $\neg$, *which result in the formulas* $\Gamma^\star$ *and in the formula* $\alpha^\star$, *then* $\Gamma^\star \vdash \alpha^\star$ *in* $\mathcal{C}_1^=$.

Proof: It suffices to note that $\neg^\star$ is a 'classical' negation. ▮

Definition 2.5.1 $\exists! x\alpha(x) =_{\mathsf{def}} \exists y \forall x(y = x \leftrightarrow \alpha(x))$

Theorem 2.5.8 *In* $\mathcal{C}_1^=$, $\vdash \forall x \exists! y(x = y)$.

Theorem 2.5.9 *If* α *does not contains the equality symbol and* $\vdash \alpha$ *in* $\mathcal{C}_1^=$, *then* $\vdash \alpha$ *in* $\mathcal{C}_1^\star$.

Proof: Let D be a formal proof of α in $\mathcal{C}_1^=$. There are only a finite number of applications of postulate (II') in D, say

$$x_1 = y_1 \rightarrow (\alpha_1(x_1) \leftrightarrow \alpha_1(y_1)), \ldots, x_k = y_k \rightarrow (\alpha_k(x_k) \leftrightarrow \alpha_k(y_k)).$$

Using variables u and v that don't appear in thje formulas of D, we denote by $K(u, v)$ the universal closure of the following formula, with respect to all free variables distinct from u and v:

$$(\alpha_1(x_1) \leftrightarrow \alpha_1(y_1)) \wedge \ldots \wedge (\alpha_k(x_k) \leftrightarrow \alpha_k(y_k)).$$

Then, if we replace subformulas of the form $x = y$ of formulas occurring in D by $K(x, y)$, we get a formal proof of α in $\mathcal{C}_1^\star$, after a few suitable adaptations. ▮

As a consequence of the above theorem, we have that if $\Gamma \cup \{\alpha\}$ is a set of formulas of $\mathcal{C}_1^\star$ such that $\Gamma \vdash \alpha$ in $\mathcal{C}_1^=$, then $\Gamma \vdash \alpha$ also in $\mathcal{C}_1^\star$.

The theorem 2.5.9 is important, for it shows that the quantificational schemes that are not deducible in $\mathcal{C}_1^\star$ remain not deducible in $\mathcal{C}_1^=$. So, $\alpha \wedge \neg\alpha \rightarrow \beta$, $\neg(\alpha(x) \wedge \neg\alpha(x)$ and $\exists x\alpha(x) \leftrightarrow \neg\forall x \neg\alpha(x)$ are not theorems of $\mathcal{C}_1^=$, since they are not theorems of $\mathcal{C}_1^\star$.

In what concerns the calculi of the hierarchy (5), we can prove several interesting results. In what follows, we sum up some of these results, without proof.

Theorem 2.5.10 *The propositions* 2.5.1, 2.5.3, 2.5.4, 2.5.6 *and* 2.5.9 *hold for* $\mathcal{C}_n^=$, $0 \leq n \leq \omega$.

Theorem 2.5.11 *If the quantificationally prime components of the formulae of* $\Gamma \cup \{\alpha\}$ *are* $\alpha_1, \dots, \alpha_m$*, then if* $\Gamma \vdash \alpha$ *in* $\mathcal{C}_0^=$*, we have that* $\alpha_1^o, \dots, \alpha_m^o, \Gamma \vdash \alpha$ *in* $\mathcal{C}_n^=$*,* $1 \leq n \leq \omega$*.*

Theorem 2.5.12 *Every calculus of the hierarchy* (5) *is strictly stronger than those following it.*

Theorem 2.5.13 *The calculi of the hierarchy* $\mathcal{C}_n^=$*,* $1 \leq n < \omega$ *is finitely trivializable, while* $\mathcal{C}_\omega^=$ *is not.*

2.6 Descriptions

In the calculi $\mathcal{C}_n^=$, $1 \leq n \leq \omega$, we can introduce the description operator ι by means of contextual definitions, as we can do in $\mathcal{C}_0^=$, in accordance with Russell's well known definition. Alternatively, the symbol ι can be introduced in the languages of the calculi as a primitive symbol, subjected to the axioms shown below, given rise to a new hierarchy of calculi of descriptions:

$$\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n, \dots, \mathcal{D}_\omega. \tag{6}$$

The postulates to be added to those of $\mathcal{C}_n^=$, $1 \leq n \leq \omega$ are the following ones (subjected to the usual restrictions; see [215]).

(D.1) $\forall x F(x) \to F(\iota y Q(y))$

(D.2) $\forall x (P(x) \leftrightarrow Q(x)) \to \iota x P(x) = \iota x Q(x)$

(D.3) $\iota x F(x) = \iota y F(y)$

(D.4) $P(\iota y Q(y)) \to \exists x P(x)$

(D.5) $\exists! x P(x) \to (\forall x ((\iota x P(x) = x) \leftrightarrow P(x))$.

It can be proven that $\mathcal{D}_n$ is a conservative extension of $\mathcal{C}_n^=$ $(0 \leq n \leq \omega)$. As above, we shall be restricted to some results of the first calculus of the hierarchy, namely $\mathcal{D}_1$. The following results deserve mentioning (as above, proofs are adaptations of the classical case):

Theorem 2.6.1 *In* $\mathcal{D}_1$*,* $\vdash \iota x F(x) = \iota x F(x)$*.*

Theorem 2.6.2 *If* $\iota x F(x)$ *is free for* x *in* $F(x)$*, then* $\vdash \exists! x F(x) \to F(\iota x F(x))$*.*

Proof: From (D.5), we have $\vdash \exists! x F(x) \to \forall x ((\iota x F(x) = x) \leftrightarrow F(x))$. Taking $\exists! x F(x)$ as hypothesis, we get $\forall x ((\iota x F(x) = x) \leftrightarrow F(x))$. Then, from (D.1), $(\iota x F(x) = \iota x F(x)) \leftrightarrow F(\iota x F(x))$, hence $F(\iota x F(x))$. So, $\vdash \exists! x F(x) \to F(\iota x F(x))$. ∎

Theorem 2.6.3 *In* $\mathcal{D}_1$*,* $\vdash \forall y (\iota x (x = y) = y)$*.*

Theorem 2.6.4 *In* $\mathcal{D}_1$*, we have:*

1. $\vdash \forall y (\iota x P(x) = y \leftrightarrow y = \iota x P(x))$
2. $\vdash \forall y \forall x (\iota x P(x) = y \wedge y = z \to \iota x P(x) = z)$
3. $\vdash \forall x \forall z (x = \iota y Q(y) \wedge \iota y Q(y) = z \to x = z)$
4. $\vdash \forall z (\iota x P(x) = \iota y Q(y) \wedge \iota y Q(y) = z \to \iota x P(x) = z)$
5. $\vdash \forall y (\iota x P(x) = y \wedge y = \iota z R(z) \to \iota x P(x) = \iota z R(z))$

6. $\vdash \iota x P(x) = \iota y Q(y) \leftrightarrow \iota y Q(y) = \iota x P(x)$

7. $\vdash \iota x P(x) = \iota y Q(y) \wedge \iota y Q(y) = \iota z R(z) \to \iota x P(x) = \iota z R(z)$

8. $\vdash F(\iota x P(x)) \wedge \exists x(x = \iota x P(x) \to F(x))$

9. $\vdash F(\iota x P(x)) \leftrightarrow \forall x(x = \iota x P(x) \to F(x))$

10. $\vdash \iota x P(x) = \iota y Q(y) \wedge \iota z Q(z) = \iota t K(t) \to \iota x P(x) = \iota t K(t)$

Theorem 2.6.5 *Let $F(x)$ be a formula and u and t two terms (variables or descriptions) free for x in $F(x)$. Then,*

$$\vdash u = v \to (F(u) \to F(v)).$$

Theorem 2.6.6 *If y does not occur free in $P(x)$ and $\iota x P(x)$ is free for x in $P(x)$, then*

$$\vdash \exists! x P(x) \to (F(\iota x P(x)) \leftrightarrow \exists y(F(y) \wedge \forall x(x = y \leftrightarrow P(x)))).$$

Proof: Let us assume $\exists! x P(x)$. So,

$$\forall x(\iota x P(x) = x \leftrightarrow P(x)).$$

Then, taken $F(\iota x P(x))$ as an hypothesis, we get

$$F(\iota x P(x)) \text{ and } \forall x(\iota x P(x) = x \leftrightarrow P(x)),$$

hence

$$F(\iota x P(x)) \wedge \forall x(x = \iota x P(x) \leftrightarrow P(x)),$$

so

$$\exists y(F(y) \wedge \forall x(x = y \leftrightarrow P(x)).$$

Consequently,

$$F(\iota x P(x)) \to \exists y(F(y) \wedge \forall x(x = y \leftrightarrow P(x))).$$

But, assuming that

$$\exists y(F(y) \wedge \forall x(x = y \leftrightarrow P(x)),$$

it results that

$$F(y) \wedge \forall x(x = y \leftrightarrow P(x)),$$

so

$$F(y) \text{ and } \forall x(x = y \leftrightarrow P(x)),$$

and hence we conclude that $y = \iota x P(x)$ and that $F(\iota x P(x))$. Then,

$$\exists y(F(y) \wedge \forall x(x = y \leftrightarrow P(x)) \to F(\iota x P(x)).$$

So,

$$\exists! x P(x) \to (F(\iota x P(x)) \leftrightarrow \exists y(F(y) \wedge \forall x(x = y \leftrightarrow P(x)))). \blacksquare$$

Theorem 2.6.7 *Let $A_1, \ldots, A_n$ the quantificationally prime components of the formulas of Γ and of the formula F. Then, if $\Gamma \vdash F$ in $\mathcal{D}_0$, then $A_1^o, \ldots, A_n^o, \Gamma \vdash F$ in $\mathcal{D}_1$ and reciprocally.*

Let us consider the set of formulas Γ of $\mathcal{D}_1$ and suppose that the finitely many descriptions that appear in these formulas are

$\iota x_1 Q_1(x_1), \iota x_2 Q_1(x_2), \ldots$

$\iota x_1 Q_2(x_1), \iota x_2 Q_2(x_2), \ldots$

$\vdots$

which we admit ordered by some order relation. We can them associate to each of such descriptions one of the variables $t_0, \ldots, t_{n-1}$, which do not appear in the formulas of Γ, in such a manner that the association is a bijection. The ι-transforms of the formulas of Γ are the formulas obtained from the members of Γ by replacing in them the descriptions by their associated variables, and identifying the variables corresponding to descriptions like $\iota x F(x)$ of D.4.

Theorem 2.6.8 *If $\vdash F$ in $\mathcal{D}_1$ and F does not contain the symbols $=$ and ι, then $\vdash F$ in $\mathcal{C}_1^\star$.*

Proof: If F can be proven in $\mathcal{D}_1$, it is deducible in $\mathcal{C}_1^{=\star}$ from a finite number of formulas of types D.1–D.5 (see page 23 above), where $\mathcal{C}_1^{=\star}$ is the calculus obtained from $\mathcal{C}_1^=$ by introducing formation rules enabling descriptions as new terms in addition to the variables. Let us denote by Δ a deduction of F in $\mathcal{C}_1^{=\star}$ from formulas of types D.1–D.5. If we substitute, in such a deduction, formulas by their ι-transforms, we see that we get a valid deduction $\Delta^\star$ in $\mathcal{C}_1^=$. But after such substitutions, the formulas of types D.1, D.3 and D.4 turn out to be axioms of $\mathcal{C}_1^=$ and $\Delta^\star$ is a deduction of F from formulas of the forms (i) $\forall x(P(x) \leftrightarrow Q(x)) \to t_i = t_j$ and (ii) $\exists! x P(x) \to \forall x(t_k = x \leftrightarrow P(x))$. If in $\Delta^\star$ we identify pairs of variables like t_i and t_j which appear free in the formulas of the type (i) above, then $\Delta^\star$ becomes a deduction $\Delta^{\star\star}$, still valid in $\mathcal{C}_1^=$; but since the formulas of the specified type are axioms of $\mathcal{C}_1^=$, they can be suppressed from the hypothesis of $\Delta^{\star\star}$. It results a deduction $\Delta^{\star\star\star}$ of F in $\mathcal{C}_1^=$ from formulas of the form (ii). However, to say that F is deducible in $\mathcal{C}_1^=$ from formulas of the above type is to say that F is deducible in $\mathcal{C}_1^\star$ from formulas of the following forms: $\forall x(x = x)$, $x = y \to (A(x) \to A(y))$ and $P(t_k)$. Therefore, taking into account theorem 2.5.9, we easily complete the proof. ∎

The calculi $\mathcal{D}_0, \mathcal{D}_2, \ldots, \mathcal{D}_n, \ldots, \mathcal{D}_\omega$ have properties similar to those made for $\mathcal{D}_1$.

Theorem 2.6.9 *The Theorems 2.6.1 to 2.6.6 hold for every calculi $\mathcal{D}_n$, $0 \leq n \leq \omega$.*

Theorem 2.6.10 *If $\vdash F$ in $\mathcal{D}_0$ and F does not contain the symbol ι, then $\vdash F$ in $\mathcal{C}_0^=$.*

Theorem 2.6.11 *The calculi $\mathcal{D}_n$, $0 \leq n \leq \omega$ are simply consistent.*[7]

Proof: By the last theorem, $\mathcal{D}_0$ is simply consistent, once $\mathcal{C}_0^=$ is. Since the remaining calculi are sub-systems of $\mathcal{D}_0$, the theorem follows. ∎

Theorem 2.6.12 *Being $0 \leq n < \omega$ and $A_1, \ldots, A_n$ the quantificationally prime components of the formulas of Γ and of F, then $\Gamma \vdash F$ in $\mathcal{D}_0$ iff $A_1^{(n)}, \ldots, A_n^{(n)}\Gamma \vdash F$ in $\mathcal{D}_0$.*

Theorem 2.6.13 *The calculi $\mathcal{D}_n$, $0 \leq n < \omega$ are undecidable.*

Theorem 2.6.14 *Every calculus of the hierarchy $\mathcal{D}_n$, $0 \leq n \leq \omega$ is strictly stronger than those following it.*

Theorem 2.6.15 *$\mathcal{D}_n$, $0 \leq n < \omega$ is finitely trivializable.*

Theorem 2.6.16 *$\mathcal{D}_\omega$ is not finitely trivializable.*

Theorem 2.6.17 *If $\Gamma \vdash \alpha$ in $\mathcal{D}_n$ and the formulas of $\Gamma \cup \{\alpha\}$ do not contain the description symbol, then $\Gamma \vdash \alpha$ in $\mathcal{C}_n^=$, $0 \leq n \leq \omega$.*

[7] This means that for no formula A, are both A and $\neg A$ provable in these calculi ([154, p. 124]).

2.7 Semantics

We begin by sketching a two-valued semantics for $\mathcal{C}_1$.[8] Here, $\mathcal{F}$ stands for the set of formulas of $\mathcal{C}_1$, Γ and Δ designate subsets of $\mathcal{F}$, while $\overline{\Gamma}$ denotes the set $\{\alpha \in \mathcal{F} \ : - \vdash \alpha\}$.

We say that a set Γ of formulas is trivial iff $\overline{\Gamma} = \mathcal{F}$; otherwise, it is non-trivial. Γ is inconsistent iff there is at least one formula α such that both α and $\neg\alpha$ belong to $\overline{\Gamma}$; otherwise, Γ is consistent. Finally, Γ is maximal non-trivial iff it is not-trivial and, for any formula α, if $\alpha \notin \Gamma$, then $\Gamma \cup \{\alpha\}$ is trivial.

Theorem 2.7.1 *If* Γ *is maximal non-trivial, then:*

1. $\Gamma \vdash \alpha \Leftrightarrow \alpha \in \Gamma$	2. $\alpha \in \Gamma \Rightarrow \neg^\star\alpha \notin \Gamma$
3. $\neg^\star\alpha \in \Gamma \Rightarrow \alpha \notin \Gamma$	4. $\alpha \in \Gamma$ or$\neg^\star\alpha \in \Gamma$
5. $\vdash \alpha \Rightarrow \alpha \in \Gamma$	6. $\alpha, \alpha^o \in \Gamma \Rightarrow \neg\alpha \notin \Gamma$
7. $\neg\alpha, \alpha^o \in \Gamma \Rightarrow \alpha \notin \Gamma$	8. $\alpha, \alpha \to \beta \in \Gamma \Rightarrow \beta \in \Gamma$
9. $\alpha^o \in \Gamma \Rightarrow \alpha \notin \Gamma$ or$\neg\alpha \notin \Gamma$	10. $\alpha^o \in \Gamma \Rightarrow (\neg\alpha)^o \in \Gamma$

Proof: We shall prove only the first property. Suppose that $\Gamma \vdash \alpha$ but $\alpha \notin \Gamma$. Then, since Γ is maximal non-trivial, $\Gamma \cup \{\alpha\} \vdash \alpha \wedge \neg^\star\alpha$. Hence $\Gamma \vdash \alpha \to (\alpha \wedge \neg^\star\alpha)$, and $\Gamma \vdash \neg^\star\alpha$. But, since $\Gamma \vdash \alpha$, then $\Gamma \vdash \alpha \wedge \neg^\star\alpha$, therefore Γ is trivial, which is absurd. ▮

Definition 2.7.1 *A valuation of* $\mathcal{C}_1$ *is a mapping* $\nu : \mathcal{F} \mapsto \{1, 0\}$ *such that:*

1) $\nu(\alpha) = 0 \Rightarrow \nu(\neg\alpha) = 1$
2) $\nu(\neg\neg\alpha) = 1 \Rightarrow \nu(\alpha) = 1$
3) $\nu(\beta^o) = \nu(\alpha \to \beta) = \nu(\alpha \to \neg\beta) = 1 \Rightarrow \nu(\alpha) = 0$
4) $\nu(\alpha \to \beta) = 1 \Leftrightarrow \nu(\alpha) = 0$ *or* $\nu(\beta) = 1$
5) $\nu(\alpha \wedge \beta) = 1 \Leftrightarrow \nu(\alpha) = \nu(\beta) = 1$
6), $\nu(\alpha \vee \beta) = 1 \leftrightarrow \nu(\alpha) = 1$ *or* $\nu(\beta) = 1$
7) $\nu(\alpha^o) = \nu(\beta^o) = 1 \Rightarrow \nu((\alpha \vee \beta)^o) = \nu(\alpha \wedge \beta)^o) = \nu((\alpha \to \beta)^o) = 1.$

Theorem 2.7.2 *If* ν *is a valuation of* $\mathcal{C}_1$, *it has the following properties:* $\nu(\alpha) = 1 \Leftrightarrow \nu(\neg^\star\alpha) = 0$, $\nu(\alpha) = 0 \Leftrightarrow \nu(\neg^\star\alpha) = 1$, $\nu(\alpha) = 0 \Leftrightarrow \nu(\alpha) = 0$ *and* $\nu(\neg\alpha) = 1$, $\nu(\alpha) = 1 \Leftrightarrow \nu(\alpha) = 1$ *or* $\nu(\neg\alpha) = 0$.

Definition 2.7.2 *(i) A valuation* ν *is* singular *if there exists at least one formula* α *such that* $\nu(\alpha) = \nu(\neg\alpha) = 1$; *otherwise,* ν *is* normal. *(ii) A formula* α *is* valid *if for every valuation* ν, $\nu(\alpha) = 1$. *(iii) A valuation* ν *is a* model *for a set* Γ *of formulas if* $\nu(\alpha) = 1$ *for any* $\alpha \in \Gamma$. *(iv) A formula* α *is a* semantical consequence *of* Γ *if every model* ν *of* Γ *is such that* $\nu(\alpha) = 1$; *in this case, we write* $\Gamma \models \alpha$. *In particular,* $\models \alpha$ *(which abbreviates* $\emptyset \models \alpha$*), means that* α *is valid.*

Theorem 2.7.3 (Soundness) $\Gamma \vdash \alpha \Rightarrow \Gamma \models \alpha$ *(in particular,* $\vdash \alpha \Rightarrow \models \alpha$*).*

Proof: As in classical logic. ▮

Lemma 2.7.1 *Every non-trivial set of formulas is contained in a maxima non-trivial set.*

Proof: By adapting the corresponding proof in classical logic. ▮

Corolary 2.7.1 *There exist maximal non-trivial inconsistent sets.*

[8]The semantics of valuations, conceived as a general semantical method, was developed by the first author of this paper in the sixties of last century, in his Logic Seminar at the Federal University of Paraná, Brazil.

Proof: It is easy to see that $\{\alpha, \neg\alpha\}$ is inconsistent but non-trivial. By the preceding Lemma, it is contained in a maximal non-trivial set of formulas, which is inconsistent. ▮

Lemma 2.7.2 *Every maximal non-trivial set* Γ *of formulas has a model.*

Proof: Define a mapping $\nu : \mathcal{F} \mapsto \{0, 1\}$ as follows: for every formula α, if $\alpha \in \Gamma$, then $\nu(\alpha) = 1$, and $\nu(\alpha) = 0$ otherwise. Then, it is easy to see that ν satisfies all the conditions in the definition of a valuation. ▮

Corolary 2.7.2 *Any non-trivial set of formulas has a model.*

Corolary 2.7.3 *There are singular (and of course normal) valuations.*

Proof: The set $\{\alpha, \neg\alpha\}$ is inconsistent but non-trivial, hence it is contained in a maximal non-trivial set, as shown above. But this set has a model, which of course is singular. ▮

Theorem 2.7.4 (Completeness) $\Gamma \models \alpha \Rightarrow \Gamma \vdash \alpha$ *(in particular,* $\models \alpha \Rightarrow \vdash \alpha$*).*

Proof: Let us suppose that $\Gamma \nvdash \alpha$; then, $\Gamma \cup \{\neg^{\star}\alpha\}$ is non-trivial and has a model ν. So, there is a model ν of Γ such that $\nu(\alpha) = 0$, which is absurd.▮

Theorem 2.7.5 *There are inconsistent (but non-trivial) sets of formulas which have models.*

Definition 2.7.3 *Let* $\Delta =_{\mathsf{def}} \{\alpha^o \in \mathcal{F} : \vdash \alpha\}$*. Then* Γ *is said strongly non-trivial if* $\Gamma \cup \Delta$ *is non-trivial.* Γ *is said to be strictly non-trivial if* $\Gamma \cup \Delta$ *is non-trivial.*

Theorem 2.7.6 *There exist sets of formulas which are non-trivial and sets of formulas which are strictly non-trivial.*

Proof: To prove the first part of the theorem, suppose that Δ is the set $\{\alpha^o \in \mathcal{F} : \vdash \alpha\}$. Then Δ is consistent, which implies that it is also non-trivial. So, Δ is contained in a maximal non-trivial set Δ'. Let Δ'' be $\Delta' - \Delta$. Then Δ'' is strongly non-trivial. ▮

Next, we shall discuss a byproduct of the semantics for $\mathcal{C}_1$ sketched above, namely, the decidability of this calculus.

2.7.1 The decidability of $\mathcal{C}_1$.

Definition 2.7.4 (Quasi-Matrix) *We call quasi-matrices the tables constructed according to the following instructions, for each formula of* $\mathcal{C}_1$ *(see [9]): given a formula* α*,*

1. Make a list of all propositional variables which appear in α, and arrange them in a line.

2. Under the propositional variables, place all possible distributions of 0's and 1's which can be attributed to them, as usual.

3. Make a list of the negations of propositional variables appearing in the formula, and calculate their truth value, in each line, as follows: (i) if a variable has value 0, its negation will have truth value 1; (ii) if the variable has value 1, bifurcate the line in which it appears, by writing in the first part the value 0 for the negation and, in the second part, the value 1 for the negation. Before a bifurcation, the truth values must be the same for the two resulting lines.

4. Complete the previous list, adding the sub-formulas of α and the negations of proper sub-formulas, and calculate, for each line, the truth value of each sub-formula of α and, if it is a proper sub-formula, calculate the value of its negation, whose proper sub-formulas and their negations had already been listed and calculated, as follows: (i) When no negations are involved, proceed as in the truth-tables for the classical propositional calculus; (ii) If any of the formulas in consideration is a negation, so of the form $\neg\alpha'$, write the truth value 1 under it in those lines where α' has value 0, and in the lines where α' has value 1, proceed as follows: (i') If α' is of the form $\neg\beta$, then check if the value of β is the same as the value of $\neg\beta$; in this case, bifurcate the line writing the value 0 in the first part and the value 1 in the second one. If the value of β is distinct from the value of $\neg\beta$, simply write 0. (ii') If α' is of the form $\beta \bowtie \gamma$, where $\bowtie \in \{\rightarrow, \vee, \wedge\}$, there are two cases to consider: (a) α' if of the form $\delta \wedge \neg\delta$ or of the form $\neg\delta \wedge \delta$. In this case, write the value 0 for the formula α'; (b) α' is neither of the form $\delta \wedge \neg\delta$ nor of the form $\neg\delta \wedge \delta$. In this case, check if the value of β if equal to the value of $\neg\beta$ or if the value of γ is equal to the value of $\neg\gamma$. In the positive case, bifurcate the line, writing the value 0 in the first part and, in the second one, write 1. If the value of β is distinct from the value of $\neg\beta$, simply write 0.

Below we shall exemplify this definition. Before that, we shall state the following lemma:

Lemma 2.7.3 $\nu : \mathcal{F} \mapsto \{0, 1\}$ *is a valuation iff:*

1. $\nu(\neg\alpha) = 0 \Rightarrow \nu(\alpha) = 1$
2. $\nu(\neg\neg\alpha) = 1 \Rightarrow \nu(\alpha) = 1$
3. $\nu(\beta^o) = \nu(\alpha \rightarrow \beta) = \nu(\alpha \rightarrow \neg\beta) = 1 \Rightarrow \nu(\alpha) = 0$
4. $\nu(\alpha \rightarrow \beta) = 1 \Leftrightarrow \nu(\alpha) = 0 \,\text{or}\, \nu(\beta) = 1$
5. $\nu(\alpha \wedge \beta) = 1 \Leftrightarrow \nu(\alpha) = \nu(\beta) = 1$
6. $\nu(\alpha \vee \beta) = 1 \Leftrightarrow \nu(\alpha) = 1$ or $\nu(\beta) = 1$
7. $\nu((\alpha \wedge \beta)^o) = 0 \Leftrightarrow \nu(\alpha^o) = 0$ or $\nu(\beta^o) = 0$
8. $\nu((\alpha \vee \beta)^o) = 0 \Leftrightarrow \nu(\alpha^o) = 0$ and $\nu(\beta^o) = 0$
9. $\nu((\alpha \rightarrow \beta)^o) = 0 \Leftrightarrow \nu(\alpha^o) = 0$ or $\nu(\beta^o) = 1$

Lemma 2.7.4 $\nu(\alpha^o) = 0 \Leftrightarrow \nu(\alpha) = \nu(\neg\alpha) = 1$

Proof: (a) $\nu(\alpha^o) = 0$ implies that $\nu(\alpha \wedge \neg\alpha) = 1$ and $\nu(\alpha) = \nu(\neg\alpha) = 1$. (b) Suppose that $\nu(\alpha) = \nu(\neg\alpha) = 1$. If $\nu(\alpha^o) = 1$, then $\nu(\alpha) = \nu(\neg\alpha) = \nu(\alpha^o) = 1$, that is, $\nu(\alpha) = \nu(\neg^\star\alpha) = 1$, and ν would not be a valuation. Hence $\nu(\alpha^o) = 0$ and, therefore, $\nu(\alpha) = \nu(\neg\alpha) = 1$. ▮

Lemma 2.7.5 $\nu : \mathcal{F} \mapsto \{0, 1\}$ *is a valuation iff the conditions 1-6 of Lemma 2.7.3 hold and:*

7i. $\nu((\alpha \rightarrow \beta)^o) = 0 \Leftrightarrow \nu(\alpha) = \nu(\neg\alpha) = 1 \,\text{or}\, \nu(\beta) = \nu(\neg\beta) = 1$

7ii. $\nu((\alpha \wedge \beta)^o) = 0 \Leftrightarrow \nu(\alpha) = \nu(\neg\alpha) = 1 \,\text{or}\, \nu(\beta) = \nu(\neg\beta) = 1$

7ii. $\nu((\alpha \vee \beta)^o) = 0 \Leftrightarrow \nu(\alpha) = \nu(\neg\alpha) = 1 \,\text{or}\, \nu(\beta) = \nu(\neg\beta) = 1$

Definition 2.7.5 *Let ν be a valuation and α a formula. Then ν_α is called the restriction of ν to the set of sub-formulas of α and the negations of proper sub-formulas of α.*

Lemma 2.7.6 *For every valuation ν and formula α, $\nu(\alpha) = \nu_\alpha(\alpha)$.*

Definition 2.7.6 *Let ν be a valuation and Γ a set of formulas. Then ν_Γ is the restriction of ν to the set Γ.*

Definition 2.7.7 *We say that a line of a quasi-matrix* correspondsto *ν_Γ if $\nu_\Gamma(\alpha)$ is the value corresponding to α in that line for every $\alpha \in \Gamma$, where Γ is the set of all formulas of the matrix.*

Lemma 2.7.7 *Given a quasi-matrix Q, then for every valuation ν there exists a line of Q which corresponds to ν_Γ, where Γ is the set of all formulas of Q.*

Proof: By induction on the number of columns of Q. ∎

Definition 2.7.8 *Let Q be a quasi-matrix for a formula α and let Γ the set of all sub-formulas and of negations of proper sub-formulas of α. Let k be a line of Q and $k(\alpha)$ the value attributed to α in k. Then we call $\Delta(\Gamma, k)$ the set of formulas such that, for every formula α,*

(I) *If $\alpha \in \Gamma$, then $\alpha \in \Delta(\Gamma, k)$ iff $k(\alpha) = 0$*

(II) *If $\alpha \notin \Gamma$, then $\alpha \in \Delta(\Gamma, k)$ iff*

a) α is atomic, or

b) α is of the form $\neg\alpha_1$ and $\alpha_1 \notin \Delta(\Gamma, k)$, or

c) α is $\alpha_1 \wedge \alpha_2$ and $\alpha_1 \in \Delta(\Gamma, k)$ and $\alpha_2 \in \Delta(\Gamma, k)$, or

d) α is $\alpha_1 \vee \alpha_2$ and $\alpha_1 \in \Delta(\Gamma, k)$ and $\alpha_2 \in \Delta(\Gamma, k)$, or

e) α is $\alpha_1 \to \alpha_2$ and $\alpha_1 \notin \Delta(\Gamma, k)$ and $\alpha_2 \in \Delta(\Gamma, k)$

Some properties of the set $\Delta(\Gamma, k)$ are the following:

1. $\neg\alpha \in \Delta(\Gamma, k) \Rightarrow \alpha \notin \Delta(\Gamma, k)$

2. $\alpha \in \Delta(\Gamma, k) \Rightarrow \neg\neg\alpha \in \Delta(\Gamma, k)$

3. $\neg^\star\alpha \in \Delta(\Gamma, k) \Leftrightarrow \alpha \notin \Delta(\Gamma, k)$

4. $\alpha \to \beta \notin \Delta(\Gamma, k) \Leftrightarrow \alpha \in \Delta(\Gamma, k)$ or $\beta \notin \Delta(\Gamma, k)$

5. $\alpha \in \Delta(\Gamma, k)$ or $\beta \in \Delta(\Gamma, k) \Leftrightarrow \alpha \wedge \beta \in \Delta(\Gamma, k)$

6. $\alpha \notin \Delta(\Gamma, k)$ or $\beta \notin \Delta(\Gamma, k) \Leftrightarrow \alpha \vee \beta \notin \Delta(\Gamma, k)$

7. $(\alpha \bowtie \beta)^o \in \Delta(\Gamma, k) \Rightarrow \alpha \notin \Delta(\Gamma, k)$ and $\neg\alpha \notin \Delta(\Gamma, k)$, or $\beta \notin \Delta(\Gamma, k)$ and $\neg\beta \notin \Delta(\Gamma, k)$, where $\bowtie \in \{\to, \wedge, \vee\}$.

Lemma 2.7.8 (A. Loparić) *For every line k of a quasi-matrix Q, there is a valuation ν such that ν_Γ corresponds to k, where Γ is the set of formulas of Q.*

Proof: Let ν be a function from $\mathcal{F}$ in $\{0, 1\}$ such that, for every $\alpha \in \mathcal{F}$, $\nu(\alpha) = 0$ if $\alpha \in \Delta(\Gamma, k)$ and $\nu(\alpha) = 1$ if $\alpha \notin \Delta(\Gamma, k)$. Then, by the above properties 1-7 of the set $\Delta(\Gamma, k)$, ν is a valuation. Since ν_Γ and k are the same, we can say that there exists a valuation ν such that ν_Γ corresponds to k. ∎

Theorem 2.7.7 (M. Fidel) *The calculus $\mathcal{C}_1$ is decidable.*

Proof: Consequence of Lemmas 2.7.6, 2.7.7 and 2.7.8. The formula α is a theorem of $\mathcal{C}_1$ iff in any quasi-matrix for α the last column has only 1's. In effect, in this case, for every valuation ν, we have that $\nu(\alpha) = \nu_\alpha(\alpha) = 1$. ▮

Let us exemplify the method presented here by showing that $\neg(\alpha \vee \beta) \rightarrow \neg\alpha \wedge \neg\beta$ is not valid in $\mathcal{C}_1$:

α	β	$\neg\alpha$	$\neg\beta$	$\alpha \vee \beta$	$\neg(\alpha \vee \beta)$	$\neg\alpha \wedge \neg\beta$	$\neg(\alpha \vee \beta) \rightarrow \neg\alpha \wedge \neg\beta$
0	0	1	1	0	1	1	1
		0	1	1	0	0	1
1	0	1	1	1	0	1	1
					1	1	1
			0	1	0	0	1
0	1	1	1	1	0	1	1
					1	1	1
			0	1	0	0	1
		0	1	1	0	0	1
					1	0	0
1	1		0	1	0	0	1
					1	0	0
		1	1	1	0	1	1
					1	1	1

The extension of the semantics of $\mathcal{C}_1$ to the systems $\mathcal{C}_n, 2 \leq n < \omega$ is immediate. All definitions and theorems are the same, with evident adaptations (for instance, $\neg^\star$ becomes $\neg^{(n)}$, and α^o becomes $\alpha^{(n)}$). In constructing quasi-matrices, when α is of the form $\beta^{(n-1)} \wedge \neg\beta^{(n-1)}$ or of the form $\neg\beta^{(n-1)} \wedge \beta^{(n-1)}$, we must write 0 for the formula α. Really, we must have $\nu(\neg\alpha) = 0$ for, if not, it results, due to clause 7 of Definition 2.7.1, that $\nu(\beta \wedge \neg^{(n)}\beta) = 1$, and then ν would not be a valuation. We remark that this clause is precisely that one which characterizes the quasi-matrices of the systems $\mathcal{C}_n, 1 \leq n < \omega$.

For instance, to see that the schema $(\alpha^{(n-1)} \wedge \neg\alpha^{(n-1)})^{(n)}$ is valid in $\mathcal{C}_n$, but not in $\mathcal{C}_m$, for $m > n$, it suffices to show that the schema $(\alpha \wedge \neg\alpha)^o$ is valid in $\mathcal{C}_1$, but not in $\mathcal{C}_2$. Really, we have in $\mathcal{C}_1$:

α	$\neg\alpha$	$\alpha \wedge \neg\alpha$	$\neg(\alpha \wedge \neg\alpha)$	$(\alpha \wedge \neg\alpha) \wedge \neg(\alpha \wedge \neg\alpha)$	$\neg((\alpha \wedge \neg\alpha) \wedge \neg(\alpha \wedge \neg\alpha))$
0	1	0	1	0	1
1	0	0	1	0	1
	1	1	0	0	1

while, in $\mathcal{C}_2$,

α	$\neg\alpha$	$\alpha \wedge \neg\alpha$	$\neg(\alpha \wedge \neg\alpha)$	$(\alpha \wedge \neg\alpha) \wedge \neg(\alpha \wedge \neg\alpha)$	$\neg((\alpha \wedge \neg\alpha) \wedge \neg(\alpha \wedge \neg\alpha))$
0	1	0	1	0	1
1	0	0	1	0	1
	1	1	0	0	1
			1	1	0
					1

Is also possible to show that $\mathcal{C}_\omega$ is decidable; see [167] and section 6.

2.7.2 Semantics for $\mathcal{C}_1^=$

The semantics sketched above can be extended to the quantificational calculi described earlier. Let us begin by defining an *interpretation* for $\mathcal{C}_1^=$ (really, for the language of $\mathcal{C}_1^=$) as a pair $I = \langle D, \rho\rangle$, where D is a non-empty set and ρ is a mapping such that: (i) ρ associates an element $\rho(a) \in D$ to each individual constant a of the language of $\mathcal{C}_1^=$; (ii) for each n-ary functional symbol f of the language of $\mathcal{C}_1^=$, ρ associates an n-ary function from D^n to D, and (iii) to each n-ary predicate symbol P, other than equality, ρ associates an n-ary relation on D; (iv) to the equality symbol $=$, ρ associates the diagonal of D, namely, the set $\Delta_D =_{\mathsf{def}} \{\langle x, x\rangle : x \in D\}$.[9]

If I is an interpretation for $\mathcal{C}_1^=$, then for each element of D we choose a new individual constant c, the name of the element (as usual, different names are chosen for different elements). This new language, as it is well known, is called the *diagram language* of $\mathcal{C}_1^=$ relatively to I ([223, p. 18]), termed $\mathrm{D}\mathcal{C}_1^=$.

If I is an interpretation for $\mathcal{C}_1^=$, a *valuation* ν of $\mathcal{C}_1^=$ is a map from the set of sentences of $\mathrm{D}\mathcal{C}_1^=$ in $\{0, 1\}$, defined as follows:

(1–7) Clauses 1-7 of Definition 2.7.1

8) $\nu(\forall x\alpha(x)) = 1$ iff for any individual constant c of $\mathrm{D}\mathcal{C}_1^=$, $\nu(\alpha(c)) = 1$

9) $\nu(\exists x\alpha(x)) = 1$ iff there exists a constant c of $\mathrm{D}\mathcal{C}_1^=$ so that $\nu(\alpha(c)) = 1$

10) $\nu(\forall x(\alpha(x))^o) = 1 \Rightarrow \nu((\forall x\alpha(x))^o) = \nu((\exists x\alpha(x))^o) = 1$

11) If α and β are congruent (see 17), then $\nu(\alpha) = \nu(\beta)$

12) $\nu(c = c') = 1$ iff $\rho(c) = \rho(c')$

13) $\nu(c = c') = 1$ and $\nu(\alpha(c)) = 1$, then $\nu(\alpha(c')) = 1$

We say that a valuation ν *satisfies* a sentence α of $\mathrm{D}\mathcal{C}_1^=$ (an in particular of $\mathcal{C}_1^=$) if $\nu(\alpha) = 1$. ν is a *model* for a set Γ of sentences if $\nu(\alpha) = 1$ for every $\alpha \in \Gamma$, and α is a *semantic consequence* of Γ, in symbols, $\Gamma \vDash \alpha$ if $\nu(\alpha) = 1$ for every model ν of Γ.

Theorem 2.7.8 (Soundness) $\Gamma \vdash \alpha \Rightarrow \Gamma \vDash \alpha$, *where* $\Gamma \cup \{\alpha\}$ *is a set of sentences.*

Proof: As in the classical case, using induction on the length of the proof of α from Γ. ▮

We say that Γ is *trivial* if $\Gamma \vdash \alpha$ for every α; otherwise, Γ is *non-trivial.* Γ is *inconsistent* if there exists α such that $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$; otherwise, Γ is *consistent*. Finally, a set Γ os sentences is a *Henkin set* if for any formula $\alpha(x)$ with just one free variable, there exists a constant c of the language of $\mathcal{C}_1^=$ such that $\Gamma \vdash \exists x\alpha(x) \to \alpha(c)$.

Theorem 2.7.9 *If* Γ *is non-trival (and a Henkin set), then it is contained in a maximal non-trivial (and also a Henkin set) set of sentences.*

Proof: As in the classical case. ▮

Theorem 2.7.10 *If* Γ *is a maximal non-trivial Henkin set of sentences, then:*

(1) $\alpha \to \beta \in \Gamma \Leftrightarrow \alpha \notin \Gamma$ *or* $\beta \in \Gamma$	(2) $\alpha \wedge \beta \in \Gamma \Leftrightarrow \alpha, \beta \in \Gamma$
(3) $\alpha \vee \beta \in \Gamma \Leftrightarrow \alpha \in \Gamma$ *or* $\beta \in \Gamma$	(4) $\Gamma \vdash \alpha \Leftrightarrow \alpha \in \Gamma$
(5) $\alpha \in \gamma \Leftrightarrow \neg^\star\alpha \notin \Gamma$	(6) $\neg\alpha, \alpha^o \in \Gamma \Rightarrow \alpha \notin \Gamma$
(7) $\alpha, \alpha^o \in \Gamma \Rightarrow \neg\alpha \notin \Gamma$	(8) $\alpha \in \Gamma$ *or* $\neg^\star\alpha \in \Gamma$
(9) $\alpha, \alpha \to \beta \in \Gamma \Rightarrow \beta \in \Gamma$	(10) $\alpha^o \in \Gamma \Rightarrow (\neg\alpha)^o \in \Gamma$

[9] As we have said before, the language may contain neither individual constants nor functional symbols.

(11) $\alpha^o, \beta^o \in \Gamma \Rightarrow (\alpha \to \beta)^o, (\alpha \wedge \beta)^o, (\alpha \vee \beta)^o \in \Gamma$
(12) $\forall x \alpha(x) \in \Gamma$ *iff for any constant c of* $\mathcal{C}_1^=$, $\alpha(c) \in \Gamma$
(13) $\exists x \alpha(x) \in \Gamma$ *iff for some constant c of* $\mathcal{C}_1^=$, $\alpha(c) \in \Gamma$
(14) $\forall x(\alpha(x))^o \in \Gamma \Rightarrow (\forall x \alpha(x))^o \in \Gamma$ *and* $(\exists x \alpha(x))^o \in \Gamma$
(15) *If* α *and* β *are congruent, then* $\alpha \leftrightarrow \beta \in \Gamma$
(16) $c = c', \alpha(c) \in \Gamma \Rightarrow \alpha(c') \in \Gamma$

Proof: As in the classical case, by using the strong negation $\neg^\star$ instead of the weak negation $\neg$. ∎

Theorem 2.7.11 *If* Γ *is a Henkin non-trivial set of sentences (either consistent or inconsistent), then* Γ *has a model.*

Proof: Consequence of the preceding theorem. ∎

Corolary 2.7.4 *Every non-trivial set of sentences of the language of* $\mathcal{C}_1^=$ *has a model.*

Theorem 2.7.12 (Completeness) $\Gamma \models \alpha \Rightarrow \Gamma \vdash \alpha$

Theorem 2.7.13 (Löwenheim-Skolem) *If* Γ *has an infinite model, then it has an infinite denumerable model.*

Other results can of course be obtained, so that it is possible to construct a paraconsistent model theory; for instance, E. A. Alves has developed such a theory for a variant of the calculus $\mathcal{C}_1$ to which he introduced an additional axiom, namely, $\neg\neg\alpha \leftrightarrow \alpha$ (we recall that $\alpha \to \neg\neg\alpha$ is not a thesis of $\mathcal{C}_1$, as we have seen –see Theorem 2.1.3) [10]. The preceding results can also be adapted to be applied to the calculi $\mathcal{C}_n$, $1 \leq n < \omega$, so as to the calculi with descriptions $\mathcal{D}_n$, $1 \leq n < \omega$.

A general theory of valuation was elaborated by N. C. A. da Costa; it can be applied not only to paraconsistent logics, but to almost any system of logic (see [71]).

2.8 Syllogism and paraconsistency

Similarly to the case of traditional syllogistic, which was interpreted within classical monadic predicate calculus, it is possible to develop a paraconsistent syllogistic. It is based on, for instance, the monadic calculus corresponding to the paraconsistent predicate logic $\mathcal{C}_1^\star$. In order to reach that, it suffices that one translate the propositions A, I, E and O into $\mathcal{C}_1^\star$: the translations are as follows, which were based on the classical setting:

Aab	$\forall x(a(x) \to b(x))$
Iab	$\exists x(a(x) \wedge b(x))$
Eab	$\forall x(a(x) \to \neg b(x))$
Oab	$\exists x(a(x) \wedge \neg b(x))$

There are two brief remarks to be made within this context. (1) The valid positive deductions in $\mathcal{C}_0^\star$, the classical predicate calculus, are also valid in $\mathcal{C}_1^\star$; that is, when no explicit negation is involved, the positive deductions of $\mathcal{C}_0^\star$ and $\mathcal{C}_1^*$ are the same. (2) In $\mathcal{C}_1^\star$ one can find paraconsistent predicates, such that, for instance, there are elements that satisfy the predicate and, at the same time, do not satisfy it; i.e., for some predicate p the following holds:

$$\exists x(p(x) \wedge \neg p(x)).$$

Thus, based on arguments rather similar to the ones found in the classical case, it is possible to verify the validity of inferences, and one changes accordingly the theories of opposition, conversion, immediate inferences and syllogism. (Each predicate within the universe of discourse has three parts: of the elements that satisfy it, of those that do not satisfy it, and of those that simultaneously satisfy it and do not satisfy it. Simple graphics supply then evidence for the validity, or for the invalidity, of certain inferences and conversions.)

Based on this approach, one can prove the following result. In the paraconsistent logic $\mathcal{C}_1^\star$, all modes of the first and of the third figures of the syllogism are valid; none of the second is valid; and of the fourth, just Bramantip and Dimaris modes are valid. It is worth mentioning that since $\mathcal{C}_1^\star$ has a strong negation, of a classical trend, and if such negation is adopted in the interpretation of syllogistic reasoning, then the classical theory is obtained. As is known, Łukasiewicz has axiomatised the theory of categorical syllogism, based on the classical propositional calculus and admitting as specific axioms certain categorical propositions, as well as some appropriate definitions. Based on the paraconsistent propositional calculus, for instance the calculus $\mathcal{C}_1$, it is also possible to formulate an axiomatics for paraconsistent syllogistic, articulated in parallel lines to the theory just outlined. Moreover, we should note that there are further extensions or modifications of the Aristotelian syllogistic that also admit paraconsistent versions, such as Hamilton's, De Morgan's and Gergone's.

3 Paraconsistent set theory

Cantor's naive theory was based mainly on two fundamental principles: the postulate of extensionality (if the sets x and y have the same elements, then they are equal), and the postulate of separation or comprehension (every property determines a set, composed of the objects that have this property). The latter postulate, in the standard (first-order) language of set theory, becomes the following formula (or scheme of formulas):

$$\exists y \forall x (x \in y \leftrightarrow F(x)). \tag{7}$$

Now, it is enough that one replaces the formula $F(x)$, in (7), by $x \notin x$ in order to derive Russell's paradox. That is, the principle of separation (7) entails inconsistency. Thus, if one adds (7) to first-order logic, conceived as the logic of a set theoretic language, a trivial theory is obtained. There are also other paradoxes, such as those of Currys and Moh Schaw-Kwei, that indicate that (7) is trivial or, more precisely, trivialises set theory, if its underlying logic is classical, even ignoring negation.[10] In other words, classical positive logic is incompatible with (7); the same holds also for several other logics, such as the intuitionistic one. Classical set theories are distinguished by the restrictions that are imposed on (7), to the effect of avoiding paradoxes. In order that the theory thus obtained does not become too weak, some further axioms, besides extensionality and separation (with due restrictions), are added, depending on the particular case in question. Thus, for instance, in Zermelo-Fraenkel (ZF), separation is formulated in the following way:

$$\forall z \exists y \forall x (x \in z \leftrightarrow (x \in \ \wedge F(x))), \tag{8}$$

[10]Let us exemplify this fact with Curry's paradox. In (7), substitute for $F(x)$ the expression $x \in x \to \alpha$, where α is a formula whatever in which y does not appear free. Then, by (7), we get $\exists y \forall x (x \in y \leftrightarrow (x \in x \to \alpha))$. Let us call c, in honour of Curry, this set y; so, $\forall x (x \in c \leftrightarrow (x \in x \to \alpha))$. But then $c \in c \leftrightarrow (c \in c \to \alpha)$, hence (i) $c \in c \to (c \in c \to \alpha)$ and (ii) $(c \in c \to \alpha) \to c \in c)$. But the law of contraction $(\gamma \to (\gamma \to \beta)) \to (\gamma \to \beta)$ holds, so (i) entails (iii) $c \in c \to \alpha$. From (ii) and (iii), we get $c \in c$. Finally, from this last sentence and from (iii), by Modus Pones, we obtain α. This shows that (7) entails triviality even without negation.

where the variables are subject to the usual conditions. In ZF, then, $F(x)$ determines the subset of the elements of the set z that have the property F (or satisfy the formula $F(x)$). In the Kelly-Morse system, on the other hand, separation is formulated as follows:

$$\exists y \forall x (x \in y \leftrightarrow (F(x) \wedge \exists z (x \in z)), \tag{9}$$

while, in Quine's system NF, the notion of stratification is employed, and the scheme of separation is written like (7), provided that $F(x)$ is stratified (besides the standard conditions regarding the variables).

However, we can ask whether it would be possible to examine the problem from a distinct viewpoint: what is needed in order to maintain the scheme (7) without restrictions (with no regard to the conditions on the variables)? The answer is immediate: one should change the underlying logic, so that (7) does not inevitably lead to trivialisation. The separation scheme, without strong restrictions, leads to contradictions. Hence, such a logic has to be a paraconsistent one. It was slowly verified that there are infinitely many ways to weaken the classical restrictions imposed on the separation scheme, each of them corresponding to distinct categories of paraconsistent logic. Furthermore, extremely weak logics have been formulated to be the underlying logics of set theories in which (7) is used without any restrictions (not related to the choice of variables).

An important point is that several paraconsistent set theories contain the classical one, in Zermelo-Fraenkel's, Kelly-Morse's or Quine's formulations. Hence, paraconsistency goes beyond the classical domain, and allows, among other things, the reconstruction of traditional mathematics (see [108], [76], [109], [184]). It is quite fair then to claim that paraconsistent theories extend the classical ones, just as Poncelet's imaginary geometry comprises the standard Euclidean geometry. Moreover, we should stress a difficulty found in the very foundations of logic. Classical elementary logic (it would, in fact, be enough to consider only part of its positive logic) and the separation postulate seen to be both evident; we may even to claim that they are equally evident or intuitive. However, they are mutually incompatible, and constitute thus a case of incompatible evidences (this fact generates a difficulty from the viewpoint of classical logic). Without presenting detailed philosophical analyses, we shall just note that classical theories adopt a particular line of approach, and paraconsistent theories, another one. Such an exploration contributes for a better comprehension of the classical position itself, a clearer understanding of negation, and a more realistic perception of the possibility of discourse, even if one partially puts aside the principle of non-contradiction.

3.1 The systems $\mathcal{NF}_n$, $1 \leq n \leq \omega$

Here we begin by describing $\mathcal{NF}_1$.[11]The underlying logic of $\mathcal{NF}_1$ is the calculus $\mathcal{C}_1^=$, that is, we assume the language of this calculus plus the propositional postulates $\to_1$ to $\neg_4$ of page 8, the postulates (I)-(VII) for the predicate calculus (see page 17), and the postulates for equality (I') and (II') (page 21). The specific postulates of $\mathcal{NF}_1$ are:

(NF1) (Extensionality) $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \to x = y)$

(NF2) (Separation) $\exists y \forall x (x \in y \leftrightarrow \alpha(x))$, where x and y are distinct variables, y does not occur free in $\alpha(x)$ and this formula is either stratified or is of the form $x \notin x$.[12]

[11]Caiero and de Souza have developed a paraconsistent version of the ML system in [55].

[12]We recall that a formula is stratified if it is possible to replace each variable occurring in it by a numeral in the following manner: we replace everywhere the same variable by the same numeral

If we add to $\mathcal{C}_1^=$ the scheme $\neg(\alpha \wedge \neg\alpha)$, we get the classical first-order predicate calculus with identity. The system NF of Quine is obtained by adjoining to this calculus the postulates (NF1) and (NF2) above, provided that this second postulate is subjected to the sole restriction that $\alpha(x)$ must be stratified; we shall denote Quine's system by $\mathcal{NF}_0$.

In order to introduce the set theories $\mathcal{NF}_n$, $1 \leq n < \omega$, we employ the calculus $\mathcal{C}_n^=$, $1 \leq n < \omega$ as their underlying logics plus the specific postulates above. Then we have:

Theorem 3.1.1 *$\mathcal{NF}_n$, $1 \leq n < \omega$ contains $\mathcal{NF}_0$.*

Proof: It follows from the fact that if α is a theorem of $\mathcal{NF}_0$, and if we replace all occurrences of $\neg$ in this formula by $\neg^{(n)}$, obtaining α', then α' is provable in $\mathcal{NF}_n$, $1 \leq n < \omega$, where $\neg^{(n)}\alpha$ is the formula $\neg\alpha \wedge \alpha^{(n)}$.▮

Theorem 3.1.2 *If $\mathcal{NF}_n$, $1 \leq n < \omega$ is non-trivial, then $\mathcal{NF}_0$ is consistent.*

Proof: Let us suppose that $\mathcal{NF}_n$, $1 \leq n < \omega$ is non-trivial and that $\mathcal{NF}_0$ is inconsistent. Let $\alpha_1, \ldots, \alpha_n$ a proof of a contradiction in $\mathcal{NF}_0$, where α_n is $\beta \wedge \neg\beta$. Then, if $\alpha'_1, \ldots, \alpha'_n$ are formulas obtained from the α_i as explained in the proof of the preceding theorem, then this last sequence would be a derivation of $\beta' \wedge \neg^{(n)}\beta'$ in $\mathcal{NF}_n$. But in this system $(\gamma \wedge \neg_{(n)}\gamma) \rightarrow \delta$ is a valid scheme, and so $\mathcal{NF}_n$ would be trivial.▮

We can prove the following

Theorem 3.1.3

1. In the hierarchy $\mathcal{NF}_n$, $1 \leq n < \omega$, every system is stronger that those which follow them.

2. If $\mathcal{NF}_1$ is non-trivial, then all $\mathcal{NF}_n$, $1 < n < \omega$ are also non-trivial.

3. $\mathcal{NF}_n$, $1 \leq n < \omega$ is inconsistent.

Let us make some comments on item 3. In $\mathcal{NF}_n$, Russell's set, that is, the set $R =_{\mathsf{def}} \{x : x \notin x\}$, does exist (see the next subsection), that is, in these systems we have $\vdash \exists y \forall x (x \in y \leftrightarrow x \notin x)$. As we shall see below, it is easy to prove that $R \in R \wedge R \notin R$; so, the systems $\mathcal{NF}_n$ are inconsistent. The other items are proved without difficulty (see [76]).

The next step is to show that if $\mathcal{NF}_0$ is consistent, then $\mathcal{NF}_1$ is non-trivial. Therefore, due to item 2 of the preceding theorem, the consistency of $\mathcal{NF}_0$ entails the non-triviality of $\mathcal{NF}_n$, $1 \leq n < \omega$. Let us define a system $\mathcal{NF}_1^*$ as follows: we keep with the propositional postulates (see page 8) $\rightarrow_1 -- \rightarrow_3$, $\wedge_1 -- \wedge_3$, $\vee_1 -- \vee_3$ only and add the following new postulates: $(\rightarrow_4^*)$ (Peirce's Law) $((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha$ and $(\rightarrow_5^*)$ $(\neg\alpha \rightarrow \beta) \rightarrow ((\neg\alpha \rightarrow \neg\beta) \rightarrow \alpha)$, where β is not atomic. This new set of postulates of course provides an axiomatization for the classical propositional logic. The remaining postulates of $\mathcal{NF}_1^*$ are those of $\mathcal{NF}_1$, except for those which turn to be redundant. For instance, since α^o is provable in $\mathcal{NF}_1^*$ when α is not atomic, it results that postulates (V) and (VI) of page 17 are provable. So, $\mathcal{NF}_1$ is weaker than $\mathcal{NF}_1^*$.

Lemma 3.1.1 *The consistency of $\mathcal{NF}_0$ entails the non-triviality of $\mathcal{NF}_1^*$.*

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so that, for each occurrence of $\in$, the numeral immediately following $\in$ is the immediate successor of the numeral immediately preceding $\in$ [140, p. 213].

Proof: Let f be a map whose domain is the set of formulas of $\mathcal{NF}_1^*$ and whose range is the set of formulas of $\mathcal{NF}_0$, defined as follows, where $\mathcal{V} =_{\mathsf{def}} \{x : x = x\}$:

1.$f(x = y) =_{\mathsf{def}} x = y$	2.$f(x \in y) =_{\mathsf{def}} x \in y$
3.$f(x \in y) =_{\mathsf{def}} x \in y$	4.$f(x \notin y) =_{\mathsf{def}} x \in \mathcal{V} \wedge y \in \mathcal{V}$
5.$f(\forall x\alpha) =_{\mathsf{def}} \forall x f(\alpha)$	6.$f(\exists x\alpha) =_{\mathsf{def}} \exists x f(\alpha)$.
7.$f(\alpha \wedge \beta) =_{\mathsf{def}} f(\alpha) \wedge f(\beta)$	8.$f(\alpha \vee \beta) =_{\mathsf{def}} f(\alpha) \vee f(\beta)$
9.$f(\alpha \to \beta) =_{\mathsf{def}} f(\alpha) \to f(\beta)$	

Then, using the preceding results, we can see that if α is a theorem of $\mathcal{NF}_0$, then $f(\alpha)$ is a theorem of $\mathcal{NF}_1^*$. Since the rules of inference of $\mathcal{NF}_1^*$ are valid in$\mathcal{NF}_0$, any theorem α of $\mathcal{NF}_1^*$ induces a theorem $f(\alpha)$ of $\mathcal{NF}_0$. Therefore, supposing that $\mathcal{NF}_0$ is consistent, $\mathcal{NF}_1^*$ cannot be trivial; for instance, $\emptyset \in \emptyset$ is not a theorem of $\mathcal{NF}_1^*$, since $f(\emptyset \in \emptyset) = \emptyset \in \emptyset$ is not provable in $\mathcal{NF}_0$.▮

Theorem 3.1.4 *If $\mathcal{NF}_0$ is consistent, then $\mathcal{NF}_1$ is non-trivial.*

Proof: The consistency of $\mathcal{NF}_0$ implies the non-triviality of $\mathcal{NF}_1$ because this system is weaker than $\mathcal{NF}_1^*$.▮

Theorem 3.1.5 *If $\mathcal{NF}_0$ is consistent, then all the inconsistent systems $\mathcal{NF}_n$, $1 \leq n < \omega$ are non-trivial.*

Proof: It suffices to note that the theories $\mathcal{NF}_n$, $1 \leq n < \omega$ are weaker than $\mathcal{NF}_1^*$.▮

Changing a little bit the proof of Theorem 3.1.4, we can prove the following result: if $\mathcal{NF}_0$ is consistent, then the system obtained from $\mathcal{NF}_n$, $1 \leq n < \omega$ by adjoining axioms guaranteeing the existence of the sets of all non-k-circular sets ($k = 1, 2, \ldots$) is non-trivial. For example, the set of all non-3-circular sets is the following set: $\{x : \neg\exists y_1 \exists y_2 \exists y_3 (x \in y_1 \wedge y_1 \in y_2 \wedge y_2 \in y_3 \wedge y_3 \in x)\}$.

$\mathcal{NF}_\omega$ is like $\mathcal{NF}_1$, except that its underlying logic is $\mathcal{C}_\omega^=$ instead of $\mathcal{C}_1^=$.

$\mathcal{NF}_\omega$ is weaker than $\mathcal{NF}_1$, so if $\mathcal{NF}_0$ is consistent, it is non-trivial.

3.2 Zermelo-Fraenkel like systems

Starting with the Zermelo-Fraenkel system, we can introduce a new hierarchy of paraconsistent set theories $\mathcal{ZF}_n$, $1 \leq n \leq \omega$, similar to the hierarchy $\mathcal{NF}_n$, $1 \leq n \leq \omega$. Of course that instead of ZF we could employ any other classical system of set theory or type theory as the first system in the hierarchy [82], [83]. Among the several versions of ZF that can be used, perhaps the best form is Church's [62], which admits the existence of the universal set.

Let us call this theory $\mathcal{ZF}_0$. The language L of $\mathcal{ZF}_0$ is that of $\mathcal{C}_1^=$, but with only one specific (binary) predicate symbol $\in$ (membership). The syntactic notions of L are obvious adaptations of those of $\mathcal{C}_0^=$. In addition, $\notin$ has its standard meaning and the description symbol ι is introduced by contextual definition, following Russell. I order to state the postulates of $\mathcal{ZF}_0$, we need some definitions.

Definition 3.2.1

1. $\{x : \alpha(x)\} =_{\mathsf{def}} \iota y \forall x(x \in y \leftrightarrow \alpha(x))$

2. $\emptyset =_{\mathsf{def}} \{x : x \notin x\}$

3. $\mathcal{V} =_{\mathsf{def}} \{x : x = x\}$

4. $x \subset y \leftrightarrow \forall z(z \in x \to z \in y)$

5. $\{x\} =_{\mathsf{def}} \{y : y = x\}$

6. $\mathcal{P}(x) =_{\mathsf{def}} \{y : y \subset x\}$

7. $\overline{x} =_{\mathsf{def}} \{y : y \notin x\}$

8. $\exists\{x : \alpha(x)\} \leftrightarrow \exists y \forall x(x \in y \leftrightarrow \alpha(x))$

We easily define other basic notions such as $x \cup y$, $x \cap y$, $\{x, y\}$, $\langle x, y\rangle$, $\cup x$, $\cap x$, relation, function. etc.

Definition 3.2.2

1. $trans(x) \leftrightarrow \forall y(y \in x \rightarrow y \subset x)$ (transitive set)

2. $conn(x) \leftrightarrow \forall y \forall z(y \in x\ z \in x \rightarrow (z = y \vee y \in z \vee z \in y))$ (connected set)

3. $wf(x) \leftrightarrow (x \neq \emptyset \rightarrow \exists y(y \in x \wedge x \cap y = \emptyset))$ (well-founded set)

4. $ord(x) \leftrightarrow trans(x) \wedge conn(x) \wedge wf(x)$ (x is an ordinal)

5. $low(x) \leftrightarrow wf(x) \wedge$ x is equipotent to y (x is a low set)

The postulates of $\mathcal{ZF}_0$ are those of $\mathcal{C}_0^{=}$ plus the following:

(P1) $\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y$ (Extensionality)

(P2) $\exists\{z : z = x \vee z = y\}$ (Pair Set)

(P3) $\exists\{z : \exists y(y \in x \wedge z \in y\}$ (Union Set)

(P4) $\exists \bigcap x$ (Intersection Set)

(P5) $\exists\{x : x$ is a finite ordinal$\}$ (Infinity)

(P6) Any formulation of the Axiom of Choice.

(P7) $low(z) \rightarrow \exists x\{x : \alpha(x) \wedge x \in z\}$ (Separation)

(P8) $\forall x \forall y \forall z((\alpha(x, y) \wedge \alpha(x, z) \rightarrow x = y) \wedge (\alpha(x, y) \wedge \alpha(z, y) \rightarrow x = z)) \wedge \forall y(y \in t \rightarrow \exists x \alpha(x, y)) \rightarrow (low(t) \rightarrow \exists v \forall x(x \in v \leftrightarrow \exists y(\alpha(x, y) \wedge y \in t)))$ (Replacement)

(P9) $low(x) \rightarrow \exists\{y : y \subset x\}$ (Power Set)

(P10) $\exists\{y : y \notin x\}$ (Complement)

$\mathcal{ZF}_0$ is really a strong set theory; the well-founded sets constitute a 'model' of usual Zermelo-Fraenkel set theory. On the other hand, the universal set does exist in $\mathcal{ZF}_0$:

$$\vdash \exists\{x : x = x\},$$

that is, the set $\mathcal{V}$ such that $\vdash \forall x(x \in \mathcal{V})$ and, in particular, $\vdash \mathcal{V} \in \mathcal{V}$. The collection of all sets, plus $\emptyset$, $\mathcal{V}$, $\subset$, $\cup$, $\cap$ and form a complete Boolean algebra.

Theorem 3.2.1 *In $\mathcal{ZF}_0$:*

a) $\vdash x \cup \overline{x} = \mathcal{V}$ b) $\vdash x \cap \overline{x} = \emptyset$ c) $\vdash \overline{\overline{x}} = x$ d) $\vdash \overline{\mathcal{V}} = \emptyset$

e) $\vdash \bigcap \mathcal{V} = \emptyset$ f) $\vdash \bigcup \mathcal{V} = \mathcal{V}$ g) $\vdash \bigcap \emptyset = \mathcal{V}$ h) $\vdash \bigcup \emptyset = \emptyset$

i) $\vdash \neg wf(\mathcal{V})$ j) $\vdash \neg low(\mathcal{V})$ k) $\vdash \overline{\emptyset} = \mathcal{V}$

l) $x \subset y \rightarrow \overline{y} \subset \overline{x}$ m) $\vdash wf(\{\mathcal{V}\})$ n) $\vdash wf(\emptyset)$ o) $\vdash low(\emptyset)$

p) $\vdash$ f *is a function* $\wedge low(dom(f)) \rightarrow low(range(f))$, where $dom(f)$ and $range(f)$ stand for the domain and the range of f respectively.

Let us call α^{wf} the formula obtained from the formula α by restricting its variables by the condition $wf(\ )$. Then we have the following result, whose proof is immediate:

Theorem 3.2.2 *If α is a closed theorem of the standard ZF, then α^{wf} is a theorem of $\mathcal{ZF}_0$.*

As Church himself notes, it would be interesting to investigate the extension of $\mathcal{ZF}_0$ by the introduction of new postulates, for example, similar to the specific postulates of Quine's NF [62]. A system of this type was already studied in [72].

Let us describe the system $\mathcal{ZF}_1$. This system is related to $\mathcal{ZF}_0$ as $\mathcal{C}_1^=$ is related to $\mathcal{C}_0^=$ (recall that this last one is the standard first-order predicate calculus with identity). So, we should have, among other things, that: (a) $\mathcal{ZF}_1$ should be partially included in $\mathcal{ZF}_0$, though the latter is also to be contained, in a certain sense, in the former; (b) $\mathcal{ZF}_1$ should be consistent, but can be used to base inconsistent but non-trivial theories; in particular, in $\mathcal{ZF}_1$ we should be able to define 'inconsistent' set-theoretical structures, such as Russell's set and Russell's relations (see below), and other more complex structures (inconsistent arithmetics, 'inconsistent' groups, etc.).

Therefore, we construct $\mathcal{ZF}_1$ as follows: its language, called L, and the logical postulates are those of $\mathcal{C}_1^=$. The basic set-theoretical concepts are analogous to those of $\mathcal{ZF}_0$, although the concepts involving negation give raise to two notions: one involving the weak negation ($\neg$) and the other involving the strong negation ($\neg^\star$); in general, the symbols for those negations will differ only by the fact that the strong versions are starred (for instance, we have two empty sets: $\emptyset =_{\mathsf{def}} \{x : \neg(x = x)\}$ and $\emptyset^\star =_{\mathsf{def}} \{x : \neg^\star(x = x)\}$).

Each specific axiom or axiom scheme of $\mathcal{ZF}_0$ originates two corresponding axioms or axiom schemes of $\mathcal{ZF}_1$, one with the strong negation and another with the weak one (we suppose that negation does occur essentially; otherwise, the postulate of $\mathcal{ZF}_0$ originates only one of $\mathcal{ZF}_1$, having the same syntactical form).

It is important to note that, for example, $\mathcal{NF}_1$ is inconsistent, while $\mathcal{ZF}_1$ is consistent, but apparently non trivial, though can be used as a basis for inconsistent but non trivial theories.

The following results can be proved without difficulty; we still remark that $\mathcal{ZF}_1$ is part of $\mathcal{ZF}_0$, but that $\mathcal{ZF}_0$ is also contained, in certain sense, in $\mathcal{ZF}_1$.

Theorem 3.2.3

1. *Let α be a sentence of L and $\alpha^\star$ the sentence obtained from α by replacing $\neg$ by $\neg^\star$. Then α is a theorem of $\mathcal{ZF}_0$ iff $\alpha^\star$ is a theorem of $\mathcal{ZF}_1$.*

2. *$\mathcal{ZF}_1$ is consistent (in connection with $\neg$ or with $\neg^\star$) iff $\mathcal{ZF}_0$ is consistent.*

3. *If α is a closed theorem of $\mathcal{ZF}_0$, then $\forall x \forall y((x \in y)^o \wedge (x = y)^o) \rightarrow \alpha$ is a theorem of $\mathcal{ZF}_1$.*

In the next subsection, we shall show how $\mathcal{ZF}_1$ can be employed as a paraconsistent base for inconsistent but non-trivial theories (and, loosely speaking, of inconsistent mathematics).

As a final remark, let us notice that inside the set theories described above we may construct semantics of $\mathcal{C}$-logics, where the syntactical metalinguistic level may be considered as classical (in the sense that all the syntactical propositions are well-behaved); see [108] where there are also further references.

3.3 Russell sets and relations

By adapting the standard notion of a mathematical structure [48], it is possible to define the concept of a paraconsistent structure [78]. Nonetheless, here we will not enter into details about the general theory of those last structures. Instead, we shall consider some particular cases, to show how a certain kind of paraconsistent mathematics can be developed.

Definition 3.3.1 (Russell set) $R =_{\mathsf{def}} \{x : x \notin x\}$

The first structure to be studied is

$$\mathfrak{R} = \langle \mathcal{V}, R \rangle, \tag{10}$$

characterized by the axiom

$$\exists R \wedge \forall x \exists \mathcal{P}(x), \tag{11}$$

where $\mathcal{V}$ is the domain (the universe set) of $\mathfrak{R}$ and R is its sole predicate; $\mathcal{P}(x)$ denotes the power set of x.

We emphasize that in what follows our logic is $\mathcal{ZF}_1$ (and that, in consequence, we may make free use of the valid schemes and rules of $\mathcal{C}_1^=$).

Theorem 3.3.1 $\vdash R \in R \wedge R \notin R$

Proof: Given the definition of R, $x \in R \leftrightarrow x \notin x$. Hence, replacing x for R, we get $R \in R \leftrightarrow R \notin R$. However, if $R \in R$, it follows that $R \notin R$ and, if $R \notin R$, then $R \notin R$. Therefore, by the excluded middle, $R \notin R$. Similarly, we prove that $R \in R$ by assuming that $R \notin R$. ▮

Theorem 3.3.2 $\vdash y \in \{x\} \leftrightarrow y = x$

Proof: In $\mathcal{ZF}_1$, we have that $\exists \{x\}$; therefore, $\vdash y \in \{x\} \leftrightarrow y = x$, by the definition of $\{x\}$. ▮

Theorem 3.3.3 $\vdash x \in R \rightarrow \{x\} \in R$

Proof: Either $\{x\} \notin \{x\}$ or $\{x\} \in \{x\}$. In the first case, $\{x\} \in R$, by the definition of R. In the second case, $\{x\} = x$ and, given the hypothesis, $\{x\} \in R$.▮

Theorem 3.3.4 $\vdash x, y \in R \rightarrow \{x, y\} \in R$

Proof: Either $\{x, y\} \notin \{x, y\}$ or $\{x, y\} \in \{x, y\}$. In the first case, $\{x, y\} \in R$, by the definition of R. In the second case, either $\{x, y\} = x$ or $\{x, y\} = y$ and, given the hypothesis, $\{x, y\} \in R$.▮

Theorem 3.3.5 $\vdash \{\{x, R\}\} \in R$

Proof: Either $\{\{x, R\}\} \in \{\{x, R\}\}$ or $\{\{x, R\}\} \notin \{\{x, R\}\}$. In the second case, it is immediate that $\{\{x, R\}\} \in R$. In the first case, by Theorem 3.3.2, it follows that $\{x, R\} = \{\{x, R\}\}$. Therefore, $x = R = \{x, R\}$ and, given that $R \in R$, by Theorem 3.3.4, $\{x, R\} \in R$. Consequently, by Theorem 3.3.3, $\{\{x, R\}\} \in R$.▮

Theorem 3.3.6 (Arruda and Batens, 1982) $\bigcup R = \mathcal{V}$.

Proof: It suffices to prove that, for every x, $x \in \bigcup R$. Let us suppose that (i) $\{x, R\} \notin \{x, R\}$. Hence, $\{x, R\} \in R$ and, by the definition of the union set, $x \in \bigcup R$. On the other hand, if (ii) $\{x, R\} \in \{x, R\}$, then either $\{x, R\} = x$ or $\{x, R\} = R$. In the second case, it follows that $x \in \bigcup R$. If $\{x, R\} = x$, we have that $\{\{x, R\}\} = \{x\}$, and given that $\{\{x, R\}\} \in R$ (Theorem 3.3.5), if follows that $\{x\} \in R$; accordingly, $x \in \bigcup R$.▮

This last theorem shows that a set theory with Russell's set has in general a universal class.

Theorem 3.3.7 (Arruda) $\ldots \subset \mathcal{P}(\mathcal{P}(R)) \subset \mathcal{P}(R) \subset R$.

Proof: If $x \in \mathcal{P}(R)$, then $x \subset R$. Now, either $x \notin x$ or $x \in x$. If $x \notin x$, then $x \in R$; if $x \in x$, given that $x \subset R$, it follows that $x \in R$. Therefore, $\mathcal{P}(R) \subset R$. Furthermore, if $x \in \mathcal{P}(\mathcal{P}(R))$, then $x \subset \mathcal{P}(R)$, and by the preceding result, $x \subset R$; hence $x \in \mathcal{P}(R)$. Thus, $\mathcal{P}(\mathcal{P}(R)) \subset \mathcal{P}(R) \subset R$. Now it is easy to complete the proof.▮

Theorem 3.3.8 *In $\mathcal{ZF}_1$ plus the axiom (11):*

1. $\vdash \emptyset \in R, \{\emptyset\} \in R, \{\{\emptyset\}\} \in R, \ldots$
2. $\vdash \exists x(x \notin R)$
3. $\vdash \mathcal{V} \notin R$
4. $\vdash x, y \in R \to \langle x, y\rangle \in R$
5. $\vdash \mathcal{V} = \mathcal{P}(\mathcal{V})$
6. $\vdash x \subset R \to x \in R$
7. $\vdash R \cup \overline{R} = \bigcup R$
8. $\vdash R \times R \subset R$
9. $\vdash \neg^\star \exists \{x : \neg^\star(x \in x)\}$

10 $\vdash \exists R \to \exists x(x \text{ is infinite})$

Given Theorem 3.3.5, it is possible to demonstrate that R is, as it were, an 'internal model' of the set theory in which we are working. Moreover, given that $\bigcup R = \mathcal{V}$, if follows that the existence of R implies the existence of infinite sets. The properties of R are by no means arbitrary. Thus, it is not possible to prove everything with regard to R without also proving, at the same time, that some classical, standard set theories are inconsistent [67], [76]. For instance, it seems not possible to prove that $(R \in R)^o$; furthermore, $\mathcal{ZF}_1$ with the extra axiom $\exists R$ is non-trivial iff $\mathcal{ZF}_0$ is consistent.

3.3.1 Russell relations

Definition 3.3.2 (Russell's Relations) *For $n = 1, 2, \ldots$ and $0 < i \leq n$ (by convention, $\langle x\rangle = x$),*

$$R_{n,i} =_{\mathsf{def}} \{\langle x_1, \ldots, x_n\rangle : \langle x_1, \ldots, x_n\rangle \notin x_i\}.$$

So, $R_{1,1} = R$. With $R_{n,i}$ assuming appropriate postulates, we can built structures similar to the above $\mathfrak{R}$. Then, we can prove that

Theorem 3.3.9

1. $\vdash R_{n,i} \in R_{n,i} \wedge R_{n,i} \notin R_{n,i}$
2. $\vdash \mathcal{V} \times \cdots \times \mathcal{V} = \bigcup R_{n,i}$, where the product on the left has n terms. In particular, $\vdash \mathcal{V} \times \mathcal{V} \subset \cup \cup R_{2,1}$

Theorem 3.3.10

1. $\vdash \forall t(\langle \emptyset, t\rangle \in R_{2,1})$
2. $\vdash \cup \cup R_{2,1} = \mathcal{V}$

Proof: As for 1., by assumption, $\exists R_{2,1}$; then $\langle \emptyset, t\rangle \in R_{2,1}) \leftrightarrow \langle \emptyset, t\rangle \notin \emptyset$. Concerning 2., since $\{\{\emptyset\}, \{\emptyset, t\}\} = \langle \emptyset, t\rangle \in R_{2,1}$, it follows that $\{\emptyset, t\} \in \cup R_{2,1}$ and $t \in \cup \cup R_{2,1}$.▮

Given the consistency of $\mathcal{NF}_0$ (cf. the preceding subsection), if we add to $\mathcal{NF}_n$, $1 \leq n < \omega$ new postulates guaranteeing the existence of Russell relations, the resulting system is non-trivial.

3.4 Paraconsistent Boolean algebra

Within various paraconsistent set theories, even in classical set theories such as Zermelo-Fraenkel, it is possible to consider intuitively a set as an ordered pair, in the classical sense, of sets that are part of a universe-class $\mathcal{V}$. Thus, a set X is a pair $\langle X_1, X_2 \rangle$, where:

(1) $x \in X$ iff $x \in X_1$

(2) $x \notin X$ iff $x \in X_2$

(3) $x \in X$ and $x \notin X$ is equivalent to $x \in X_1$ and $x \in X_2$

Given that the principle of the excluded middle is supposed maintained in certain paraconsistent set theories, it should be the case that $X_1 \cup X_2 = \mathcal{V}$. If $X_1 \cap X_2 = \emptyset$, a classical set is obtained. Let us consider then the collection of the sets just constructed on $\mathcal{V}$, which shall be denoted by $\mathfrak{V}$. An element of $\mathfrak{V}$ is called a paraconsistent set, or a p-set. In what follows we shall outline an algebra of p-sets $\mathfrak{V}$. We will suppose that the p-sets are embedded in a classical set theory, for instance, ZF.

Let us suppose that $X = \langle X_1, X_2 \rangle$ and $Y = \langle Y_1, Y_2 \rangle$; then

Definition 3.4.1

(i) $X \sqcup Y =_{\mathsf{def}} \langle X_1 \cup Y_1, X_2 \cap Y_2 \rangle$

(ii) $X \sqcap Y =_{\mathsf{def}} \langle X_1 \cap Y_1, X_2 \cup Y_2 \rangle$

(iii) $\mathbf{1} =_{\mathsf{def}} \langle \mathcal{V}, \emptyset \rangle$

(iv) $\mathbf{0} =_{\mathsf{def}} \langle \emptyset, \mathcal{V} \rangle$

(v) $\overline{X} =_{\mathsf{def}} \langle X_2, X_1 \rangle$

(vi) $X \sqsubset Y =_{\mathsf{def}} X_1 \subset Y_1 \wedge Y_2 \subset X_2$

Theorem 3.4.1

$\vdash X \sqcup X = X$	$\vdash X \sqcap X = X$	$\vdash \overline{\overline{X}} = X$
$\vdash X \sqcup Y = Y \sqcup X$	$\vdash X \sqcap Y = Y \sqcap X$	$\overline{\mathbf{1}} = \mathbf{0}$
$\vdash (X \sqcup Y) \sqcup Z = X \sqcup (Y \sqcup Z)$	$\vdash (X \sqcap Y) \sqcap Z = X \sqcap (Y \sqcap Z)$	$\vdash \overline{\mathbf{0}} = \mathbf{1}$
$\vdash \mathbf{1} \sqcup X = \mathbf{1}$	$\vdash \mathbf{1} \sqcap X = X$	$\vdash \mathbf{0} \sqcap X = \mathbf{0}$
$\vdash \mathbf{0} \sqcup X = X$	$\vdash X \sqcup \overline{X} \sqsubset \mathbf{1}$	$\vdash \mathbf{0} \sqsubset X \sqcap \overline{X}$
$\vdash X \sqsubset X$	$\vdash \mathbf{0} \sqsubset X$	$\vdash X \sqsubset \mathbf{1}$
$\vdash X \sqsubset Y \wedge Y \sqsubset X \rightarrow X = Y$	$\vdash X \sqsubset Y \wedge Y \sqsubset Z \rightarrow X \sqsubset Z$	
$\vdash X \sqsubset X \sqcup \overline{X}$	$\vdash X \sqcap \overline{X} \sqsubset X$	

Definition 3.4.2 *The structure* $\mathfrak{P} = \langle \mathfrak{V}, \sqcup, \sqcap, ^{-}, \mathbf{1}, \mathbf{0} \rangle$ *is called a paraconsistent Boolean algebra.*

Through the employment of this structure it is possible to formalize several paraconsistent patterns of reasoning, just as with classical Boolean algebras one can put in algebraic terms various classical inferences. Moreover, one can verify that the paraconsistent logical mechanism considered here does not exclude classical logic, but extends it in some sense; though under another viewpoint, it can be embedded into traditional logical structures. Of course, such remarks are valid for particular categories of paraconsistent structures; however, they are of extreme relevance in order to corroborate the fact that both paraconsistent logic as well as paraconsistent mathematics, as far as we understand them, do not destroy either the traditional logic, or standard mathematics, but only complement them and, in certain cases, extend them.

The structure of the paraconsistent Boolean algebra clearly is richer than the classical one. Thus, for instance, one can introduce two operators, π_1 and π_2, such that, given a p-set X, $\pi_1(X) = X_1$ and $\pi_2(X) = X_2$, where X_1 and X_2 are in another Boolean algebra, the classical algebra of the subsets of $\mathcal{V}$ etc.

When the structure $\mathfrak{P}$, in the above definition, is such that, for every $X = \langle X_1, X_2 \rangle$, it is the case that $X_1 \cap X_2 = \emptyset$, one obtains a Boolean algebra that essentially is the usual algebra of the subsets of $\mathcal{V}$.

In this way, it is possible to construct a general theory of paraconsistent structures (algebraic, topological, of order etc.), obtaining thus a generalization of the traditional theory of structures, such as Bourbaki's. Moreover, paraconsistent structures, such as those described in this section, have been applied to several areas, such as computer science, artificial intelligence and logic programming,as we shall see below (see, e.g., [226], [38], [39], and [151]). This provides a significant motivation for their study.

it is worthwhile to note that paraconsistent structures were employed in quantum mechanics by Dalla Chiara and Giuntini (see [115], [116], [117]).

3.5 Paraconsistent mathematics

In this subsection we outline a paraconsistent formulation of the elementary differential and integral calculus, in which it is true that what we may call the l'Hospital Principle:

> "Two finite quantities which differ by an infinitely small quantity are equal."

The paraconsistent nature of such a principle is clear, though we are not trying to advance an erudite exegesis of the l'Hospital's works (see [169], [214]). To begin with, we describe two (classical) algebraic structures $\mathcal{A}$ and $\mathcal{A}^\star$.

The ring $\mathcal{A}$ The ring $\mathcal{A}$ is described as follows. Let $\mathbb{R}$ denote the field of real numbers and a an element of a fixed open interval I ($I \subset \mathbb{R}$). An *infinitesimal variable* is a real valued function defined on I, which has limit zero in a (we could use right or left limits). The expression 'infinitesimal variable' is here employed inspired by the terminology of the classical French treatises, for instance those by Picard and Gousart, as well as by that of Cauchy. The elements of $\mathcal{A}$ are ordered pairs $\langle r, f \rangle$, where $r \in \mathbb{R}$ and f is an infinitesimal variable.

If $\langle r, f \rangle$ and $\langle s, g \rangle$ belong to $\mathcal{A}$, then $\langle r, f \rangle = \langle s, g \rangle$ if $r = s$ and $f = g$. Addition is defined as follows: $\langle r, f \rangle + \langle s, g \rangle =_{\mathsf{def}} \langle r+s, f+g \rangle$. By an abuse of language, we make $0 = \langle 0, 0 \rangle$, where the second zero in $\langle 0, 0 \rangle$ is the identically zero function in I. We also put $-\langle r, f \rangle =_{\mathsf{def}} \langle -r, -f \rangle$, and $\langle r, f \rangle - \langle s, g \rangle =_{\mathsf{def}} \langle r, f \rangle + (-\langle s, g \rangle) = \langle r-s, f-g \rangle$.

Further definitions are: $\langle r, f \rangle \cdot \langle s, g \rangle =_{\mathsf{def}} \langle rs, rg + fs + fg \rangle$ and $1 =_{\mathsf{def}} \langle 1, 0 \rangle$. Furthermore, if $r \neq 0$, we put $\langle r, f \rangle^{-1} =_{\mathsf{def}} \langle r^{-1}, -f/r(f+r) \rangle$ and $\langle s, g \rangle \div \langle r, f \rangle =_{\mathsf{def}} \langle s, g \rangle \cdot \langle r, f \rangle^{-1}$.

The elements of $\mathcal{A}$ with the operation $+$ constitute a commutative group, and with $\times$, a commutative semi-group with unity; on the other hand, multiplication is distributive in relation to addition. Therefore, $\mathcal{A}$ is a commutative ring with unity. The elements of $\mathcal{A}$ are called *hyper-reals*. If we identify $\langle r, 0 \rangle$ with r, then the field $\mathbb{R}$ is contained in $\mathcal{A}$ as a sub-ring. We write $\langle r, f \rangle$ as $r + \langle 0, f \rangle$. A pair such as $\langle 0, f \rangle$ is called an *infinitesimal*, and will be usually denoted by small Greek letters. Any element of $\mathcal{A}$, $\langle r, f \rangle$, then, is the sum of a standard real and an infinitesimal: $\langle r, f \rangle = r + \varepsilon$, where $\varepsilon = \langle 0, f \rangle$.

Division can be extended to infinitesimals. In fact, let us suppose that $\kappa = \langle 0, f\rangle$, $\lambda = \langle 0, g\rangle$ and $\lim f/g = r$ in $\mathcal{A}$. Then, we put $\kappa/\lambda = \langle r, f/g - r\rangle$ (the variable g is supposed not to assume the value 0). Therefore, making $h = f/g - r$, it is easy to check that the infinitesimal $\varepsilon = \langle 0, h\rangle$ is such that $\kappa = \lambda \cdot \varepsilon$. When $\lim f/g = \infty$ or does not exist, the quotient κ/λ is not defined.

We introduce in $\mathcal{A}$ a relation of inequality, $<$, so that $\langle r, f\rangle < \langle s, g\rangle$ if $r < s$ or, in the case that $r = s$, if $f < g$ (the variable f is less than the variable g in all points of I).

The set of all hyper-reals of the form $\langle r, f\rangle$ is called *the monad* of $r \in \mathbb{R}$. The *order* of an infinitesimal ε in relation to another infinitesimal κ is defined with no difficulty. Given a function $f : \mathbb{R} \mapsto \mathbb{R}$, it can be extended such that $dom(f) = \mathcal{A}$ under the hypothesis that $\lim_{x\to t} f(x)$ exists and is finite for any $t \in \mathbb{R}$; in fact, we put $f(r + \varepsilon) = b + \delta$, where $b = \lim_{x\to r} f(x)$ and δ is an infinitesimal obviously defined.

The basic concepts of the differential calculus can then be defined in terms of infinitesimals. For example, we have:

$$\lim_{x\to r} f(x) = b$$

means that $f(r + \varepsilon) = b + \delta$ for every infinitesimal ε (δ is also an infinitesimal), where $r \in \mathbb{R}$. Similarly, the derivative of f is defined as follows:

$$f(r + \varepsilon) - f(r) = f'(r) \cdot \varepsilon + \delta,$$

for any infinitesimal ε, δ being an infinitesimal or order higher than that of ε. The properties of limits, derivatives etc. are easily proved from the properties of infinitesimals.

The quasi-ring $\mathcal{A}^\star$ The quasi-ring $\mathcal{A}^\star$ is obtained from $\mathcal{A}$ by the introduction of infinite 'numbers'. An *infinite variable* is a standard real-variable function, defined in I, and divergent at $a \in I$.

The pair $\langle v, 0\rangle$ is called an infinite *hyper-real* number when v is an infinite variable. The (finite) elements of $\mathcal{A}$ and the infinite numbers form $\mathcal{A}^\star$. Two infinite numbers $\langle v, 0\rangle$ and $\langle u, 0\rangle$ are equal if $v = u$. The operations of $\mathcal{A}^\star$ are defined as in $\mathcal{A}$, but extended as follows:

Addition

1. If $\langle v, 0\rangle$ is infinite and $\langle k, f\rangle$ is finite, then : $\langle v, 0\rangle + \langle k, f\rangle = \langle k, f\rangle + \langle v, 0\rangle = \langle v + k, 0\rangle$.
2. If $\langle v, 0\rangle$ and $\langle u, 0\rangle$ are both infinite, then $\langle v, 0\rangle + \langle u, 0\rangle = \langle u, 0\rangle + \langle v, 0\rangle = \langle v + u, 0\rangle$, whenever $\lim(v + u) = \infty$; $\langle v, 0\rangle + \langle u, 0\rangle = \langle u, 0\rangle + \langle v, 0\rangle = \langle k, f\rangle$ when $\lim(v + u) = k$ (standard real), where $f = k - (u + v)$. Otherwise, $\langle v, 0\rangle + \langle u, 0\rangle$ and $\langle u, 0\rangle + \langle v, 0\rangle$ are not defined.

Subtraction By definition, the opposite of an infinite $\langle v, 0\rangle$ is $\langle -v, 0\rangle$. Then, the difference of two elements of $\mathcal{A}^\star$ is defined as the sum of the first with the opposite of the second.

Multiplication

1. If $\langle v, 0\rangle$ and $\langle u, 0\rangle$ are infinite, then $\langle v, 0\rangle \cdot \langle u, 0\rangle = \langle u, 0\rangle \cdot \langle v, 0\rangle = \langle uv, 0\rangle$.
2. If $\langle v, 0\rangle$ is infinite and $\langle k, f\rangle$ is finite, but not an infinitesimal, then $\langle v, 0\rangle \cdot \langle k, f\rangle = \langle k, f\rangle \cdot \langle v, 0\rangle = \langle vk, 0\rangle$
3. If $\langle v, 0\rangle$ is infinite and $\langle 0, f\rangle$ is an infinitesimal, then $\langle v, 0\rangle \cdot \langle 0, f\rangle = \langle 0, f\rangle \cdot \langle v, 0\rangle = \langle vf, 0\rangle$, when $\lim(vf) = \infty$; if $\lim(vf) = 0$, the product is equal to $\langle 0, vf\rangle$. Otherwise, the product is not defined.

Division Let $\langle v, 0\rangle$ ne an infinite number satisfying obvious conditions; $\langle v, 0\rangle^{-1}$ is the infinitesimal $\langle 0, v^{-1}\rangle$. If $\langle 0, f\rangle$ is an infinitesimal satisfying appropriate conditions, then $\langle 0, f\rangle^{-1}$ is the infinite $\langle f^{-1}, 0\rangle$. The quotient of two elements of $\mathcal{A}^\star$ is the product of the first by the inverse of the second.

The relation $<$ can be extended to $\mathcal{A}^\star$ as it is clear. However, $\mathcal{A}$ and $\mathcal{A}^\star$ are not archimedean structures. By means of infinitesimals and infinities one can express in $\mathcal{A}^\star$ the basic ideas and results of the infinitesimal calculus (for analogous views, see [221], [162], [163]). In particular, we may get something like the de l'Hospital principle on curves, according to which a smooth curve may be analysed into an infinite number of infinitesimal straight lines ([169], [214]).

The classical theory of infinitesimals and infinite quantities, as well as Du Bois Reymond's theory of orders of magnitude concerning the asymptotic behaviour of functions (cf. [47, Note II]), are translatable in the language of $\mathcal{A}^\star$. The same is true in connection with other topics of pure and applied mathematics, for instance, the theory of differential equations and Fourier series. Loosely speaking, $\mathcal{A}^\star$ is a model of a theory with infinitesimals and infinite quantities. It seems worth noticing that natural suppositions about the orders of infinitesimals and infinities give raise to propositions that are undecidable in Zermelo-Fraenkel set theory [124].

In order to describe a paraconsistent model for the differential and integral calculus, we start with the language L in which we can treat the central notions of this calculus. L is essentially Manin's $\mathrm{L_2Real}$ [172, p. 109], conveniently extended by the introduction of names, in the sense of Shoenfield [223], for all elements of $\mathcal{A}$ (or of $\mathcal{A}^\star$). So, L is composed of the following primitive symbols: (1) individual variables; (2) variables for functions of one variable; (3) individual constants: the names of the elements of $\mathcal{A}$ (or of $\mathcal{A}^\star$); (4) the symbols $+$ and $\times$ for binary operations; (5) two binary relation symbols: $\equiv$ and $<$; (6) the connectives: $\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$ and $\neg$; (7) the quantifiers: $\forall$ and $\exists$; (8) parentheses.

Terms of L: (a) the individual variables and the individual constants and the function variables are terms; (b) if F is a function variable and t is a term, then $F(t)$ is a term; (c) if t_1 and t_2 are terms, then so are $t_1 + t_2$ and $t_1 \times t_2$.

Formulas of L: If t_1 and t_2 are terms, then $t_1 \equiv t_2$ and $t_1 < t_2$ are atomic formulas; the remaining formulas are defined as usual, but quantification of function variables is allowed. The common syntactic notions such as those of bound and free variables, sentence etc. are defined in the usual way.

The following clauses define when ν is a paraconsistent valuation of L:

1. Names denote the corresponding elements of $\mathcal{A}$ (or of $\mathcal{A}^\star$).

2. $\nu(t_1 < t_2) = 1$, where $t_1 < t_2$ is an atomic sentence, if $t_1 < t_2$ is true in $\mathcal{A}$ (or in $\mathcal{A}^\star$). Otherwise, $\nu(t_1 < t_2) = 0$.

3. $\nu(t_1 \equiv t_2) = 1$, iff $t_1 - t_2$ is infinitesimal with respect to ε, ε being an infinitesimal.

4. $\nu(t_1 \not\equiv t_2) = \nu(\neg(t_1 \equiv t_2)) = 1$, where $t_1 \equiv t_2$ is an atomic sentence, if $t_1 \neq t_2$ in $\mathcal{A}$ (or in $\mathcal{A}^\star$); otherwise, $\nu(t_1 \not\equiv t_2) = 0$.

5. Similarly, we define the value of ν for any sentence of L, replacing $\equiv$ and $\not\equiv$ in the sentence by convenient symbolic combinations, respectively as in 3 and 4.

If, as it was remarked above, a sentence F is undecidable in usual set theory plus the axioms of $\mathcal{A}$ (or of $\mathcal{A}^\star$), its value is chosen arbitrarily, in such a way that ν be the characteristic function of a maximal non-trivial set of sentences. Therefore,

ν is really a valuation in L. It is immediate to verify that one may have, for some terms t_1 and t_2, that $\nu(t_1 \equiv t_2) = \nu(t_1 \not\equiv t_2) = 1$.

In fact, ν constitutes a paraconsistent valuation and determines a *model* of the infinitesimal calculus in which the de l'Hospital principle about finite quantities that differ by an infinitely small increment holds. Moreover, in the case of $\mathcal{A}^\star$, the model is such that even de l'Hospital second principle, on smooth curves, happens to be valid (when conveniently interpreted).

The method here delineated to obtain ν is analogous to those of Mortensen [184], and of synthetic differential geometry [41], [42]. For the treatment of functions of several variables, we have to strengthen L by the introduction of functional variables [220]. Summarizing this sub-section, we may say that ν is a paraconsistent structure, which we will call the *de l'Hospital structure*. On the paraconsistent version of the infinitesimal calculus here outlined, see [108]. Further developments could start with the paradoxes of Burali-Forti and of Cantor, which naturally motivate the definition of interesting paraconsistent structures. The development of paraconsistent arithmetics, on other hand, offers no difficulty at all.

4 Jaśkowski's Logic

4.1 Jaśkowski's Discussive Logic

Following a suggestion of J. Łukasiewicz, Stanislaw Jaśkowski (1906-1965) was the first logician to construct a system of paraconsistent propositional calculus [144], [145] (see [146], [87]). Jaśkowski motivated his *discussive* logic (sometimes also referred to as *discursive* logic) by the consideration of the following questions: (i) the problem of the systematization of theories which contains contradictions, as in dialectics; (ii) the study of theories where there are contradictions originated by vagueness, and (iii) the direct study of some empirical theories whose postulates or basic assumptions are contradictory (see [15], [16]).

Later, da Costa and Dubikajtis extended the discussive propositional calculus to first and higher-order predicate calculi [88], [89]; see also [155], [156], [157]. Recently, discussive logic has been applied to the theory of pragmatic truth (see section 4.3), to the foundations of physics [87] (see also below) and also in the philosophy of science (for a general account, see [95]), as well as to several other areas which we cannot treat by limitation of space. In this section we shall sketch the main ideas related to Jaśkowski's discussive logic.

Let us call $\mathcal{J}$ the discussive propositional calculus whose language $\mathcal{L}$ and notations are those one of the modal system S5. We pose: $\Diamond\Gamma =_{\mathsf{def}} \{\Diamond\alpha : \alpha \in \Gamma\}$. So, $\mathcal{J}$ can be semantically defined as

$$\Gamma \models_{\mathcal{J}} \alpha \text{ iff } \Diamond\Gamma \models_{S5} \Diamond\alpha,$$

where the notation has an obvious meaning. It is immediate that

Theorem 4.1.1

(1) $\models_{\mathcal{J}} \alpha$ *iff* $\models_{S5} \Diamond\alpha$

(2) $\Gamma \models_{\mathcal{J}} \alpha$ *iff there are* $\gamma_1, \ldots, \gamma_n$ in Γ *such that* $\models_{S5} \Diamond\gamma_1 \wedge \ldots \wedge \Diamond\gamma_n \to \Diamond\alpha$

Corolary 4.1.1 $\Gamma \models_{\mathcal{J}} \alpha$ *iff there is a finite set* $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ *such that* $\{\gamma_1, \ldots, \gamma_n\} \models_{\mathcal{J}} \alpha$.

Corolary 4.1.2 *If* $\models_{S5} \alpha$*, then* $\models_{\mathcal{J}} \alpha$.

Proof: Since if $\models_{S5} \alpha$, then $\models_{\mathcal{J}} \Diamond\alpha$.▌

Due to the preceding definitions and results, we can see that $\mathcal{J}$ accomplishes Jaśkowski's main intensions. $\mathcal{J}$ has several axiomatizations [74], [89]; the one presented here was introduced in [87]. The postulates are:

(J1) If α is an axiom of S5, then $\Box\alpha$.

(J2) $\Box\alpha, \Box(\alpha \rightarrow \beta) / \Box\beta$

(J3) $\Box\alpha / \alpha$

(J4) $\Diamond\alpha / \alpha$

(J5) $\Box\alpha / \Box\Box\alpha$

Lemma 4.1.1 *If $\vdash_{\mathcal{J}} \alpha$ means that α is provable in $\mathcal{J}$, then: if $\vdash_{\mathcal{J}} \alpha$, then $\models_{S5} \Diamond\alpha$.*

Proof: By induction on the length of the given derivation of α in $\mathcal{J}$.▌

Lemma 4.1.2 *If $\models_{S5} \alpha$, then $\vdash_{\mathcal{J}} \Box\alpha$.*

Proof: By induction on the length of a given derivation of α in S5.▌

Theorem 4.1.2 $\vdash_{\mathcal{J}} \alpha$ *iff* $\models_{\mathcal{J}} \alpha$.

Proof: If $\vdash_{\mathcal{J}} \alpha$, then by Lemma 4.1.1, $\models_{S5} \Diamond\alpha$. So, by definition, $\models_{\mathcal{J}} \alpha$. Conversely, if $\models_{\mathcal{J}} \alpha$, then by definition $\models_{S5} \Diamond\alpha$. So, by Lemma 4.1.2, $\vdash_{\mathcal{J}} \Box\Diamond\alpha$. By postulate (J3), $\vdash_{\mathcal{J}} \Diamond\alpha$, and by postulate (J4), $\vdash_{\mathcal{J}} \alpha$.▌

Definition 4.1.1 *We write $\Gamma \vdash_{\mathcal{J}} \alpha$ iff there are $\gamma_1, \ldots, \gamma_n$ such that $\vdash_{\mathcal{J}} \Diamond\gamma_1 \wedge \ldots \wedge \Diamond\gamma_n \rightarrow \Diamond\alpha$.*

Theorem 4.1.3 $\Gamma \vdash_{\mathcal{J}} \alpha$ *iff* $\Gamma \models_{\mathcal{J}} \alpha$.

Proof: Immediate consequence of the above definition and Theorem **??**.▌

Theorem 4.1.4 *Modus Ponens, that is, the rule $\alpha, \alpha \rightarrow \beta / \beta$, is not valid in $\mathcal{J}$.*

Proof: The uniform predicate calculus $\mathcal{U}$ is a subcalculus of the monadic calculus (that is, that first-order predicate calculus which deals with unary predicates only –[61]) in which there is only one individual variable, say x. There is an obvious bijection between the set of formulae of $\mathcal{U}$ and the language of S5: given a formula α of $\mathcal{U}$, we obtain the corresponding formula α' of the language of S5 by replacing any subformula $P_i(x)$ of α by p_i and any universal quantification $\forall x$ by $\Box$. We can show (see [87]) that if α is a formula of $\mathcal{U}$ and α' its corresponding formula in S5, then $\models_{\mathcal{U}} \alpha$ iff $\models_{S5} \alpha'$. To prove the theorem, then, is enough to note that $\exists x\alpha(x)$ and $\exists x(\alpha(x) \rightarrow \beta(x))$ does not imply that $\exists x\beta(x)$ in $\mathcal{U}$.▌

Theorem 4.1.5 *(a) The rules $\alpha, \beta / \alpha \wedge \beta$ and $\alpha, \neg\alpha / \beta$ are not valid in $\mathcal{J}$; (b) The deduction theorem $\Gamma, \alpha \vdash_{\mathcal{J}} \beta \Rightarrow \Gamma \vdash_{\mathcal{J}} \alpha \rightarrow \beta$ is not true in $\mathcal{J}$.*

These results show that $\mathcal{J}$ can be used to deal with inconsistent set of premisses while avoiding triviality, that is, it is a paraconsistent.

Definition 4.1.2

(1) [Discussive Implication] $\alpha \rightarrow_d \beta =_{\mathsf{def}} \Diamond\alpha \rightarrow \beta$

(2) [Discussive Conjunction] $\alpha \wedge_d \beta =_{\mathsf{def}} \Diamond\alpha \wedge \beta$

(3) [Discussive Equivalence] $\alpha \leftrightarrow_d \beta =_{\mathsf{def}} (\alpha \to_d \beta) \wedge_d (\beta \to_d \alpha)$

(4) [Impossibility] $\nabla\alpha =_{\mathsf{def}} \neg\Diamond\alpha$

Theorem 4.1.6 *The connectives* $\to_d$, $\vee$, $\wedge_d$, $\leftrightarrow_d$ *and* $\leftrightarrow_d$ *have all the classic properties of* $\to$, $\vee$, $\wedge$, $\leftrightarrow$ *and* $\neg$ *respectively.*

This shows that the classical propositional calculus is naturally embedded in $\mathcal{J}$. However,the following rules are not valid in this calculus:

Theorem 4.1.7 *The following formulas and rules are not valid in* $\mathcal{J}$.

(1) $\alpha \to_d (\beta \to_d \alpha \wedge \beta)$

(2) $\alpha \to_d (\neg\alpha \to_d \beta)$

(3) $(\alpha \wedge \beta \to_d \gamma) \to_d (\alpha \to_d (\beta \to_d \gamma))$

(4) $\Gamma, \alpha \models_{\mathcal{J}} \beta$ and $\Gamma, \alpha \models_{\mathcal{J}} \neg\beta \Rightarrow \Gamma \models_{\mathcal{J}} \neg\alpha$

(5) $(\alpha \leftrightarrow_d \neg\alpha) \to_d \beta$

(6) $(\alpha \to_d \neg\alpha) \to_d \neg\beta$

Theorem 4.1.8 (1) $\mathcal{J}$ is decidable.

(2) $\mathcal{J}$ has no finite characteristic matrix, but has the finite model property.

Proof: The decidability of $\mathcal{J}$ follows from that of S5. In fact, being α a formula of $\mathcal{J}$, use Definition 4.1.2 to obtain a formula with $\Diamond$, $\vee$, $\neg$, $\wedge$, $\to$ and $\leftrightarrow$ by eliminating the discussive connectives. Such a formula is a formula β of S5. Now, since β is a thesis of $\mathcal{J}$ iff $\Diamond\beta$ is a thesis of S5, use a decision procedure for S5 to verify if $\Diamond\beta$ is either or not a thesis of S5. As for (2), it follows from the fact that S5 has the finite model property, so as that it has no a finite characteristic matrix (for details, see [74]).∎

$\mathcal{J}$ has (at least) two equivalent semantics. One based on Kripke structures, and another based on the notion of Hanle's algebra. Speaking briefly, a Hanle algebra is a structure $\mathcal{H} = \langle A, -, \wedge, \star\rangle$, where $\langle A, -, \wedge\rangle$ is a Boolean algebra and $\star$ is an unary operator over A such that $\star 1 = 1$ and $\star x = 0$ for all $x \neq 0$, where 0 and 1 are respectively the first and the last elements of the Boolean algebra. We can prove that the following sentences are equivalent, where α is a formula of S5: (i) α is a theorem of S5; (ii) α is valid in every Henle algebra, and (iii) α is valid in every finite Henle algebra [87]. From these statements it is easy ti derive a semantics for $\mathcal{J}$. Notwithstanding, in what follows we shall make reference only to the semantic based on Kripke structures.

A Kripke semantics for $\mathcal{J}$ can be constructed by the same steps we take to build a model theory for S5, and the corresponding results can be proved.

Definition 4.1.3 *Let* Γ *be a set of formulas of the language of* $\mathcal{J}$. *Then,*

(1) $\overline{\Gamma} =_{\mathsf{def}} \{\alpha : \Gamma \vdash_{\mathcal{J}} \alpha\}$

(2) *If* $\overline{\Gamma}$ *is the set of all formulas, then* Γ *is trivial; otherwise, it is non-trivial.*

(3) *If there is a formula* α *such that* $\Gamma \vdash_{\mathcal{J}} \alpha$ *and* $\Gamma \vdash_{\mathcal{J}} \neg\alpha$, *then* Γ *is inconsistent; otherwise, it is consistent.*

(4) *If there is a formula* α *such that* $\Gamma \vdash_{\mathcal{J}} \alpha$ *and* $\Gamma \vdash_{\mathcal{J}} \nabla\alpha$, *then* Γ *is strongly inconsistent.*

Theorem 4.1.9 *There are inconsistent but non-trivial sets of formulas.*

Proof: If p is a propositional variable, then $\{p, \neg p\}$ is inconsistent, but non-trivial.▮

A Kripke structure is an ordered pair $K = \langle W, \upsilon \rangle$ where W is a non-empty set whose elements are called *worlds*, and υ is a mapping $\upsilon : W \times P \mapsto \{0, 1\}$, where P is the set of propositional variables of the language of $\mathcal{J}$. As usual, if $\upsilon(w, p) = 1 (= 0)$, we say that p is *true* (*false*) in the world w. Furthermore, if α is a formula, then we say that K, w *force* α, and write $K, w \Vdash \alpha$, according to the following recursive definition (let us suppose for brevity that we have chosen $\vee$, $\neg$ and $\Box$ as the primitive connectives of our language):

(i) $K, w \Vdash p$ iff $\upsilon(w, p) = 1$

(ii) $K, w \Vdash \beta \vee \gamma$ iff $K, w \Vdash \beta$ or $K, w \Vdash \gamma$

(iii) $K, w \Vdash \neg\beta$ iff $K, w \nVdash \beta$

(iv) $K, w \Vdash \Box\beta$ iff for every $t \in W$, $K, t \Vdash \beta$

Definition 4.1.4 *If* K *is a Kripke structure, then* K *is a model of* Γ *if for every* γ *there is a world* $w \in W$ *such that* $K, w \Vdash \gamma$.

Theorem 4.1.10

(1) Γ *has a model iff it is non-trivial.*

(2) *There are inconsistent sets of formulas which have models.*

Proof: Just apply Kripke semantics for $\mathcal{J}$.▮

These results reinforce that $\mathcal{J}$ is a paraconsistent logic.

We can define first-order discussive calculi with necessary equality as follows (which can be extended to higher-order systems, as already mentioned). Let us call $\mathcal{J}^*$ the calculus whose language is that of the quantificational system S5Q$^=$ of Hughes and Creswell [143]. Then, we define

$$\Gamma \models_{\mathcal{J}^*} \alpha \text{ iff } \Diamond\Gamma \models_{S5Q^=} \Diamond\alpha.$$

In particular, $\models_{\mathcal{J}} \alpha$ iff $\models_{S5Q^=} \Diamond\alpha$. Then, all the above results can be applied to $\mathcal{J}^*$. It is possible to present a sound and complete axiomatization for $\mathcal{J}^*$ as follows:

(J1*) If α is an axiom of S5Q$^=$, then $\Box\alpha$

(J2*) Rules similar to (J2) to (J5) of $\mathcal{J}$ (see page 46)

(J3*) $\Box(\alpha \to \beta(x)) / \Box(\alpha \to \forall x \beta(x))$, where x is not free in α.

Then, it is easy to see that within $\mathcal{J}^*$, $\to_d$, $\wedge_d$, $\vee$, $\leftrightarrow_d$, ∇, $\forall$ and $\exists$ have all the classical properties of $\to$, $\wedge$, $\vee$, $\leftrightarrow$, $\neg$, $\forall$ and $\exists$ respectively. Furthermore, it is not difficult to justify that: (i) S5Q$^=$ is contained in $\mathcal{J}^*$; (ii) both $\mathcal{J}^*$ and S5Q$^=$ are not decidable; (iii) both have algebraic semantics relative to which they are sound and complete; (iv) there are inconsistent but non-trivial sets of formulas of $\mathcal{J}^*$, and finally (v) a set of formulas Γ is non-trivial iff there is a Kripke structure for $\mathcal{J}^*$ which is a model of Γ. Then, there are inconsistent sets of formulas which do have models.

Another and equivalent way of building a first-order discussive calculus is as follows, which is suitable for certain applications, as we shall say below. This calculus, which we call $\mathcal{J}^{**}$ (it is essentially the same as $\mathcal{J}^{*}$), the language is also the same as that one of S5Q$^=$, but if α is a formula, then we write $\uplus\alpha$ to denote the formula composed by preceding α by any sequence of universal quantifiers so that all variables of $\uplus\alpha$ are bound.

To define the calculus $\mathcal{J}^{**}$, we pose that

$$\vdash_{\mathcal{J}^{**}} \alpha \text{ iff } \vdash_{S5Q^=} \Diamond \uplus \alpha.$$

Furthermore, we say that α is a syntactical consequence of Γ in $\mathcal{J}^{**}$, that is, $\vdash_{\mathcal{J}^{**}} \alpha$, iff there are $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\Diamond \uplus (\gamma_1 \wedge \ldots \wedge \gamma_n \rightarrow \alpha)$ is valid in S5Q$^=$. The postulates of $\mathcal{J}^{**}$ are (with the common restrictions):

(J1**) If α is an instance of a classical tautology, then $\Box \uplus \alpha$ is an axiom.

(J2**) $\Box \uplus \alpha, \Box \uplus (\alpha \rightarrow \beta) / \Box \uplus \beta$

(J3**) $\Box \uplus (\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta))$

(J4**) $\Box \uplus (\Box\alpha \rightarrow \alpha)$

(J5**) $\Box \uplus (\Diamond\alpha \rightarrow \Box\Diamond\alpha)$

(J6**) $\Box \uplus (\forall x \alpha(x) \rightarrow \alpha(t))$

(J7**) $\Box \uplus \alpha / \alpha$

(J8**) $\Box \uplus \alpha / \Box \uplus \Box\alpha$

(J9**) $\Diamond \uplus \alpha / \alpha$

(J10**) $\Box \uplus (\alpha \rightarrow \beta(x)) / \Box \uplus (\alpha \rightarrow \forall x \beta(x))$

(J11**) Vacuous quantification may be introduced in any formula.

(J12**) $\Box \uplus (x = x)$

(J13**) $\Box \uplus (x = y \rightarrow (\alpha(x) \rightarrow \alpha(y))$

Theorem 4.1.11 *In $\mathcal{J}^{**}$, the connectives $\rightarrow_d$, $\wedge_d$, $\vee$, ∇, $\leftrightarrow_d$, $\forall$ and $\exists$ have all the standard properties of classical (material) implication, conjunction, disjunction, negation, (material) equivalence, and of the universal and existential quantifiers respectively.*

Thus, classical first-order logic is contained in $\mathcal{J}^{**}$. Furthermore, when we restrict the formulas to the *stable* ones only, that is, those formulas α such that $\Box(\alpha \leftrightarrow_d \Diamond\alpha)$ is true, then $\mathcal{J}^{**}$ reduces in a certain sense to classical first-order logic.

4.2 Application to the foundational analysis of physical theories

A physical theory T can be roughly characterized as follows. We start from a language L by means of which we express the postulates of T. These postulates or axioms may be divided up into three levels: the logical axioms, say those of first-order predicate calculus, the mathematical axioms, say those of a set theory like Zermelo-Fraenkel, and the specific, or physical, axioms of T. It is clear that this schema is quite general, and should be suitable adapted in some particular cases (see [95]). Anyway, it is useful for some analysis of physical theories, mainly if we are to consider the possibilities of inconsistencies in such a frameworks. For instance, in certain situations it may be more adequate to use a calculus like $\mathcal{J}^{**}$ of the preceding section instead of standard first-order logic. This is the case we would like to make reference here, although without all the technical details.

According to Dalla Chiara and Toraldo do Francia [114], the language of a physical theory T is interpreted in a set-theoretical structure of the form

$$\mathcal{A} = \langle M, S, \langle Q_0, \ldots, Q_n \rangle, \rho \rangle,$$

where

(1) M is an instance of a mathematical species of structures in the sense of Suppes-Bourbaki (see [**?**]).

(2) S is a set of physical situations, that is, a set of physical states assumed by a physical system in a certain interval of time.

(3) Each Q_i, $0 \leq i \leq n$, denotes an operationally defined quantity whose domain of definition is some subset of S. In general, Q_0 represents time.

(4) ρ is a function that associates to each term employed to characterize S, and in particular to each Q_i, a set-theoretical entity in M.

When we try to measure a physical quantity Q_i of a physical system in a state $s \in S$, in a certain time t_j, we usually consider that its 'acceptable values' $q_i(t_j)$ lie in an interval $[q_i - \epsilon, q_i + \epsilon]$ of the real number line, where the length ϵ depends on the specific measurement technique and on the nature of the involved quantity. In a certain sense, *all values* in the interval are 'appropriate values' for the measurement of the quantity Q_i of the physical system in a state $s \in S$. For instance, in measuring the table where we are working just now, we should accept (for all 'practical purposes') any value in the interval 1.20 ± 10^{-3} meters.

So, let $\alpha(t, q_i(t_j))$ be a formula of L whose only free variables are t and t_j (which stand for time). Dalla Chiara and Toraldo di Francia consider the case of partial formulas, that is, formulas which are not defined for all values of its variables and parameters, but here we shall restrict our attention to 'total' formulas only. Then, we say that $\alpha(t, q_i(t_j))$ is *true* with respect to a situation s, written $\vDash_s \alpha$, if there are values t^0 of t in the considered time interval and q_i^0 of Q_i in the corresponding interval, such that $\alpha(t^0, q_i^0)$ is true in M in the standard (Tarskian) sense. We also say that $\alpha(t, q_i(t_j))$ is true in $\mathcal{A}$, and write $\mathcal{A} \models \alpha(t, q_i(t_j))$, if $\alpha(t, q_i(t_j))$ is true in M for every $s \in S$. Paraconsistency enters in Dalla Chiara and Toraldo di Francia's approach whenever we get $\overline{t^0}$ and $\overline{q_i^0}$ also in the intervals such that $\neg\alpha(\overline{t^0}, \overline{q_i^0})$ is also true in M. Let us explain this case with an example.

Let us take Newton's second law $f = m.a$. The three physical variables appearing in this equation correspond to three deterministic physical quantities *force* (F), *mass* (M) and *acceleration* (A), which are the physical quantities to be measured, whose acceptable range of values for a certain physical situation s lie respectively within three intervals $[f_1, f_2] \subseteq \mathbb{R}$, $[m_1, m_2] \subseteq \mathbb{R}$ $[a_1, a_2] \subseteq \mathbb{R}$, each one of them expressing a certain precision ϵ for the measurements. Then $\vDash_s f = m.a$ when there exist three real numbers $p_1 \in [f_1, f_2]$, $q_1 \in [m_1, m_2]$ and $r_1 \in [a_1, a_2]$ such that $p_1 = q_1.r_1$.

However, due to the imprecision ϵ, there are also other three real numbers p_2, q_2 and r_2, each one in the respective interval, so that $p_2 \neq q_2.r_2$, and these numbers are also *acceptable values* for the measurements of the corresponding physical quantities. So, strictly speaking, $\vDash_s \neg(F = m.a)$ too, that is, the negation of Newton's law should also be true with respect to the same physical situation s. This way, we may have, for a sentence α and physical situation $s \in E$, both $\vDash_s \alpha$ and $\vDash_s \neg\alpha$, but of course not $\vDash_s \alpha \wedge \neg\alpha$, for this last case would entail the existence of three real numbers p', q' and r' belonging to the respective intervals such that $\vDash_s p' = q'.r' \wedge p' \neq q'.r'$, which is impossible [114, p. 66].

This definition of truth reflects a kind of *empirical truth*, having interesting consequences pointed out by its authors, like the non truth-functionality of the logical connectives, in the sense that the truth of a conjunction is not equivalent to the simultaneous truth of both conjuncts. Here, it is interesting to note the 'paraconsistent aspect' of this definition of truth, for we can have both $\vDash_s \alpha$ and $\vDash_s \neg\alpha$. In specifying the underlying logic of Dalla Chiara and Toraldo di Francia's approach, we may use the postulates of a Jaśkowski's discussive logic, say $\mathcal{J}^{**}$, as logical axioms of such physical theories (alternatively, we could use a paraclassical logic instead –see section 7.3).

4.3 Application to Partial Truth

In Mikenberg, da Costa and Chuaqui [180], the mathematical concept of pragmatic truth (today called 'partial truth') was introduced. An infinitary logical system was presented to treat this concept, and some applications of it are made in logic and in algebra. An application of this concept in the foundations of the theory of probability was studied in [77], and some of its extensions to inductive logic and to the philosophy of science are discussed, respectively, in da [90], [92]. This framework was also employed in order to examine some issues involved in the theory of acceptance ([93]), as well as in the modelling of natural reasoning ([94]). For a general discussion, see [95]. The wide range of applications of the notion of partial truth motivates an investigation of the logic of partial truth. In this section, we will show that there are important connections between this logic and Jaśkowski's discussive logic. We will study these connections in the context of the concept of 'pragmatic structures', which we shall present below. As we shall see, such structures can be treated as worlds of a Kripke structure, and in this setting the necessity operator in modal logic corresponds to the notion of 'pragmatic validity', and the possibility operator to the notion of partial truth. Two systems are then put forward to formalize these notions. One of the main results of the present section is that the logic of partial truth is paraconsistent. The philosophical import of this result, which justifies the application of partial truth to inconsistent settings, is then discussed.

A remark on our terminology is important here. We call the kind of truth defined in this section *partial truth*. Originally, it was called 'pragmatic truth', owing to its connections with the pragmatic conception of truth, as developed by philosophers like James, Dewey and particularly Peirce (cf. [180], [77], and [95]). However, our piece is not exegetical. The sole point we would like to emphasize is that our definition was heuristically inspired by some passages of pragmatic thinkers, such as Pierce, when he wrote that, "consider what effects, that might conceivably have practical bearings, we conceive the object of our conceptions to have. Then, our conception of these effects is the whole of our conception of the object" [194, p. 31]. In our opinion, the definition of partial, or pragmatic truth investigated in this section captures, at least in part, the common concept of a theory *saving the appearances*, usually by means of partially fictitious constructions (see [230], and [52]).

Let us suppose that we are interested in studying a certain domain of knowledge Δ in the field of empirical sciences, for instance the physics of particles. We are, then, concerned with certain real objects (in the physics of particles, with some configurations in a Wilson chamber, some spectral lines, etc.), whose set we denote by A_1. Among the objects of A_1, there are some relations that interest us and that we model as partial relations R_i, $i \in I$, every relation having a fixed arity. The relations R_i are partial relations, that is, each R_i, supposed of arity r_i, is not necessarily defined for all r_i-tuples of elements of A_1. More formally, an n-place partial relation R can be viewed as a triple $\langle R_1, R_2, R_3 \rangle$, where R_1, R_2, and R_3 are mutually disjoint sets, with $R_1 \cup R_2 \cup R_3 = D^n$, and such that R_1 is the set of n-tuples that belong to R; R_2 the set of n-tuples that do not belong to R; and finally R_3 of those n-tuples for which it is not defined whether they belong or not to R. (Note that when R_3 is empty, R is an usual n-place relation that can be identified with R_1.)

The reason for using partial relations is that they are supposed to express what we do know, or what we accept as true, about the actual relations linking the elements of A_1. Then, the partial structure $\langle A_1, R_i \rangle_{i \in I}$ encompasses, so to say, what we know or accept as true about the actual structure of Δ. However, in order to systematize our knowledge of Δ, it is convenient to introduce in our structure $\langle A_1, R_i \rangle_{i \in I}$ some *ideal* objects (in the physics of particles, quarks for example). The set of these new objects will be denoted by A_2. It is understood that $A_1 \cap A_2 = \emptyset$, and we put $A = A_1 \cup A_2$. This way, the modelling of Δ involves new partial relations R_j, $j \in J$, some of which extend the relations R_i, $i \in I$. Furthermore , there are some sentences (closed formulas) of the language L, in which we talk about the structure $\langle A, R_k \rangle_{k \in I \cup J}$ $(I \cap J = \emptyset)$ that we accept as true or that are true (in the sense of the correspondence theory of truth). This occurs, for instance, with sentences expressing true decidable propositions (a proposition whose truth or falsehood can be *decided*), and with some general sentences which express laws or theories already accepted as true. Let us denote the set of such sentences, dubbed *primary*, by $\mathcal{P}$ (this set may be empty).

Then, taking into account the above informal discussion, we are led to suggest that a simple pragmatic structure be regarded as a set-theoretic structure of the form

$$\mathfrak{A} = \langle A_1, A_2, R_i, R_j, \mathcal{P} \rangle_{i \in I, j \in J},$$

which we call a *simple pragmatic structure* (sps), where the involved elements satisfy the preceding conditions. Alternatively, we can write simply

$$\mathfrak{A} = \langle A, R_k, \mathcal{P} \rangle_{k \in K}$$

for a sps, where $A = A_1 \cup A_2$ and the R_k are partial relations defined on A, and $\mathcal{P}$ is a set of sentences of the language L of the same similarity type as that of $\mathfrak{A}$, and which is interpreted in $\mathfrak{A}$. We remark that for some k, R_k may be empty.

Let L be a first-order language with equality, but without function symbols. The symbols of L are, then, logical symbols (connectives, individual variables, the quantifiers, and the equality symbol), auxiliary symbols (parentheses), a collection of individual constants, and a collection of predicate symbols. To interpret L in a sps $\mathfrak{A}$ is to associate with each individual constant of L an element of A, the universe of $\mathfrak{A}$, and with each predicate symbol of L of arity n a relation R_k, $k \in K$, of the same arity. It is supposed that every predicate of the family R_k, $k \in K$ is associated with a predicate symbol.

Definition 4.3.1 *Let L and $\mathfrak{A} = \langle A, R_k, \mathcal{P} \rangle_{k \in K}$ be respectively a language and a sps in which L is interpreted. Let $\mathfrak{B}$ be a total structure, that is a usual structure (whose relations of arity n are defined for all n-tuples of elements of its universe), and we suppose that L is also interpreted in $\mathfrak{B}$. Then, $\mathfrak{B}$ is said to be* $\mathfrak{A}$-normal *if the following properties are verified:*

(1) *The universe of* $\mathfrak{B}$ is A.

(2) *The (total) relations of* $\mathfrak{B}$ *extend the corresponding partial relations of* $\mathfrak{A}$.

(3) *If* c *is an individual constant of* L, *then in both* $\mathfrak{A}$ *and* $\mathfrak{B}$, c *is interpreted by the same element.*

(4) *If* $\alpha \in \mathcal{P}$, *then* $\mathfrak{B} \models \alpha$.

Given a pragmatic structure $\mathfrak{A}$, it may happen that there are no $\mathfrak{A}$-normal structures, but it is possible to provide a system of necessary and sufficient conditions for the existence of such structures (see [180]). One condition of this system is as follows. For each partial relation R_k in $\mathfrak{A}$, we construct a set M_k of atomic sentences and negations of atomic sentences such that the former corresponds to n-tuples that satisfy R_k, and the latter to n-tuples that do not satisfy R_k (such sentences correspond to n-tuples in the 'anti-extension' of R_k). Let M be the set $\bigcup_{k \in K} M_k$. Therefore, a sps $\mathfrak{A}$ admits an $\mathfrak{A}$-normal structure only if the set $M \cup \mathcal{P}$ is consistent. In what follows we shall always suppose that our sps satisfy the relevant conditions; in other words, given any sps $\mathfrak{A}$, the set of $\mathfrak{A}$-normal structures is not empty.

Definition 4.3.2 *Let* L *and* $\mathfrak{A}$ *be respectively a language and a sps in which* L *is interpreted. We say that a sentence* α *of* L *is* pragmatically true, or partially true *in the sps* $\mathfrak{A}$ *according to* $\mathfrak{B}$ *if*

(1) $\mathfrak{B}$ *is an* $\mathfrak{A}$*-normal structure, and*

(2) $\mathfrak{B} \models \alpha$, *that is,* α *is true in* $\mathfrak{B}$ *in conformity with the Tarskian definition of truth.*

That is, we say that α is *pragmatically (or partially) true* in the sps $\mathfrak{A}$ if there exists an $\mathfrak{A}$-normal $\mathfrak{B}$ in which α is true in the standard Tarskian sense. If α is not pragmatically (partially) true in the sps $\mathfrak{A}$ according to $\mathfrak{B}$ (is not pragmatically true in the sps $\mathfrak{A}$), we say that α is *pragmatically (partially) false* in the sps $\mathfrak{A}$ according to $\mathfrak{B}$ (is pragmatically (partially) false in the sps $\mathfrak{A}$).

Given a sps $\mathfrak{A}$, it is natural to consider its $\mathfrak{A}$-normal structures as the worlds of a Kripke structure for S5 with quantification, i.e. we have a universe and several structures, defined in such a universe, in which the language L can be interpreted, and where every world is accessible to every world (cf. [143]). It is also natural to extend the language L of the sps $\mathfrak{A}$ to a modal language, by the adjunction of the modal operator $\Box$ to its primitive symbols. The operator $\Box$ which in modal logic represents the notion of necessity, corresponds in the present situation to *pragmatic validity* (in a sps $\mathfrak{A}$). Analogously, the possibility symbol $\Diamond$, definable in terms of $\Box$ and negation, corresponds to *pragmatic truth* (in a sps $\mathfrak{A}$). Thus, we are led to extend the semantics of L in an obvious way, such that the symbols $\Box$ and $\Diamond$ will represent the concepts of pragmatic validity and of pragmatic truth, respectively. Moreover, since the universes of all 'worlds' belonging to a sps are the same, it is reasonable that equality behaves, in the cases of pragmatic truth and of pragmatic validity, as necessary equality.

Among the pragmatically valid formulas –that is, those formulas α such that $\Box\alpha$ is a theorem of S5 with quantification and necessary equality, there are the logically pragmatically true formulas –that is, those formulas α such that $\Box\Diamond\alpha$ or, equivalently, $\Diamond\alpha$ is a theorem of the same system. From now on, in order to simplify the language, the former class of formulas will be called *strictly pragmatically valid* and the latter will be called *pragmatically valid.* The first class of formulas coincides with the set of theorems of S5 with quantification and necessary equality; the second, with Jaśkowski's logic associated with the same system.

Therefore, the logical system which will be denoted by PV and which systematizes the notion of strict pragmatic validity has a language L^* whose primitive symbols are those of a standard formalization of the first-order predicate calculus with equality and individual constants, plus the symbol $\Box$ (for simplicity, function symbols are excluded). The defined symbols are introduced as usual, and the common conventions in the writing of formulas, and in the formulation of postulates (axiom schemes and primitive rules of inference) etc. are employed without explicit mention. The postulates of PV are the following:

(1) If α is an instance of a (propositional) tautology, then α is an axiom.

(2) $\alpha,\ \alpha \to \beta \,/\, \beta$

(3) $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$

(4) $\Box\alpha \to \alpha$

(5) $\Diamond\alpha \to \Box\Diamond\alpha$

(6) $\forall x\alpha(x) \to \alpha(t)$

(7) $\alpha \,/\, \Box\alpha$

(8) $\alpha \to \beta(x) / \alpha \to \forall x\beta(x)$

(9) $x = x$

(10) $x = y \to (\alpha(x) \to \alpha(y))$

In the postulates above, the symbols have clear meanings. In particular, in Axiom Scheme 6, t is either a variable free for x in $\alpha(x)$ or an individual constant. This system is essentially S5 with quantification and necessary equality. We define the concept of deduction as in [141]. Their basic idea is essentially that one can only use the generalization rule $\alpha / \forall\alpha$ in a step k of a deduction when a subsequence of the deduction up to k is a proof of α. This restriction to the generalization rule is exactly similar to the one adopted with regard to the necessitation rule, $\alpha / \Box\alpha$. Then, the usual derived rules, such as the deduction theorem, remain valid.

The semantics of PV can be easily developed: the basic (strict) semantical concepts of pragmatic truth, pragmatic falsehood, pragmatic validity, pragmatic invalidity, pragmatic semantic consequence, etc. offer no difficulties in being formulated (and maintain the spirit of Definition 4.3.2) (for more details on these points, see [110], [111]).

We have the following theorem, whose proof can be obtained by the methods of [143], [128]:

Theorem 4.3.1 *Let* Γ *be a set of formulas of* L^* *and* α *be a formula of the same language. Then,* $\Gamma \vdash \alpha$ *if, and only if,* $\Gamma \models \alpha$.

That is, α is a *syntactic consequence* of Γ in PV iff α is a *strict pragmatic semantic consequence* of Γ.

The logic of strict pragmatic validity, which we have just sketched, can be extended to higher-order (modal) languages, for example by an adaptation of some ideas presented in [128, Chap. 3]. Moreover, one can also develop a metatheoretical study of this logic, for instance, by adopting different modal systems as basic (S4, for instance), by distinguishing between frames and models etc. Instead of pursuing this line here, we will now consider a different system in order to study the logic of pragmatic validity. This system, as we will see, is constructed, as it were, in terms of PV, which was presented here mainly as an auxiliary construction. We will then show that the logic of pragmatic validity is paraconsistent.

Let us call PT (for 'pragmatic truth') a system whose language is the same as that of PV; the underlying intuition is that of constructing a system in which $\vdash \alpha$ means that $\Diamond\alpha$ is strictly pragmatically valid. As before, if α is a formula of L^*, which the language of PV or of PT, we write $\uplus\alpha$ to denote the formula composed by preceding α by any sequence of universal quantifiers, so that all variables of $\uplus\alpha$ are bound.

Clearly, in order that α be pragmatically valid in the sense intended, we must have that $\vdash \alpha$ in PT if, and only if, $\vdash \Diamond \uplus \alpha$ in PV. So, PT is a kind of Jaśkowski's discussive logic associated to PV (we recall that, given a modal system M, the Jaśkowski's logic associated to M is the set of all formulas α such that $\Diamond\alpha$ is a thesis of M).

PT can be axiomatized as $\mathcal{J}^{**}$ of the preceding section (see postulates J1** to J13** of page 48). The definitions of proof and of (formal) theorem are the usual ones. We proceed to show that that postulates really provide an axiomatization for PT.

Lemma 4.3.1 *If α is a theorem of our proposed axiomatization for PT, then $\Diamond \uplus \alpha$ is a theorem of PV.*

Proof: By induction on the length of the proof of α in the proposed axiomatization for PT. Let $\alpha_1, \ldots, \alpha_n$, where α_n is α, be a (formal) proof of α in the proposed axiomatization for PT. Then, α_i, $1 \leq i \leq n$, is an axiom or is obtained by the application of one of the rules. If a_i is an axiom, then it has the form $\Box \uplus \beta$, where β is an axiom of PV (observe that PV is S5 with quantification and necessary equality). Therefore, $\Box \uplus \beta$ is a theorem of PV, and so $\Diamond \uplus \Box \uplus \beta$, i.e. $\Diamond \uplus \alpha_i$ is also a theorem of PV. Suppose that α_i is a consequence of two preceding formulas by Rule J2**. Then α_i is $\Box \uplus \beta$, obtained from the premises $\Box \uplus \gamma$ and $\Box \uplus (\gamma \rightarrow \beta)$. By the induction hypothesis, $\Diamond \uplus \Box \uplus \gamma$ and $\Diamond \uplus \Box \uplus (\gamma \rightarrow \beta)$ are provable in PV. Consequently, $\Box \uplus \gamma$ and $\Box \uplus (\gamma \rightarrow \beta)$ are also provable in PV, and so is $\Box \uplus \beta$. But if $\Box \uplus \beta$ is a theorem of PV, then, $\Diamond \uplus \Box \uplus \beta$, i.e. $\Diamond \uplus \alpha_i$, is also a theorem. The other rules are similarly treated.▮

Lemma 4.3.2 *If α is a theorem of PV, then $\Box \uplus \alpha$ is a theorem of the proposed axiomatization for PT.*

Proof: By induction on the length of the proof of α in PV. Let $\alpha_1, \ldots, \alpha_n$, where α_n is α, be a proof of α in PV. If α_i, $1 \leq i \leq n$, is an axiom of PV, then $\Box \uplus \alpha$ a is a theorem of the proposed axiomatization for PT, as is easy to see. If α_i is obtained by an application of modus ponens (Rule J2**), from γ and $\gamma \rightarrow \alpha_i$, we have, by the induction hypothesis, that $\Box \uplus \gamma$ and $\Box \uplus (\gamma \rightarrow \alpha_i)$ are provable in the proposed axiomatization. Then, by Rule J2**, $\Box \uplus \alpha_i$ is also provable. Rule J8** is treated analogously.▮

Theorem 4.3.2 *Postulates J1**–J13** characterize PT; that is, we have:*

$$\vdash \alpha \textit{ in } PT \textit{ iff } \vdash \Diamond \uplus \alpha \textit{ in } PV.$$

Proof: Let us suppose that α is a theorem of the proposed axiomatization for PT; then, by Lemma 4.3.1, $\Diamond \uplus \alpha$ is a theorem of PV. Conversely, assume that $\Diamond \uplus \alpha$ is a theorem of PV. So, by Lemma 4.3.2, $\Box \uplus \Diamond \uplus \alpha$ is a theorem of the proposed axiomatization for PT. Therefore, by Rule J7**, $\Diamond \uplus \alpha$ is a theorem of PT, and so by Rule J9**, α is also a theorem of PT.▮

Definition 4.3.3 *In PT we say that the formula αa is a syntactic consequence of a set of formulas Γ (in symbols, $\Gamma \vdash \alpha$) if there exist $\gamma_1, \ldots, \gamma_n$ in Γ, such that*

$$(\Diamond\gamma_1 \wedge \ldots \wedge \Diamond\gamma_n) \rightarrow \Diamond\alpha$$

is a theorem of PT (or, equivalently,

$$\Diamond \uplus ((\Diamond\gamma_1 \wedge \ldots \wedge \Diamond\gamma_n) \rightarrow \Diamond\alpha)$$

is a theorem of PV). When $n = 0$, the first formula above reduces, by convention, to α (and $\emptyset \vdash \alpha$ means, thus, that $\vdash \alpha$).

Definition 4.3.4 *A pragmatic theory is a set T of sentences (closed formulas of PT), such that if $\gamma_1, \ldots, \gamma_n$ are in T and $\{\gamma_1, \ldots, \gamma_n\} \vdash \alpha$, then α is also in T.*

It follows that if T is a pragmatic theory and α is a (closed) theorem of PT, then $\alpha \in T$. Let $\mathcal{S}$ be the set of all sentences of PT and T be a pragmatic theory. Using the already introduced terminology, we say that T is *trivial* (*overcomplete*) if $T = \mathcal{S}$; otherwise, T is *non-trivial*. Furthermore, T is *inconsistent* if there is at least one sentence α such that $\alpha \in T$ and $\neg\alpha \in T$, where $\neg$ is the symbol of negation of PT; otherwise, T is *consistent*. Then, we can prove the following result:

Theorem 4.3.3 *There exist pragmatic theories which are inconsistent but non-trivial*

Proof: Let c and M be respectively any individual constant and a monadic predicate symbol of PT. The theory whose (nonlogical) axioms are $M(c)$ and $\neg M(c)$ is inconsistent. But it is nontrivial, because the corresponding theory of PV, whose (nonlogical) axioms are $\Diamond M(c)$ and $\Diamond\neg M(c)$, is consistent. In effect, it is easy to construct a Kripke model for PV in which both $\Diamond M(c)$ and $\Diamond\neg M(c)$ are true. However, in no Kripke model for PV the formula $\Diamond(M(c) \wedge \neg M(c))$ is true.▮

By considering the definitions of the discussive connectives $\rightarrow_d$ and $\wedge_d$ as put in Definition 4.1.2, we can also prove that, in PT, $\rightarrow_d$, $\wedge_d$, $\vee$, $\forall$ and $\exists$ satisfy all the schemes and rules of classical positive logic. Really, if we consider a valid primitive scheme (or rule) of classical positive logic, and replace in it implication by discussive implication and conjunction by discussive conjunction, we obtain a valid scheme (or rule) of PT, as is easily seen.

Theorem 4.3.4 *If T is a pragmatic theory, then $\alpha \in T$ iff there exist $\gamma_1, \ldots, \gamma_n$ are in T such that*

$$(\gamma_1 \wedge_d \ldots \wedge_d \gamma_n) \rightarrow_d \Diamond\alpha$$

is a theorem of PT.

We remark that in some of the applications of the theory developed above, it is sometimes convenient to employ an alternative definition of syntactic consequence. For instance, in certain applications in the foundations of physics, instead of Definition 4.3.3, it is more appropriate to adopt the following alternative:

Definition 4.3.5 *In PT, the sentence α is said to be a proper syntactic consequence of a set of sentences Γ if there is $\gamma_1, \ldots, \gamma_n$ are in Γ such that $\Diamond(\gamma_1 \wedge \ldots \wedge \gamma_n)$ and $\Box((\gamma_1 \wedge \ldots \wedge \gamma_n) \rightarrow \alpha)$ are theorems of PT (or of PT and some extra axioms).*

The philosophical significance of the above formal account can be seem through a consideration of inconsistency in our belief systems [110]. If one focuses on Theorem 4.3.3, it can be seen that a pragmatic theory can contain contradictory theorems without reducing to triviality. This means that PT belongs to the class of paraconsistent logics. In the case of pragmatic (partial) truth, this is not an unreasonable situation: contradictory propositions may, of course, both be pragmatically true [95, Chap. 5]. Thus partial truth can be used to provide the epistemic framework for characterizing inconsistent belief systems.

More precisely, we can formulate a position according to which 'belief that p' is not to be understood as 'belief that p is true' in the correspondence sense. Such an account has been exposed in [95]; when it comes to representational structures such as scientific theories, 'belief that p' is to be understood as 'belief that p is pragmatically or partially true'. This allows for the accommodation of inconsistency by acknowledging that it is not a permanent feature of reality to which theories must correspond, but is rather a temporary aspect of such theories which may nevertheless be extremely fruitful in a heuristic sense. On this account it is not the 'logic of science', in the sense of the underlying logic of deduction and inference, which is paraconsistent, but rather the appropriate 'logic of truth'.

The logic of pragmatic truth as delineated above has also been developed to serve as a 'logic of scientific acceptance' (ibid.). The nature of acceptance is relatively little discussed within the philosophy of science. Those accounts that do consider it tend to divide between two extremes: those that identify acceptance and belief and those that separate the two entirely. The former typically regard belief in terms of the correspondence view of truth, whereas the latter fall prey to the accusation of some conventionalism. An alternative is to retain the connection between belief and acceptance whilst rejecting truth-as-correspondence. On this view, to accept a theory is to be committed, not to believing it to be true per se, but to holding it as if it were true, for the purposes of further elaboration, development and investigation. Thus acceptance involves belief that the theory is partially or pragmatically true only and this, we believe, corresponds to the fallibility of scientists themselves.

Linking acceptance and pragmatic truth in this way restores a formal similarity between 'truth', taken generally, and acceptance with regard to deductive closure. So, it has been argued, for example, that acceptance differs from truth in that whereas the latter is deductively closed, in the sense that what one deduces from a set of truths is also true, the former generally is not (see [228]). This is correct if closure is understood only in classical terms. However, what the above formal analysis shows is that acceptance, understood within the framework of pragmatic truth, may be regarded as closed under the Jaśkowski's discussive system. To put it more precisely: although there is no closure under classical conjunction and material implication, one can define discussive forms of implication and conjunction as above, with respect to which acceptance can indeed be regarded as closed. This is a result of both general and particular significance. Our contention is that inconsistency can be accommodated within an appropriate framework under which the set of propositions which we accept is closed under implication. The appropriate framework is precisely that which we have presented in this book and the form of implication is, of course, discussive. Shifting perspective again from the specific to the more general, it is the failure to consider such non-classical systems which undercuts the claim that 'logic' is not specially relevant to reasoning. Within the framework of pragmatic truth we can accommodate inconsistency while still retaining a sense of deductive closure. In this manner the relevance of logic to reasoning –especially scientific reasoning– is restored (see [95]).

5 Annotated Logics

Reasoning about inconsistency is of importance also in computer science, data base theory and in artificial intelligence. For instance, in constructing a knowledge base about a certain domain D of knowledge, we generally consult n experts, say $E_1, \ldots, E_n$ of that field of knowledge. Each expert contributes with facts and rules which form bases $S_1, \ldots, S_n$ of sets of sentences, so that the whole knowledge base can ne taken as the set $S_1 \cup \ldots \cup S_n$. However, experts may disagree and hence this last set may be inconsistent, having no models and consequently deemed meaningless by the first order model theory.

In 1987, V. S. Subrahmanian devised a kind of paraconsistent logic, termed 'annotated logic', that was suitable for representing databases and knowledge bases that contain inconsistencies [226]. Later, Blair and Subrahamanian [40] developed further this framework endowing it with a fixed-point theory, model theory and proof theory. Later, they extended earlier results to allow logic programming over a complete lattice of truth-values, and have also extended accordingly the fixed-point theory and proof theory [39]. Kifer and Subrahmanian [151] generalized annotated logic in such a way that a framework for logic programming based on a concept of 'bilattice', which was developed before by Fitting, could be captured by the extended annotated logic framework. Kifer and Li [149] showed how annotated logic can be used as a foundation for reasoning in the presence of inconsistency; Kifer and Lozinskii [150] demonstrated an embedding of classical logic in annotated logic, and showed the connections between these logics and non-monotonic logics, having also devised a mechanical proof procedure for annotated logic. Kifer and Wu [152] showed how annotated logics serve as a foundation for object-oriented databases; Kifer and Krishnaprasad [148] showed how annotated logics can be used as a foundation for inheritance networks.

The foundational study of annotated logic was suggested by da Costa, Subrahmanian and Vago [113], who developed a family of propositional calculus, denoted $\mathsf{P}\mathcal{T}$, so as their first order counterpart, $\mathsf{Q}\mathcal{T}$, a work which was extended in [106] to annotated set theory and further results on $\mathsf{Q}\mathcal{T}$.

5.1 The annotated logic $\mathsf{Q}\mathcal{T}$

$\mathsf{Q}\mathcal{T}$ is a first order logic defined as follows. $\mathcal{T}$ is an arbitrary, but fixed complete lattice. The least element of $\mathcal{T}$ is denoted by $\bot$, while the greatest element is denoted by $\top$; furthermore, $\neg$ is taken to be an unary operator from $\mathcal{T}$ to $\mathcal{T}$. The language L of $\mathsf{Q}\mathcal{T}$ is a first order language without equality whose primitive symbols are the following:

1. Connectives: $\rightarrow$ (implication), $\vee$ (disjunction), $\wedge$ (conjunction) and $\neg$ (negation)

2. Individual variables: a denumerably infinite set of variable symbols

3. Individual constants: an arbitrary family of constant symbols

4. Quantifiers: $\forall$ (for all) and $\exists$ (exists)

5. Function symbols: for each natural number $n > 0$, a collection of function symbols of rank n

6. Predicate symbols: for each $n \geq 0$, a family of predicate symbols of rank n.

7. Auxiliary symbols: parentheses and comma.

Terms of L are introduced as usual. An (ordinary) *atom* is an expression of the form $P(t_1, \ldots, t_n)$; if P a predicate symbol of rank n and $\lambda \in \mathcal{T}$, an *annotated predicate* is a pair $\langle P, \lambda \rangle$, which we will denote simply by P_λ.

Given an annotated predicate P_λ (which we will sometimes call simply 'predicate') of rank n and n terms $t_1, \ldots, t_n$, an *annotated atom* is an expression of the form $P_\lambda(t_1, \ldots, t_n)$. The notion of *formula* is introduced in the standard way. We remark that the symbol $\neg$ is being used here in two distinct ways: firstly as a mapping from $\mathcal{T}$ to $\mathcal{T}$ and secondly, as an unary connective of L. The right meaning of $\neg$ shall be given by the context.

Definition 5.1.1 *An interpretation I for the language L is a 4-tuple*

$$I = \langle D, \eta_I, \zeta_I, \chi_I \rangle,$$

where:

(i) D is a non-empty set, called the *domain* of I.

(ii) η_I maps individual constants of L to D.

(iii) ζ_I assigns to each function symbol f of rank n a mapping from D^n to elements of D.

(iv) χ_I assigns to each predicate symbol P of rank n a function $\chi_I(P)$ from D^n to $\mathcal{T}$.

Definition 5.1.2

(i) Being I an interpretation for L, then a *variable assignment* ν for L with respect to I is a map from the set of individual variable symbols of L to D.

(ii) The *denotation* $d_{I,\nu}(t)$ of a term t of L with respect to an interpretation I and variable assignment ν is defined inductively as follows:

(a) If t is a constant symbol, then $d_{I,\nu}(t) = \eta_I(t)$

(b) If t is an individual variable, then $d_{I,\nu}(t) = \nu_I(t)$

(c) If t is $f(t_1, \ldots, t_n)$, then $d_{I,\nu}(t) = \chi_I(d_{I,\nu}(t_1), \ldots, d_{I,\nu}(t_n))$

(iii) An annotated atom $P_\lambda(t_1, \ldots, t_n)$ will be denoted as $P(t_1, \ldots, t_n) : \lambda$; an expression of the form $\underbrace{\neg \ldots \neg}_{k\,times}(A : \mu)$, where A is an ordinary atom is called a *hyper-literal* of order $k, k \geq 0$, and abbreviated $\neg^k(A : \mu)$.

Definition 5.1.3 *If I and ν are as above, A is an ordinary atom and α, β and γ are any formulas whatsoever, then:*

(i) If A is an ordinary atom $P(t_1, \ldots, t_n)$, then

$$I, \nu \models (A : \mu) \text{ iff } \chi_I(P)(d_{I,\nu}(t_1), \ldots, d_{I,\nu}(t_n)) \geq \mu.$$

(ii) $I, \nu \models \underbrace{\neg \ldots \neg}_{k\,times}(A : \mu)$ iff $I, \nu \models \underbrace{\neg \ldots \neg}_{k-1\,times}(A : \neg\mu)$. Here we shall pay attention to the two senses in which $\neg$ is being used here; inside the atom, that is, in $(A : \neg\mu)$, what appears is the map from $\mathcal{T}$ to $\mathcal{T}$, while outside the atom, that is, in the first occurrence of $\neg$ in the expression $\neg(A : \neg\mu)$, it is the negation symbol that should be considered. Condition 2 is saying that, in reducing negations, we are exchanging values in the lattice according to the map $\neg$.

(iii) $I, \nu \models \alpha \wedge \beta$ iff $I, \nu \models \alpha$ and $I, \nu \models \beta$.

(iv) $I, \nu \models \alpha \vee \beta$ iff $I, \nu \models \alpha$ or $I, \nu \models \beta$.

(v) $I, \nu \models \alpha \rightarrow \beta$ iff $I, \nu \models\models \alpha$ or $I, \nu \not\models \beta$.

(vi) If F is not a hyper-literal, then $I, \nu \models \neg F$ iff $I, \nu \not\models F$.

(vii) $I, \nu \models \forall x \alpha$ iff for all variable assignment ν' which agrees with ν in what respect all variables distinct from x, that is, $\nu'(y) = \nu(y)$ for all $y \neq x$, then $I, \nu' \models \alpha$.

(viii) $I, \nu \models \exists x \alpha$ iff for some variable assignment ν' which agrees with ν in what respect all variables distinct from x, that is, $\nu'(y) = \nu(y)$ for all $y \neq x$, then $I, \nu' \models \alpha$.

(ix) $I \models \alpha$ iff for all variable assignment ν associated with I, $I, \nu \models \alpha$.

Definition 5.1.4

1. *Let $\Gamma \cup \{\alpha\}$ be as et of formulas. We write $\vDash \alpha$, and say that α is* valid *(in $\mathcal{QT}$) if, for every interpretation I, $I \vDash \alpha$. We say that α is a* semantic consequence *of Γ iff for any interpretation I such that $I \vDash \beta$ for all $\beta \in \Gamma$, it is the case that $I \vDash \alpha$.*

2. *If α, β are formulas of L, then $\alpha \leftrightarrow \beta =_{\mathsf{def}} (\alpha \to \beta) \wedge (\beta \to \alpha)$.*

3. *If α is a formula, then $\sim \alpha =_{\mathsf{def}} (\alpha \to (\alpha \to \alpha)) \wedge \neg(\alpha \to \alpha)))$; $\sim \alpha$ is called* the strong negation *of α in $\mathcal{QT}$.*

4. *A formula is called* complex *if it is not a hyper-literal.*

Now we shall describe an axiomatic system which we call $\mathcal{A}$, whose underlying language is L. In the postulates below, α, β and γ denote any formula whatsoever, φ and ψ denote complex formulas and θ is an annotated atom.

The postulates of our system are:

$(\to_1)$ $\alpha \to (\beta \to \alpha)$

$(\to_2)$ $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$

$(\to_3)$ $\alpha,\ \alpha \to \beta \,/\, \beta$

$(\to_4)$ $(\alpha \to \beta) \to \alpha) \to \alpha$

$(\wedge_1)$ $\alpha \wedge \beta \to \alpha$

$(\wedge_2)$ $\alpha \wedge \beta \to \beta$

$(\wedge_3)$ $\alpha \to (\beta \to \alpha \wedge \beta)$

$(\vee_1)$ $\alpha \to (\alpha \vee \beta)$

$(\vee_2)$ $\beta \to (\alpha \vee \beta)$

$(\vee_3)$ $(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$

$(\neg_1)$ $(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi))$

$(\neg_2)$ $\varphi \to (\neg\varphi \to \alpha$

$(\neg_3)$ $\varphi \vee \neg\varphi$

$(\exists_1)$ $\alpha(t) \to \exists x \alpha(x)$

$(\exists_2)$ $\frac{\alpha(x) \to \beta}{\exists x \alpha(x) \to \beta}$

$(\forall_1)$ $\forall x \alpha(x) \to \alpha(t)$

($\forall_2$) $\frac{\alpha\to\beta(x)}{\alpha\to\forall x\beta(x)}$

(τ_1) $(\theta :\bot) \wedge \neg^k(\theta : \mu) \leftrightarrow \neg^{k-1}(\theta : \neg\mu)$

(τ_2) $(\theta : \neg\mu) \to (\theta : \neg\lambda)$, where $\lambda \leq \mu$

(τ_3) If $\alpha \to (\theta : \mu_j)$ for every $j \in J$, then $\alpha \to (\theta : \mu)$, where $\mu = \sup\{\mu_j : j \in J\}$

The postulates ($\exists_1$)-($\forall_2$) are subjected to the usual restrictions. If $\mathcal{T}$ is a complete lattice, the supremum in rule (τ_3) is well defined. When $\mathcal{T}$ is finite, rule (τ_3) can be replaced by the scheme $(\theta : \mu_1) \wedge \ldots \wedge (\theta : \mu_n) \to (\theta : \mu)$, where $\mu = \sup\{\mu_j : 1 \leq j \leq n\}$.

We easily define the syntactic concepts related to the axioms above; in particular, the concept of syntactic consequence $\vdash$ is defined as usual, but the notion of deduction is not finitary if $\mathcal{T}$ is infinite. The propositional counterpart of $\mathsf{Q}\mathcal{T}$ can also de developed, as shown in [113].

Theorem 5.1.1 (Soundness) *Let $\Gamma \cup \{A\}$ be a set of formulas of $Q\mathcal{T}$. Then $\Gamma \vdash \alpha$ implies that $\Gamma \models \alpha$, that is, the axiomatic is sound with respect to the semantics of $Q\mathcal{T}$.*

Proof: By induction on the length of the deduction (if $\mathcal{T}$ is infinite, we use transfinite recursion).▮

Definition 5.1.5 *Suppose that Γ is a set of formulas such that the set of annotated constants occurring in Γ is finite (Γ itself may be infinite). In this case, Γ is said to have the finite annotation property.*

We remark that if $\mathcal{T}'$ is a substructure of $\mathcal{T}$, then $\mathcal{T}'$ is closed under the operations of $\mathcal{T}$.

Theorem 5.1.2 (Finitary Completeness) *Let $\Gamma \cup \{A\}$ be a set of formulas of $Q\mathcal{T}$. Then if $\mathcal{T}$ is finite or if $\Gamma \cup \{\alpha\}$ has the finite annotated property, then $\Gamma \models \alpha$ entails $\Gamma \vdash \alpha$.*

Proof: By extending the proof of the propositional fragment of $\mathsf{Q}\mathcal{T}$ presented in [113].▮

When $\mathcal{T}$ is infinite, it seems that the completeness can be obtained by adding to the axioms an extra infinitary rule (see [106] for further indications). The system $\mathsf{Q}\mathcal{T}$ is a non-classical logic which is both paraconsistent and paracomplete (see [**?**]).

5.1.1 Another axiomatization of $\mathsf{Q}\mathcal{T}$.

A different axiomatization of $\mathsf{Q}\mathcal{T}$ can be obtained by adjoining to the language of classical first order predicate calculus a symbol for the paraconsistent weak negation subjected to suitable axioms. To exemplify, let C be an axiomatic systematization of the first order predicate calculus without equality whose symbol of negation is $\sim$. The remaining primitive symbols of C are as those of $\mathsf{Q}\mathcal{T}$. We still suppose that the atomic formulas of C are annotated atoms, as above. Furthermore, our language has a primitive symbol $\neg$ for the weak negation. Let us denote by $\mathcal{A}'$ the axiomatic system obtained from C by adding to it the axioms ($\neg_1$), ($\neg_2$), ($\neg_3$), (τ_1), (τ_2) and (τ_3) plus the following rule:

[Rule] If α ad β are formulas such that β is obtained from α by replacement of a sub-formula of the form $\sim\alpha$ by $\alpha \to ((\alpha \to \alpha) \wedge \neg(\alpha \to \alpha))$ or by the replacement of a sub-formula of the latter form by one of the first, then we infer $\alpha \leftrightarrow \beta$.

Theorem 5.1.3 *The axiom system $\mathcal{A}$ and $\mathcal{A}'$ are equivalent, so both characterize $\mathsf{Q}\mathcal{T}$.*

Proof: Any postulate of $\mathcal{A}$ is a postulate of $\mathcal{A}'$ and item 3 of Definition 5.1.4 corresponds to a rule in $\mathcal{A}'$. Conversely, any postulate of $\mathcal{A}'$ is a postulate or definition of $\mathcal{A}$ or is provable in $\mathcal{A}$, as it is easy to show.▮

Let X be a non-empty set. A *normal structure based on* X is a function $f : X \times X \mapsto \mathcal{T}$. Let us denote by $\mathsf{Q}\mathcal{T}^2$ the logic $\mathsf{Q}\mathcal{T}$ obtained by suppressing all function symbols and all predicate symbols, with the exception of one binary predicate symbol which we represent by $\in$. Then $\mathsf{Q}\mathcal{T}^2$ is a dyadic predicate logic whose atoms are annotated by $\mathcal{T}$; these atoms have the form $\in_\lambda (a, b)$, where a and b are terms and $\lambda \in \mathcal{T}$. This atom shall be written $a \in_\lambda b$.

Theorem 5.1.4 *$\mathsf{Q}\mathcal{T}^2$ is sound with respect to the semantics of normal structures. If $\mathcal{T}$ is finite or if we consider only sets of formulas having the finite annotation property, then $\mathsf{Q}\mathcal{T}^2$ is also complete.*

Proof: Consequence of Theorems 5.1.1 and 5.1.2.▮

5.2 Annotated set theory

Theorem 5.1.4 above shows that normal structures are of importance to annotated logics, particularly when $\mathsf{Q}\mathcal{T}$ is developed as in the previous section. In this section we shall extend annotated logic to set theory, and we shall be concerned with normal structures.

Let ZF be a standard formulation of the Zermelo-Fraenkel set theory. The language of annotated set theory, termed AZF, is obtained from the language of ZF by adding two individual constants $\mathcal{T}$ and $\mathcal{U}$. The following axioms are also added to those of ZF:
(AZF.1) $\mathcal{T}$ is a complete lattice, where we denote by $\leq$ an arbitrary, but fixed ordering; we shall use $\perp$ and $\top$ to stand for its least and greatest elements respectively.

(AZF.2) $\mathcal{T} \subseteq \mathcal{U}$ and $\forall x(x \in \mathcal{U} \to x \subseteq \mathcal{U})$, that is, $\mathcal{U}$ is transitive.

In the most applications, it is usually enough to postulate that $\mathcal{T}$ is a set endowed with a reflexive binary relation with unique least and unique greatest elements.

Definition 5.2.1

(1) We say that $\mathcal{E}$ is a normal structure, *or a* normal function *based on $\mathcal{U}$ if it is a mapping from $\mathcal{U} \times \mathcal{U}$ into $\mathcal{T}$. We write $x \in_\lambda y$ instead of $\mathcal{E}(x, y) = \lambda$.*

(2) If $x \in \mathcal{U}$, then:

(2.1) $x^{(\lambda,\mathcal{E})} =_{\mathsf{def}} \{y : y \in \mathcal{U} \wedge y \in_\lambda x\}$

(2.2) $x^{[\lambda,\mathcal{E}]} =_{\mathsf{def}} \{y : y \in \mathcal{U} \wedge \exists\mu(\mu \in \mathcal{T} \wedge \mu \leq \lambda \wedge y \in_\mu x)\}$

(2.3) $\mathcal{F}_x =_{\mathsf{def}} \{f : f : x \mapsto \mathcal{T} \wedge \exists\mathcal{E}(\mathcal{E}$ is a normal function $\wedge \forall\lambda\forall y(\lambda \in \mathcal{T} \wedge y \in \mathcal{U}) \to (f(y) = \lambda \leftrightarrow y \in_\lambda x))\}$

(3) If $x, y \in \mathcal{U}$ and $\lambda \in \mathcal{T}$, then $x =_{\lambda,\mathcal{E}} y =_{\mathsf{def}} \forall z \in \mathcal{U}(z \in_\lambda x \leftrightarrow in_\lambda y)$

(4) A set $x \neq \emptyset$ is strongly transitive*if it is transitive and $\forall y(y \in x \to \mathcal{P}(y) \in x)$, where $\mathcal{P}(y)$ is the power-set of y.*

(5) x is called a universe *iff it is strongly transitive and for every function $f : x \mapsto x$ such that when $y \in x$, then $\bigcup ran(f) \in x$, where $ran(f) =_{\mathsf{def}} \{z : \exists t(t \in y \wedge f(t) = z\}$.*

Theorem 5.2.1 *If x is a universe, then x is a standard model of all axioms of* ZF*, with the possible exception of the axiom of infinity. If $\omega \in x$, then x is a complete universe (ω is the set of natural numbers).*

Set theoretical constructs are used in handling with normal structures based on $\mathcal{U}$. It seems that the more such constructs exist, the better. So, in most cases, it is useful to take $\mathcal{U}$ as an universe, that is, as a model of ZF if $\omega \in \mathcal{U}$. For instance, we have the following result:

Theorem 5.2.2 *If $\mathcal{U}$ is an universe, $\lambda \in \mathcal{T}$, and $x \in \mathcal{U}$, then $x^{[\lambda,\mathcal{E}]} \in \mathcal{U}$, $x^{(\lambda,\mathcal{E})} \in \mathcal{U}$, and $\mathcal{F}_x \in \mathcal{U}$. Furthermore, $\{y : F(y) \wedge y \in_\lambda x\} \in \mathcal{U}$, where $F(y)$ is any formula of* ZF*, and $\forall x \forall y(x, y \in \mathcal{U} \rightarrow (\forall \mathcal{E} \forall \lambda(\mathcal{E}$ is a normal function $\wedge\ \lambda \in \mathcal{T}) \rightarrow x =_{\mathcal{E},\lambda} y) \leftrightarrow x = y)$.*

Proof: Immediate, since $\mathcal{U}$ is a model of ZF (except by the axiom of infinity).∎

We may introduce a weak negation $\neg$ in AZF without difficulty; for instance, it may apply to hyper-literals only, and then we should postulate that

$$\neg^k(x \in_\lambda y) \leftrightarrow \neg^{k-1}(x \in_{\neg\lambda} y).$$

5.2.1 Fuzzy sets

The concept of fuzzy set can be subsumed by AZF. If $\mathcal{U}$ is a set, then a *fuzzy set* of $\mathcal{U}$ is a function $u : \mathcal{U} \mapsto [0, 1]$. Let us denote by $\mathcal{F}_\mathcal{U}$ the set of all fuzzy sets of $\mathcal{U}$. We say that two fuzzy sets $u, v \in \mathcal{F}_\mathcal{U}$ are *equal* iff for every $x \in \mathcal{U}$, $u(x) = v(x)$. Let us still use $\mathbf{1}_u$ and $\mathbf{0}_u$ to denote the fuzzy sets of $\mathcal{U}$ so that, for all $x \in \mathcal{U}$, $\mathbf{1}_u(x) = 1$ and $\mathbf{0}_u(x) = 0$. Furthermore, if $u, v \in \mathcal{F}_\mathcal{U}$ and $x \in \mathcal{U}$, we put $(u \sqcup v)(x) =_{\mathsf{def}} \sup u(x), v(x)$, $(u \sqcap v)(x) =_{\mathsf{def}} \inf u(x), v(x)$, $\bar{u} =_{\mathsf{def}} 1 - u(x)$.

Then, it is easy to prove that $\langle \mathcal{F}_\mathcal{U}, \sqcup, \sqcap \rangle$ is a complete lattice having the infinite distributive property and that $\langle \mathcal{F}_\mathcal{U}, \sqcup, \sqcap, ^- \rangle$ is an algebra which in general is not Boolean. A fuzzy set u of $\mathcal{U}$ can be identified with a normal structure $\tilde{u}$ based on the set $\mathcal{U} \cup [0, 1]$, such that $\mathcal{T} = \{\top, \bot\}$ and

$$\tilde{u} = \begin{cases} \top & \text{if } x \in \mathcal{U} \wedge y \in [0,1] \wedge y = u(x), \\ \bot & \text{otherwise} \end{cases}$$

Hence, AZF as the theory of normal structures encompasses that of fuzzy sets. It is clear that if $\mathcal{U}$ is a universe, the definition of fuzzy sets in terms of normal structures can be simplified. In the same vein, the theory of flou sets and of L-sets [191] can also be obtained by extending the concept of normal structures. Further developments in relating annotated logics and fuzzy logics can be found in [106], [2].

5.3 Applications

One of the most interesting traits of annotated logic is its wide field of application. In this section we shall provide an idea of the use of annotated logics to base some formalisms for reasoning about inconsistent knowledge bases.[13] At the section 7.2.3, we will sketch another use of annotated logics. Although the technical details cannot be given here in full, our references are in some extend updated and provide links for other applications mentioned below.

Expert systems and knowledge bases about a domain D are usually constructed by programmers who, in general, know little about D. So, in building an expert

[13]This section is partially based on [102].

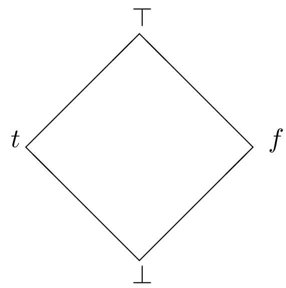

Figure 1: The Lattice FOUR

system, say in medicine, we need to consult (in general) several experts in the particular field we are interested in (say cardiology) and induce them to provide us with adequate knowledge, based on their own previous experience, for we to form our knowledge base. It is also common to accept that the information provided by the experts may be expressed in a suitable form in a certain logic language.

But the experts usually disagree. For instance, given the same observable symptoms, doctor d_1 may believe that the patient has a virus infection, doctor d_2 may conclude that it has an allergic reaction, while doctor d_3 may say that the patient either has a viral infection or an allergy, but not both. It is clear that if we had used the opinions of these three doctors in our knowledge base, we would be conduced to inconsistency. The important point is that often scientists disagree and have conflicting opinions for very good reasons, so inconsistencies like that one originated in our previous sample are to be regarded as *natural*. Then, in constructing knowledge bases, we sometimes should take into account that inconsistencies may be present, and taken seriously.

Here we shall present a general framework for logic programming over a set of truth values which has the structure of a complete lattice. We will emphasize two special cases of this framework, one which presents a four-valued logic that allows us to reason in the presence of inconsistency and another one which shows us how to reason in the presence of both inconsistency and/or uncertainty.

Let us assume that we have a fixed set $\mathcal{T}$ of truth-values which is a complete lattice under an ordering $\leq$ defined on $\mathcal{T}$. Let us denote the least upper bound and the greatest upper bound of the subsets $S \subset \mathcal{T}$ respectively by $\sqcup S$ and $\sqcap S$. For instance, consider the lattice FOUR shown in Figure 1. Here, t and f represent the classical truth values 'true' and 'false' respectively, while $\bot$ denotes 'unknown' and $\top$ means 'inconsistent' or 'over-defined'.

Intuitively, the use of a lattice like FOUR may be useful for a classically inconsistent theory T axiomatized by $\{p, \neg p, q\}$ may have a model, viz. the interpretation that assigns $\top$ to p and t to q. Due to the definition of satisfaction given below (see Definition 5.3.3), it can be shown that $\neg q$ is not a logical consequence of T, which of course is not the case in classical logic.

Another important complete lattice is SQUARE, that is, the set $SQ = [0,1] \times [0,1]$ of truth values, where $[0,1] \subset \mathbb{R}$, endowed with the ordering below. Here, the assignment of a truth value $[\mu_1, \mu_2]$ to p means that the degree of belief in p is μ_1, while the degree of disbelief in p is μ_2. The ordering is the following:

$$[\mu_1, \mu_2] \leq [\rho_1, \rho_2] \text{ iff } \mu_1 \leq_{\mathbb{R}} \rho_1 \text{ and } \mu_2 \leq_{\mathbb{R}} \rho_2,$$

where $\leq_{\mathbb{R}}$ is the ordinary 'less than or equals' defined on the reals. In SQ, $[0,0]$ intuitively denotes absolute lack of belief, $[0,1]$ denotes complete disbelief, $[1,0]$ denotes complete belief and $[1,1]$ denotes absolutely inconsistent beliefs.

Let $(A : \mu)$ be an annotated atom over $\mathcal{T}$ (see Definition 5.1.2). If α_1 and α_2 are first-order expressions (terms or atoms), then a *substitution* θ of variable symbols for terms is called a *unifier* of α_1 and α_2 iff the application of θ to α_1, denoted $\alpha_1\theta$, yields the same expression as $\alpha_2\theta$. A *most general unifier* (mgu for short) of any two syntactic expressions α_1 and α_2 is a unifier θ such that for any unifier ϑ of the expressions α_1 and α_2, there is a substitution γ such that $\theta\gamma = \vartheta$. If α_1 and α_2 are unifiable terms of atoms, then they posses a mgu (see ibid.).

Definition 5.3.1 *If $L_0, L_1, \ldots, L_n$ are annotated atoms over $\mathcal{T}$, then $L_0 \Leftarrow L_1 \wedge \ldots \wedge L_n$ is an annotated clause over $\mathcal{T}$. L_0 is called the head of the annotated clause, while $L_1 \wedge \ldots L_n$ is its body. Often we will refer to annotated clauses just as clauses.*

Definition 5.3.2 *An annotated logic program (ALP) over $\mathcal{T}$ is a finite set of annotated clauses over $\mathcal{T}$.*

5.3.1 Semantics

Let us define a semantics for ALPs. I doing that, we shall consider only those interpretations whose domain of discourse are the set of ground terms of the language (Herbrand interpretations). An interpretation I of an ALP P over $\mathcal{T}$ is a mapping $I : Bp \mapsto \mathcal{T}$, where Bp is the Herbrand base of P, that is, the set of variable free atoms expressible in the language of P. The ordering $\leq$ is extended to interpretation in a natural way, viz.,

$$I_1 \leq I_2 \text{ iff } (\forall A \in Bp)(I_1(A) \leq I_2(A)).$$

The orderings $\geq$, $<$ and $>$ are defined in the usual way. We also assume the existence of a function $\neg : \mathcal{T} \mapsto \mathcal{T}$.

Definition 5.3.3 (Satisfaction) *An interpretation I is said to satisfy*

1. the formula α iff it satisfies every closed instance of α.

2. the variable free annotated atom $(A : \mu)$ iff $I(A) \geq \mu$.

3. the variable free annotated hyper-literal $\neg(A : \mu)$ iff $I(A) \geq \neg(u)$.

4. the variable free formula $\alpha_1 \wedge \alpha_2$ iff I satisfies α_1 and α_2.

5. the variable free formula $\alpha_1 \vee \alpha_2$ iff I satisfies al_1 or α_2.

6. the variable free formula $\alpha_1 \Leftarrow \alpha_2$ iff either I satisfies al_1 or does not satisfy α_2.

7. the variable free formula $\alpha_1 \Leftrightarrow \alpha_2$ (that is, $(\alpha_1 \Leftarrow \alpha_2) \wedge (\alpha_2 \Leftarrow \alpha_1)$ iff I satisfies $\alpha_1 \Leftarrow \alpha_2$ and $\alpha_2 \Leftarrow \alpha_1$.

8. the closed formula $\exists x\alpha$ iff there is some variable free term t such that I satisfies $\alpha[x/t]$ (the result of replacing all free occurrences of x in α by t).

9. the closed formula $\forall x\alpha$ iff for every variable free term t, I satisfies $\alpha[x/t]$.

When I satisfies α, we write $I \models \alpha$, and $I\neg \models \alpha$ when it does not. In this section, we shall write $(\forall)\alpha$ and $(\exists)\alpha$ to denote $\forall x_1 \ldots \forall x_n\alpha$ and $\exists x_1 \ldots \exists x_n\alpha$ respectively, where $x_1, \ldots, x_n$ are the free variables of α.

Lemma 5.3.1 *If I is an interpretation, then*

1. $I \models \neg(A : \mu)$ iff $I \models (A : \neg\mu)$.
2. $I \models (\exists)(\neg(A : \mu)$ iff $I \models (\exists)(\alpha : \neg(\mu))$.

Theorem 5.3.1 *Suppose that P is an ALP over $\mathcal{T}$.Let P' be the ALP obtained from P by replacing all annotated literals of the form $(\neg A : \mu)$ by $(A : \neg(\mu))$. Then I is a model of P iff I is a model of P'.*

Without loss of generality, we may assume that ALPs contain no negated literals, an assumption we shall make throughout this paper. Associated with every ALP P over $\mathcal{T}$ there is a function $\mathcal{T}_p$ from the class of Herbrand interpretations to the class of Herbrand interpretations defined as follows:

$$\mathcal{T}_p(I)(A) =_{\mathsf{def}} \sqcup\{\mu : (A : \mu) = \beta_1 \wedge \ldots \wedge \beta_n \text{ is a ground instance of an annotated clause in } P \text{ and} I \models \beta_1 \wedge \ldots \wedge \beta_n\}.$$

Theorem 5.3.2 *Suppose that P is an ALP over $\mathcal{T}$ (where $\mathcal{T}$ is a complete lattice under $\leq$) and that $\mathcal{T}_p$ is as above. Then I is a model of P iff $\mathcal{T}_p(I) \leq I$.*

Theorem 5.3.3 *Suppose that P is an ALP over $\mathcal{T}$ as above. Then $\mathcal{T}_p$ is monotonic, that is, $I_1 \leq I_2$ entails $\mathcal{T}_p(I_1) \leq \mathcal{T}_p(I_2)$.*

The monotonicity of $\mathcal{T}_p$ guarantees, by the Tarski-Knaster theorem, that $\mathcal{T}_p$ has a least fixed point that coincides with the least pre-fixed point of $\mathcal{T}_p$ (here, I is a pre-fixed point of $\mathcal{T}_p$ iff $\mathcal{T}(I) \leq I$).

Theorem 5.3.4 *P has a least model that is identical the the least fixed point of $\mathcal{T}_p$.*

As $\mathcal{T}$ is a complete lattice, it possesses a least element and a greatest element which we shall denote respectively by $\perp$ and $\top$. Then, associated to every $\mathcal{T}$ there exist two distinguished interpretations, denoted by $\triangle$ and ∇ respectively, that assign the truth value $\perp$ and $\top$ respectively to every element $A \in Bp$, where Bp is the Herbrand base of P.

Definition 5.3.4 *If P is an ALP over $\mathcal{T}$, then the upward iteration of $\mathcal{T}_p$ is defined, for all ordinals λ, as*

$$\begin{cases} \mathcal{T}_p \uparrow 0 =_{\mathsf{def}} \triangle \\ \mathcal{T}_p \uparrow \lambda =_{\mathsf{def}} \sqcup_{\alpha<\lambda}\mathcal{T}_p(\mathcal{T}_p \uparrow \alpha) \end{cases}$$

Theorem 5.3.5 *$\mathcal{T}_p \uparrow \omega$ is identical to the least fixed point of $\mathcal{T}_p$.*

Definition 5.3.5 *A model I of the ALP P over $\mathcal{T}$ is supported iff $I(A) = \sqcup\{\mu : (A : \mu) \Leftarrow (\beta_1 : \mu_1) \wedge \ldots (\beta_n : \mu_n)$ is a ground instance of an annotated clause in P and $I \models \beta_1 : \mu_1) \wedge \ldots (\beta_n : \mu_n)\}$.*

Theorem 5.3.6 *I is a fixed point of $\mathcal{T}_p$ iff I is a supported model of P.*

Definition 5.3.6 *The downward iteration of $\mathcal{T}_p$ is defined as follows, where λ is any ordinal:*

$$\begin{cases} \mathcal{T}_p \downarrow 0 =_{\mathsf{def}} \nabla \\ \mathcal{T}_p \downarrow \lambda =_{\mathsf{def}} \sqcap_{\alpha<\lambda}\mathcal{T}_p(\mathcal{T}_p \uparrow \alpha) \end{cases}$$

Definition 5.3.7 *The ALP is canonical iff $\mathcal{T}_p \downarrow \omega$ is a fixed point of $\mathcal{T}_p$.*

Definition 5.3.8 *Suppose C_1 and C_2 are the annotated clauses given below:*

$$\begin{cases} (A^1 : \mu^1) \Leftarrow (B_1^1 : \rho_1^1) \wedge \ldots \wedge (B_n^1 : \rho_n^1) \\ (A^2 : \mu^2) \Leftarrow (B_1^2 : \psi_1^2) \wedge \ldots \wedge (B_m^2 : \psi_m^2), \end{cases}$$

then C_1 is semantically equivalent to C_2 iff there is a substitution Θ such that $A^1\Theta = A^2$ and

$$\{(B_1^1\Theta : \rho_1^1), \ldots, (B_n^1\Theta : \rho_n^1)\} = \{(B_1^2 : \psi_1^2), \ldots, (B_m^2 : \psi_m^2)\}.$$

Definition 5.3.9 *Suppose that C_1 and C_2 are such that their variables are renamed so that they do not share common variables. Then, an ALP P is closed iff for every pair C_1, C_2 of annotated clauses in P satisfying the previous condition, of the form*

$$\begin{cases} (A^1 : \mu^1) \Leftarrow (B_1^1 : \rho_1^1) \wedge \ldots \wedge (B_n^1 : \rho_n^1) \\ (A^2 : \mu^2) \Leftarrow (B_1^2 : \psi_1^2) \wedge \ldots \wedge (B_m^2 : \psi_m^2), \end{cases}$$

such that A^1, A^2 are unifiable via the mgu Θ and μ^1 and μ^2 are incomparable (i.e., $\mu^1 \not\leq \mu^2$ and $\mu^2 \not\leq \mu^1$), it is the case that

$$((A^1 : \sqcup\{\mu^1, \mu^2\} \Leftarrow (B_1^1 : \rho_1^1) \wedge \ldots \wedge (B_n^1 : \rho_n^1) \wedge (B_1^2 : \psi_1^2) \wedge \ldots \wedge (B_m^2 : \psi_m^2))\Theta$$

is semantically equivalent to some annotated clause in P. The closure *of an ALP P, denoted $CL(P)$, is the closed ALP obtained by repeatedly adding to P all clauses C obtained from annotated clauses C_1, C_2 whose heads are unifiable and whose heads's annotations are incomparable.*

Every ALP P can be extended to a closed ALP $CL(P)$ by adding a finite number of new annotated clauses, as it is easy to show. Moreover, we have:

Theorem 5.3.7 *Suppose P is any ALP over a complete lattice $\mathcal{T}$ of truth values. Then,*

(1) $\mathcal{T}_p = \mathcal{T}_{CL(P)}$

(2) *Hence, P and $CL(P)$ have the same models, i.e., they are logically equivalent.*

(3)*If I is a supported model of $CL(P)$, and A is a variable free atom such that $I(A) = \lambda \neq \perp$, then there is a single annotated clause in P having a ground instance of the form*

$$(A : \mu) \Leftarrow (B_1 : \psi_1) \wedge \ldots \wedge (B_n : \psi_n)$$

such that $\lambda \leq \mu$ and $I \models (B_1 : \psi_1) \wedge \ldots \wedge (B_n : \psi_n)$.

Definition 5.3.10 *P is canonical iff $\mathcal{T}_p \downarrow \omega$ is the greatest fixed point of $\mathcal{T}_p$.*

Theorem 5.3.8 *If P is canonical, then $\mathcal{T}_p \downarrow \omega$ is the greatest supported model of P.*

5.3.2 Executing queries

Some interesting applications of the above formal developments are found when we analyze the interactions between an user who ask queries to a knowledge base. In discussing cases of this type, we might assume that all ALPs are closed, which suffices for such purposes (cf. [102]). In this kind of application, the knowledge base is expressed as an ALP over a suitable chosen complete lattice $(\mathcal{T}, \leq)$ of truth values. Within such a framework, experts may use expressions such as *p is likely to be false* or even, more precisely, *p is false with a 90 % certainty* (op. cit., where examples are given).

Other applications of annotated logics will be mentioned in the next section.

6 Developments in paraconsistent logic

6.1 Some carried out developments

Ever since the sixties, when paraconsistent logic was indeed established as a logic stricto sensu,[14] several developments were made, most of then in connection with the $\mathcal{C}$-systems.[15]

Just to give an idea (although not complete) of some of these developments,[16] let us recall *some* of the most important facts. In 1969, M. Fidel proved the decidability of $\mathcal{C}_n$, $1 \leq n \leq \omega$ by algebraic methods [125] (see also [121]). Another decision method for $\mathcal{C}_n$, $1 \leq n \leq \omega$ was presented by D. Marconi in 1980 using semantic tableaux [175]. In 1987, W. Carnielli, having systematized finite many-valued logics through tableaux, approached $\mathcal{C}_1$ by such a method, also showing that $\mathcal{C}_1$ is decidable [57].

In the seventies, da Costa, Alves, Loparić and Arruda studied the semantic counterpart of the calculi $\mathcal{C}_n$, $1 \leq n \leq \omega$, later extended to the calculi $\mathcal{C}_n{}^\star$ and $\mathcal{D}_n$, $1 \leq n \leq \omega$ [20]. Da Costa and Alves' work was mentioned earlier (see section 2.7). It should be remarked that, although the subject is not discussed here, new hierarchies of calculi, constructed between $\mathcal{C}_n$ and $\mathcal{C}_{n+1}$, $n \geq 0$, were also introduced; their semantics was also studied by Alves in [9]. At the section 2.7.1, we have mentioned that Alves has proved the decidability of da Costa's propositional systems by the method of quasi-matrices; later, Loparić [165], [166] presented a two valued semantics and a decision method for $\mathcal{C}_\omega$ (see [167]).

In [21], Arruda and da Costa axiomatized some paraconsistent systems which are also relevant logics. In their systems, $\alpha \wedge (\alpha \to \beta) \to \beta$, the rules of contraction of negations, namely, $\alpha \to (\alpha \to \beta) / \alpha \to \beta$ and $(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$, and the deduction theorem do not hold. These systems (termed P and $P^\star$), so as their quantificational counterparts, are not finitely trivializable, not decidable by finite matrices, but can be extended to modal and tense logics, as shown by these authors. Routley and Loparić studied the semantic aspects of P, so as some of its 'dialectical' extensions [216]. Arruda and da Costa [19] have also developed the systems J_n, $1 \leq n \leq 5$ (at propositional and predicate levels), in which modus ponens is not valid, which were studied also by M. Bunder [54].

Based on Jaśkowski's ideas, D'Ottaviano and da Costa introduced a three valued propositional logic J_3 with two designated values, which turn out to be paraconsistent [122]. Further, D'Ottaviano presented an axiomatization of J_3 [118], [119], and made the connections between this calculus and other ones, like intuitionistic logic and Łukasiewicz's three-valued logic. The system J_3 was further studied by D'Ottaviano herself in other papers, where she introduced the concept of J_3-theories, to which several results of model theory were adapted and proved (see also [120]).

The algebrization of Jaśkowski's logic was studied by Kotas [157] (according to the axiomatics given by da Costa, Dubikajtis and Kotas himself), proving that the studied system is not decidable by finite matrices. Other studies related to Jaśkowski's logics were presented in [127], [158], [159], [160], [174] and Pinter's system for dealing with "inherent ambiguity" [201], which is a slight modification of a Jaśkowski logic.

Concerning the algebraic study of paraconsistent logic, one of the problems is that the only congruence relation in $\mathcal{C}_1$ is the identity relation, as shown by

[14] That is, with the development of (at least) first order predicate calculus. This turning point is acknowledged in general; see for instance [227].

[15] Further details on these developments and reference to the authors who have contributed to the subject can be found in [15] and [16], [100], [121].

[16] For further information, see the just mentioned references.

Mortensen [183]. Notwithstanding this fact, some approaches to algebrization of the $\mathcal{C}$-systems or to their extensions were presented in [69], [70], [101], [222], [81], [164], so as in some of Béziau's works referred to below.

The fact that the $\mathcal{C}$-systems do not enable substitutivity by equivalents is discussed by I. Urbas [229], who attempted to remedy this situation by extending these systems by the addition of new rules. Then he proved that such extensions do not conduce to systems distinct from classical logic, having arrived to the conclusion that new hierarchies should be constructed, were adequate equivalence relations were met. The same problem regarding the impossibility of defining such equivalences in da Costa's systems is discussed by Peña in [196, pp. 284ff].

Important contributions to PL were made by J. -Y. Béziau in a series of works (our references mentions some of the most recent ones). He started studying PL in the 80s, with particular interest in the semantics of $\mathcal{C}_1$, having reformulated it. He also axiomatized a sequent system for $\mathcal{C}_1$, something which were already tried to be done in the sixties by A. Raggio (see [121]), and extended $\mathcal{C}_1$ to a stronger system $\mathcal{C}_1+$ by exchanging the axioms $\alpha^o \wedge \beta^o \to (\alpha \odot \beta)^o$, where $\odot \in \{\wedge, \vee, \to\}$, by $\alpha^o \vee \beta^o \to (\alpha \odot \beta)^o$, so getting more theorems (as some De Morgan laws), and has studied a non-truth functional semantics for such a system [31]. Béziau also investigated a general theory of negation [32], and has pointed out that both classical first-order logic and the modal system S5 can be viewed as paraconsistent systems [35]. These last results lead him to the study of a new theory of opposition, where a polyhedron replaces the traditional square of opposition [37].

The extension of firs-order paraconsistent logic to set theory was already discussed in section 3, a work which was initiated by da Costa [65], and continued by Arruda [13] (see [121] for further references and historical details; for an updated work, see [108]). Higher-order PL corresponding to the $\mathcal{C}$-systems was presented in [11] and in [82], [83]. Semantics for paraconsistent systems containing descriptions and Hilbert's ϵ-symbol was discussed in [1] and in [231]. These authors have also shown how to develop a general theory of v.b.t.o.'s within the calculi $\mathcal{C}_n^=$.

The mentioned developments are only part of what have been done in the field of paraconsistent logic. A richer source of information can be found at the web, although until now there is not a general introductory book on the subject (the first introductory book on PL is Grana's [130]; see also [131, 132, 133, 134] for other of his ideas on the subject); there are so many different works related to these logics that it is practically impossible to summarize all of them in just one paper. So, let us only comments on some other lines of research.

An interesting system of paraconsistent logic was developed by Rescher and Brandom [213]. There is a formulation of dialectics by Rescher, which has some links with paraconsistent logic (see [212]; see also [211]).

6.2 A taxomomy of $\mathcal{C}$-systems

Recently, the $\mathcal{C}$-logics were studied from an alternative perspective. In 'A Taxonomy of $\mathcal{C}$-systems', Carnielli and Marcos (cf. [58])present an elaborate study of the foundations of paraconsistent logics. The authors base their investigation on showing how several classes of paraconsistent logics can be distinguished from the point of view of general abstract logics. This permits them to propose a new discriminating logical account of (various versions of) some logical principles, as the Principle of Non-Contradiction and various forms of the Pseudo-Scotus (also known as Principle of Explosion; see 3). The logics of formal inconsistency (LFIs), are then introduced as a large class of paraconsistent logics in which the concepts of consistency and inconsistency are internalizable. With this they can present a novel account of the notion of consistency, and formally distinguish thus between the notions of contradictoriness and of inconsistency. While studying the general features of some important subclasses of LFIs namely, the $\mathcal{C}$-systems and the dC-systems, they show that most paraconsistent logics in the literature can be seen as $\mathcal{C}$-systems ($\mathcal{C}$-logics), and explore their properties and shortcomings. In a continuation to the work done in [58], Carnielli, Coniglio and Marcos obtain in [59] further improvements on [58] (in a few cases even correcting the proofs presented there). While [58] founded the distinction between contradictoriness, inconsistency, consistency and non-contradictoriness, semantic and proof-theoretic aspects of the logics of formal inconsistency are stressed in [59]. The main LFIs and one of its primary subclasses, the $\mathcal{C}$-systems, are surveyed; one of the most relevant features of LFIs is their ability to encode classical logic, in the sense of being able to reproduce classical reasoning, despite constituting subsystems of classical logic. The dC-systems, a particular subclass of the C-systems, are carefully discussed; some particular cases constituting da Costa's $\mathcal{C}_n$, and Jaśkowski's D2. By adding convenient new axioms to the dC-systems they show that it is possible to introduce a large family of logics by controlling the propagation of consistency. Exploring this possibility permits the definition of thousands of new logics. The paper [59] emphasizes the semantic meaning of LFIs, discussing in detail the valuation semantics for LFIs and the possible-translations semantics. Another kind of semantics, the society semantics, is also addressed. Modal extensions of LFIs and their Kripke semantics are also treated. Those extensions are philosophically significant since they permit us to avoid a non-epistemic version of the so-called knowability paradox: The knowability paradox threatens any normal modal logic which expands KT by the alethic thesis according to which any true proposition is possibly a necessary truth. In the modal logics with classical basis, this produces the collapsing of the modal operator. On the other hand, the modal paraconsistent logics augmented with the alethic thesis do not suffer from the same. In this way, modal extensions of LFIs constitute modal logics which are on the one hand weak enough so as to be free from the danger of the knowability paradox, but on the other hand rich enough so as to allow for the encoding of all classical modal reasoning. Some first-order LFIs are also treated, and an entire subsection of [59] is dedicated to proof systems, especially to tableaux, and tableau proof systems for several LFIs are provided. Several other issues, as the difficulties of algebraizing LFIs are also studied in both papers.

Paraconsistent logic and related systems have been applied to several fields, which constitute, as already remarked, what is perhaps its main important present day characteristics. Once more we cannot make justice to all developments; it is practically impossible to someone to update the literature on the subject or even to follow all lines of research. Anyway, let us mention some of the applications we are acquainted with, at least for the reader to make an idea of the variety of possibilities offered by this kind of logics, so as to provide (at least part of) the relevant references. Some of the sections below are more developed than others, and this does not intend to man that we consider them as more important.

6.3 Other directions

Several directions, other than those which followed the lines suggested by the development of da Costa's $\mathcal{C}$-systems were also proposed to deal with inconsistencies. For instance, D. Batens studied paraconsistent systems related to dialectics [28]. In [29], he considers the case of inconsistency-adaptive logics, i.e. logics in which abnormalities are inconsistencies and presents two first-order inconsistency-adaptive logics, from the proof-theoretic and model-theoretic points of view (see also [30]).

In 1979, Lorenzo Peña introduced a new kind of PL, having some characteristics of fuzzy logics [195] (see also [196, 197], where his ideas are also referred to).

Paraconsistent logic progressed in Australia and New Zeeland in part due to the effords of R. Routley, based on its links with relevant logic. In these logics (see [12], [219, 217]), schemas and rules like $\alpha \to (\neg\alpha \to \beta)$ and $\alpha, \neg\alpha \vdash \beta$ are not valid, so that these systems can also be used to base inconsistent but not trivial theories. Marconi [174] gave a precise version of of the relation between paraconsistent logic and relevant logic). ¿From our point of view, most of the relevant systems are paraconsistent.

Another interesting field of possible applications of paraconsistent logic is dialectics. From the historical perspective, dialectics uses a source of some systems of paraconsistent logic, specially because most of dialectical tendencies mantain that there are 'real contradictions', i.e., that the extant world is contradictory. A basic antology on this subject is that of Marconi [174], which appeared 1m 1979. In this antology one may find papers by L. S. Rogosowski, R. K. Meyer, N. Rescher, L. Apostel and N. C. A. da Costa.

We mention here the researches of da Costa and R. G. Wolf on the underlying logic of dialectics, dialectics conceived according to the interpretation of McGill and Parry [179]. The resulting systems, involving propositional and predicate levels, present some formal traits analogous to the ones of the $\mathcal{C}$-logics, but they are strictly stronger than those logics (for details, see [104] and [105]).

Other contributions came from Australia with the works of Mortensen [183], Bunder [53, 54], and Priest [204, 205]; see also [206] for Priest's view on paraconsistent logics.

In [5], S. Akama reappraises D. Nelson's work on inconsistent systems from the 50ths [192, 193], by posing Nelson as a forerunner of PL, and presents an interesting study on Nelson's 'constructive' PL.[17] Other developments of PL, more related to applications, will be mentioned in the next section.

As already said, the field of paraconsistency has exploded into so many directions that the world congresses on this field are today completely justified. Paraconsistency is a world phenomena.

7 Applications

7.1 Technology

Annotated logics, initiated with Blair and Subrahmanian in the 80s and already mentioned above, have been developed and applied to other fields like robot control [188], air traffic control [189], control systems for autonomous machines, defeasible deontic reasoning [187], information systems [6] and medicine. Here, we cannot do justice to all developments which have been achieved in recent years, so we suggest to the interested reader to have a look on the papers listed in our references. Anyway, let us give at least a brief sketch of some of the recent developments in this area, at least to mention some relevant bibliography.

A programming language termed PARALOG was implemented in [8], which is a paraconsistent version of PROLOG, and was used in the construction of several computational systems form planing data, vision systems and in representing inconsistencies [7].

Further, digital circuits, inspired in annotated logics, were introduced in [3] for dealing with incompatible signals; the authors guess that this device can be useful for developing more general electric circuits, so as in logistic and decision procedures. Hardware devices are also under construction, motivated by paraconsistent ideas. The so called *para-analyser* enables scientists to handle uncertainty, inconsistencies

[17] Akama says that "da Costa did not appear to be familiar with Nelson's system" [5]. But let us remark that Nelson's [193] is mentioned in the references of da Costa's seminal work [65].

and paracompleteness [4]; several other related devices have also been constructed, leading to the first 'paraconsistent robot', Emmy [224], [225].

Interesting applications are being developed in medicine: the recognition of cancer cells, the Alzheimer illness, in disfunctions in speaking and so on. These applications are in fact new, but open the door for a very useful aspect of applied paraconsistent logics. Still in engineering, non-monotonic and defeasible forms of reasoning have been approached by paraconsistent logics, leading to the development of softwares that are being used in traffic control (trains, aircrafts, cars) [185, 186]; the hardware counterpart was presented as a chip in [190].

7.2 Informatics

The systems constructed for specific applications in engineering, mentioned in the last section, use computer devices and cannot be included in this section. However, in this section we shall make reference to other uses of paraconsistent logics in informatics.

7.2.1 Epistemic inconsistencies and the like

Paraconsistent logic has also been used in connection with the problem of providing models to 'real life reasoning' within the context of artificial intelligence. The reaching and interest of this problem leads to the general problem of 'practical reasoning', which is treated under the label of the logic of appearance. In all these themes, we find contributions by T. Pequeno, A. Buchsbaum, A. T. C. Martins and contributors (see [51], [178], [200], [60], [198]). For instance, the notion of epistemic inconsistency, referring to contradictory views about the same situation is introduced in [199]. These contradictions reflect the incompleteness (or vagueness) of our knowledge about it. The association of this phenomenon with non-monotonic reasoning is also taken into account. A logical system and the corresponding semantics, aiming at to make precise this notion and to enable reasoning on these inconsistent views, without triviality, may be found in [198].

In [49], a proof method for automation of reasoning within paraconsistent logic, namely, da Costa's calculus $\mathcal{C}_1^\star$ is presented. The method is analytical, using a specially designed tableau system. Really, the authors present two tableau systems; the first, with a small number of rules in order to be mathematically suitable for their purposes, is used to prove the soundness and the completeness of the method. The another one, which is equivalent to the former, is a system of derived rules designed to enhance computational efficiency. A prototype based on this second system was also effectively implemented.

7.2.2 Other paraconsistent fuzzy systems

In addition to the systems relating paraconsistent logic and fuzzy logic mentioned earlier, in [24], [25] and [26], Barreto and Ebecken presented some applications in artificial intelligence where paraconsistent logic is used in the the construction of paraconsistent knowledge bases, implemented in fuzzy shells. It was shown that it is possible to build paraconsistent knowledge bases at Matlab Fuzzy Logic Toolbox and that fuzzy shells are inconsistency-tolerant.

It is important to note that the 'defuzzification method', which plays an important role in such paraconsistent fuzzy systems must be chosen according with the nature of the knowledge in question. In [27] this aspect is explored and the authors describe a defuzzification method in the interpretation of a paraconsistent knowledge base. Although these works treat only partial results in the handling of inconsistency in Artificial Intelligence (AI), they advance that also problems of coherence and normalization of inconsistent and paraconsistent knowledge bases should be investigated and that semantic aspects should be considered. In [23], the relationship between paraconsistent knowledge bases and possibilistic logic is explored. Using possibilistic logic, it is possible to handle inconsistent information, considering them as paraconsistent pieces of knowledge with some given degree of paraconsistency in which any produced conclusion is involved. This solution seems to solve partially the problem of normalization and the problem of knowing if the paraconsistent propositions existing in the knowledge bases affect the obtained conclusions in deduction processes where they are used. When handling paraconsistent knowledge bases with possibilistic logic, it seems to be possible to say that to assign degrees of paraconsistency to the obtained conclusions is an adequate solution. These degrees of paraconsistency give an idea of in what extension the conclusion are affected by the used paraconsistent knowledge.

Further, in [24], the proof of consistency between possibilistic resolution and Zadeh's approximate reasoning theory is outlined. This is an interesting result for AI because it provides a theoretical base for the improvement of an efficient methodology for reasoning in the presence of inconsistency. A peculiar aspect that should be noted is that this methodology approaches a possible model of human knowledge, where qualitative analysis is needed. Some examples of applications are: medical diagnosis, juridical decisions, business decisions.

7.2.3 The matrix connection method and paraconsistency

W. Bibel's matrix connection method [44], [45] (an alternative procedure for theorem proving, other than the usual resolution technique) was also adapted and implemented to the particular case of some annotated propositional paraconsistent logics [147], [161].

Let us give here the main ideas from these last two papers to detail at least some of the connections between annotated logics and computation. We shall be working once more in a propositional language with standard connectives and other symbols. Let A be a non-empty finite ordered set of propositional symbols. To each element $a \in A$, we associate a non-empty finite lattice $\mathcal{T}_a$. The elements of $\mathcal{T}_a$ are termed *annotated constants*, and denoted by μ, ν, etc.

Definition 7.2.1 *A (*ground*)* literal *is a triple* (a, μ, p) *where* $a \in A$, $\mu \in \mathcal{T}_a$ *and* $p \in \{0, 1\}$. p *is the* polarity *of the literal. We will write either* $\sim L$ *or* $\sim L_\mu$ *for denoting the literal* $(a, \mu, 1)$*, while either* L *or* L_μ *denotes the literal* $(a, \mu, 0)$*. In the general, literals will be denoted by* K, L, M*, etc.*

Let R be an alphabet of *occurrences* or *positions*. The elements of R are denoted by r.

Definition 7.2.2 *By induction, we define the concepts of (*propositional*)* matrices *over* (A, R)*, denoted by the letters* D, E, F*, so as their* size $\sigma(F)$*, their* positions $\Omega(F) \subset R$ *and the* depth $\delta(r)$ *of* r *in the matrix* F*, for any* $r \in \Omega(F)$*:*

a) *For any literal* L *and for any* $r \in R$*, the pair* $(L, r) = L^r$ *is a matrix with* $\sigma(L^r) = 0$*,* $\Omega(L^r) = \{r\}$ *and* $\delta(L^r) = 0$.

b) *If* $F_1, \ldots F_n$*,* $n \geq 0$ *are matrices such that* $\Omega(F_i) \cap \Omega(F_j) = \emptyset$ *for* $i \neq j$ *and* $1 \leq i, j \leq n$*, then the set* $F = \{F_1, \ldots F_n\}$ *is a matrix where:*

(i) $\sigma(\emptyset) = 0$ *for* $n = 0$ *and* $\sigma(F) = 1 + \sum_{i=1}^{n} \sigma(F_i)$ *for* $n > 0$;

(ii) $\Omega(F) = \Omega(F_1) \cup \ldots \cup \Omega(F_n)$;

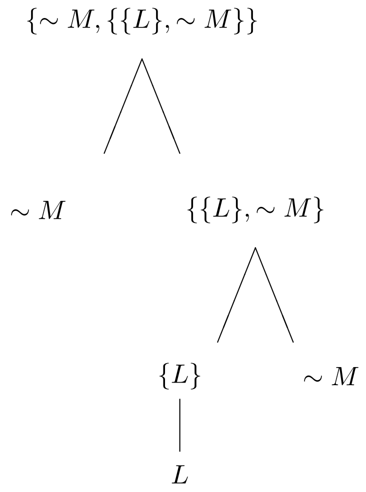

Figure 2: Representation of a matrix by a tree

(iii) $\delta(r) = m + 1$ *for any* $r \in \Omega(F_i)$, $1 \leq i \leq n$, *where* m *is the depth of* r *in* F_i.

According to this definition, the atomic parts of the matrices are ground literals, and in general a matrix is a nested set of occurrences of literals.

Example 7.2.1 *Let us consider* $A = (a, b, c, d)$ *associated with the elements of the lattice FOUR (see figure 5.3), and* $R = \{0, 1, 2, 3\}$ *be an alphabet of positions. Then* $L = (a, \perp, 0)$ *and* $\sim M = (c, f, 1)$ *are ground literals, while* $\{\{L^0\}, \{\sim M^1\}\}$ *and* $\{\sim M^0, \{\{L^1\}, \sim M^2\}\}$ *are matrices over* (A, R).

A matrix can also be viewed as a *tree*, where some leaves are associated to literals. Figure 2 present the tree corresponding to the second matrix of the above example.

Definition 7.2.3 *Let* F *be a matrix and* $l, m \in \{0, 1\}$. *The set of* formulas $\tilde{F}$ *represented by* F *with respect to* (l, m) *is inductively defined as follows:*

(i) *if* F *is a literal* $F = L^r$ *and* $l = 0$ *then* $\tilde{F} = L$;

(ii) *if* F *is a literal* $F = L^r$ *and* $l = 1$ *then* $\tilde{F} = \sim L$;

(iii) *if* $F = \{F_1, \ldots, F_n\}$, $n \geq 0$ *and if* $m = 1$, *then* $\tilde{F} = \wedge(\tilde{F_1}, \ldots, \tilde{F_n})$, *where the* $\tilde{F_i}$ *are formulas represented by* F_i *with respect to* $(l, 0)$, $i = 1, \ldots n$;

(iv) *if* $F = \{F_1, \ldots, F_n\}$, $n \geq 0$ *and if* $m = 0$ *then* $\tilde{F} = \vee(\tilde{F_1}, \ldots, \tilde{F_n})$, *where* $\tilde{F_i}$ *are formulas represented by* F_i *with respect to* $(l, 1)$, $i = 1, \ldots n$.

Definition 7.2.4 *A formula* $\tilde{F}$ *is* positively represented *by a matrix* F *if it is represented by* F *with respect to* $l = m = 0$; $\tilde{F}$ *is* negatively represented *if it is represented by* F *with respect to* $l = m = 1$. *A* propositional formula *is any formula represented by some matrix. Formulas are also denoted by* D, E, F, *but this causes no confusion with the above notation.*

We use the following notation:

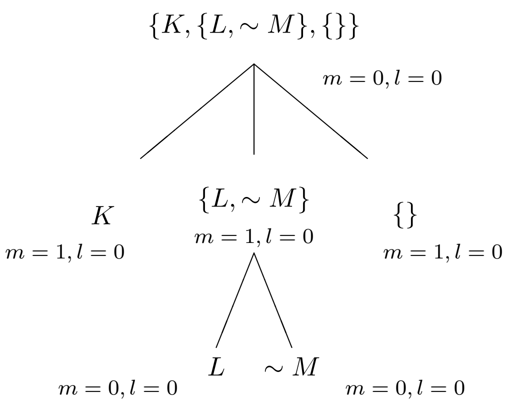

Figure 3: Positive representation of $F = \{K, \{L, \sim M\}, \{\}\}$

1. If $n = 0$, $\wedge(F_1, \ldots, F_n)$ is abbreviated by $\mathbf{T}$, while $\vee(F_1, \ldots, F_n)$ is abbreviated by $\mathbf{F}$;

2. If $n = 1$, $\wedge(F)$ and $\vee(F)$ are both abbreviated by F;

3. If $n \geq 2$, then $\wedge(F_1, \ldots, F_n)$ is a *conjunction*, while $\vee(F_1, \ldots, F_n)$ is a *disjunction*;

4. For any literal L, the formula $\neg^k L$ is called a *hyper-literal*; if L is a hyper-literal, then $\neg^k L = (a, \neg^k(\mu), p)$, where $\neg : \mathcal{T}_a \longrightarrow \mathcal{T}_a$ denotes some fixed function (that gives the meaning to the negation), and k is a multiplicity factor (a natural number);

5. If $F = \wedge(F_1, \ldots, F_n)$ and $n \geq 0$, then $\sim F = \vee(\sim F_1, \ldots, \sim F_n)$;

6. If $F = \vee(F_1, \ldots, F_n)$ and $n \geq 0$, then $\sim F = \wedge(\sim F_1, \ldots, \sim F_n)$.

7. For any formula F, $\sim\sim F = F$;

8. Any formula $\sim F \vee G$ is abbreviated by $F \rightarrow G$;

9. Any formula $(F \rightarrow G) \wedge (G \rightarrow F)$ may be written as $F \leftrightarrow G$;

10. We also take the convention that the order of precedence decreases in the sequence $\sim, \wedge, \vee, \rightarrow, \leftrightarrow$. Any parentheses, which are redundant on the basis of this convention, may be deleted.

We remark that if F and G are formulas, then $\neg(F \rightarrow G)$ and $\neg \sim F$ for example, are not formulas.

According to the above conventions, every well-formed formula (defined in the standard way) determines an unique matrix; notwithstanding, a matrix may represent more than one formula.

Example 7.2.2 *Let $F = \{K, \{L, \sim M\}, \{\}\}$ be a matrix. The tree in figure 3 is the positive representation of F.*

Example 7.2.3 *The formulas $K \wedge L \rightarrow M$, $K \rightarrow\sim L \vee M$, $L \wedge K \rightarrow M$, are all represented by $\{\sim K^1 , \sim L^2 , M^3\}$.*

The results presented in [44], chapter 2, remain applicable here, such as the following:

- If a formula $\tilde{F}$ is positively represented by a matrix F, then $\sim \tilde{F}$ is negatively represented by F;
- If two formulas $\tilde{F}_1$ and $\tilde{F}_2$ are positively represented by the same matrix F, then $\tilde{F}_1$ and $\tilde{F}_2$ are logically equivalent in the sense of annotated logics.

These results justify the use of matrices instead of formulas. The following example justifies the name *matrix* employed firstly by Bibel [44] and used also here (see also [147]).

Example 7.2.4 *Let* $\tilde{F}$ *be the formula*

$$(K\wedge \sim L \rightarrow\sim N) \wedge M \wedge \neg L \rightarrow (\sim N\wedge \sim K)$$

, where K*,* L*,* M *and* N *are literals; if we put* $\tilde{F}$ *in the disjunctive normal form (as usual), we have:*

$$(K\wedge \sim L \wedge N)\vee \sim M\vee \sim \neg L \vee (\sim N\wedge \sim K)$$

This formula may be represented by a bi-dimensional arrangement, where the literals, placed in a fixed column, are connected by "$\wedge$*", while the columns are connected by "*$\vee$*", as follows:*

$$F = \begin{bmatrix} K & & & \sim N \\ \sim L & \sim M & \sim \neg L & \\ N & & & \sim K \end{bmatrix}$$

Concerning semantics, let us give a definition:

Definition 7.2.5 *An* interpretation $\mathcal{M}$ *is a function that associates an element of the chosen lattice to every propositional symbol. By denoting* $\mathcal{M}(a) = \mu_a$*, we may write* $(a, b, c, \ldots) \mapsto (\mu_a, \mu_b, \mu_c \ldots)$*.*

Then we can define the 'truth value' $\mathcal{M}(F)$ of a matrix F as follows:

- if F is a literal $(a, \mu, 0)$, then $\mathcal{M}(F) = \mathbf{T} = \{\emptyset\}$ iff $\mathcal{M}(a) \geq \mu$, otherwise $\mathcal{M}(F) = \mathbf{F} = \emptyset$.
- if F is a literal $(a, \mu, 1)$, then $\mathcal{M}(F) = \mathbf{F} = \emptyset$ iff $\mathcal{M}(a) \geq \mu$, otherwise $\mathcal{M}(F) = \mathbf{T} = \{\emptyset\}$.
- if F is a matrix $F = \{F_1, \ldots, F_n\}$, $n \geq 0$, then $\mathcal{M}(F) = \bigcup_{k=1}^{n} \mathcal{M}(F_k)$ when $m = 0$ and $\mathcal{M}(F) = \bigcap_{k=1}^{n} \mathcal{M}(F_k)$ when $m = 1$.

We write $\mathcal{M}$ `sat` $\tilde{F}$ (and also $\mathcal{M}$ `sat` F) iff $\mathcal{M}(F) = \mathbf{T}$ for a matrix F which represents $\tilde{F}$.

Definition 7.2.6 *A matrix* F *is* valid *iff* $\mathcal{M}(F) = \mathbf{T}$ *for every interpretation. It is called* contradictory *iff* $\mathcal{M}(F) = \mathbf{F}$ *for every interpretation.*

We remark that F is valid iff $\sim F$ is contradictory.

Other semantical concepts can be introduced and the corresponding results follow, but we will not present all the details here.

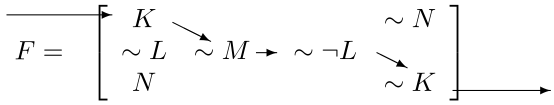

Figure 4: The path $\{K, \sim M, \sim \neg L, \sim K\}$ through the matrix F

Paths, Connections, and Validity

Definition 7.2.7 *A* path *through a matrix F is a set of occurrences of literals, defined as follows:*

1. *if $F = \emptyset$, then the only path through F is $\emptyset$;*
2. *if $F = L^r$, then the only path through F is the set $\{L^r\}$;*
3. *if $F = \{F_1, \ldots, F_m, F_{m+1}, \ldots, F_{m+n}\}$, $m, n \geq 0$, $m + n \geq 1$ for m literals $F_1, \ldots, F_m$ and for n matrices which are not literals $F_{m+1}, \ldots, F_{m+n}$, then for any matrix $E_i \in F_{m+i}$ and for any path p_i through $E_i, 1 \leq i \leq n$, the set $\bigcup_{j=1}^{m} \{F_j\} \cup \bigcup_{i=1}^{n} p_i$ is a path through F.*

Example 7.2.5 *Let F be the matrix of the Example 7.2.4; a path through F is a path in the matrix from left to right, constrained to pass by the literals (to be interpreted as 'gates') as shown in the Figure 4.*

Definition 7.2.8 *The two literals $L = (a, \mu, p)$ and $M = (a, \nu, q)$ are* complementary *iff:*

- *$p = 0$, $q = 1$ and $\nu \geq \mu$, or*
- *$p = 1$, $q = 0$ and $\mu \geq \nu$.*

Definition 7.2.9 Connections *are paths which have complementary literals as elements.*

Example 7.2.6 *Let $\mathcal{T}_a$ be the lattice FOUR, $\sim K = (a, t, 1)$, $\sim L = (a, f, 1)$, $M = (a, \top, 0)$, and the matrix $F = \{\sim K, \sim L, M\}$. The (singleton) path through F has not complementary literals. Really, neither $\sim K$ nor $\sim L$ are complementary to M. Alternatively, it suffices to see that $\top \not\leq t$ and $\top \not\leq f$ and, hence, M cannot connect by neither $\sim K$ nor $\sim L$.*

Definition 7.2.10 *We say that an ordered n-tuple $(\mu_1, \ldots, \mu_n), n > 1$, of non-comparable elements of $\mathcal{T}_a$ is a* decomposition *of μ if $\mu = \sqcup\{\mu_1, \ldots, \mu_n\}$ and there are no non-comparable elements $\{\mu'_1, \ldots, \mu'_n\}$ of $\mathcal{T}_a$ such that $\mu'_i < \mu_i$ and $\mu = \sqcup\{\mu'_1, \ldots, \mu'_n\}$. We remark that this definition is efficient for small discrete lattices.*

Example 7.2.7 *Consider the complete lattice FOUR. Then the only decomposition of the element $\top$ are the elements $\{t, f\}$ and hence, the matrix of the last example now is $F = \{\sim K, \sim L, \{M_1, M_2\}\}$, where $M_1 = (a, t, 0)$ and $M_2 = (a, f, 0)$. It is to be noted that K is complementary to M_1 and that L is complementary to M_2. Alternatively, note that $t \leq t$ and $f \leq f$.*

Theorem 7.2.1 (Soundness and Completeness) *: A matrix F is valid iff every path through F has a connection.*

Proof: : Adapted from [44, pp. 30-31], by using induction over the size of σ in the matrix F. ∎

Checking the validity of a formula The theorem proving technique, in our setting, consists in developing the following items:

a) Firstly we construct a matrix containing all the premises (or knowledge base) and a goal.

b) Then we check this matrix for the validity.

Let us suppose that we have a set of formulas $\Gamma = \{F_1, \ldots, F_n\}$ and a query G. Then, in order to investigate if G is a semantical consequence of Γ, we should verify that the matrix provided by $(\bigwedge_{i=1}^{n} F_i) \rightarrow G$ is valid.

The paraconsistent case appears in this procedure due to our definition of complementary literals. In this case the existence of both a literal $L = (a, \mu, p)$ and its 'negation' $\neg L = (a, \neg(\mu), p)$, is not a sufficient condition to assure the existence of complementary literals in the path (see the definition 7.2.8). This exemplifies that the underlying ideas of the general "paraconsistence program" [73] is included here. We notice that in order to obtain a proof of the query, it is necessary that all paths of the matrix $\sim (\bigwedge_{i=1}^{n} F_i) \ \vee G$ do have connections, which imply the existence of complementary literals in every path.

A case study Consider the construction of a simple medical system, aimed at diagnosing three diseases K, L and M. Let us suppose that there are two different symptoms, denoted by N and O. The intended usage of this system may be briefly described as follows:

(i) The core part of the system is the knowledge provided by a doctor (DOC_1).

(ii) When we intend to apply this knowledge to a specific patient, say Paul, then the pathologists, X-ray technicians and other professionals who conduct medical tests on Paul add the results of these tests to the knowledge base.

(iii) In order to use the system, we submit a goal to the program in a similar way as it is done in PROLOG.

In the common procedures, such a system would work by keeping the main knowledge base described above in one file, while each patient's records are maintained in a separate file (or possibly as a record in a given file). Then the main knowledge base and this file (or the record as the case may be) are merged together to form a current knowledge base to be used in the program built to diagnose the patient's disease.

We assume that our system is written in the form of a finite set of annotated formulas over FOUR. Suppose now that DOC_1 provided us the following five rules (formulas).

(F_1) $K_t \rightarrow L_f$
(F_2) $L_t \rightarrow K_f$
(F_3) $K_t \rightarrow M_t$
(F_4) $N_t \rightarrow K_t$
(F_5) $O_t \rightarrow L_t$

Intuitively, the doctor is telling that:

- An individual cannot have both diseases K and L (F_1 and F_2).
- If an individual has the disease K, then he has the disease M. (F_3)
- If an individual has the symptom N, then he has the disease K. (F_4)
- If an individual has the symptom O, then he has the disease L. (F_5)

$$\begin{bmatrix} K_t & L_t & K_t & N_t & O_t & \sim N_t & K_t \\ \sim L_f & \sim K_f & \sim M_t & \sim K_t & \sim L_t & & L_f \end{bmatrix}$$

Figure 5: The matrix of the formula $F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5 \wedge N_t \rightarrow (K_t \wedge L_f)$

In order to exemplify the use and behavior of the program, we describe three situations. The first one is similar to a query to a PROLOG program, while the other two explore the capacity of of our method in handling inconsistencies:

Case 1: Suppose that the pathologist tell us that Paul was tested positively for the symptom N and that we want to know, for example, if Paul has the disease K but not L.

To answer this query we must verify if the matrix of the formula

$$F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5 \wedge N_t \rightarrow (K_t \wedge L_f)$$

is valid.

The formula above is transformed firstly into its disjunctive normal form and then into its matrix form. The result of this process is shown in figure 5. Now we must check if all paths in this matrix are complementary. For example, the first path $\{K_t, L_t, K_t, N_t, O_t, \sim N_t, K_t\}$ has as complementary literals N_t and $\sim N_t$. It is easy to see that all paths in this matrix have complementary literals. Since the matrix is valid, we can conclude that Paul has the disease K but not L.

Case 2: Let us suppose now that the pathologist determined that Paul was tested positively for symptoms N and O, and that we want to know if Paul has both diseases K and L.

To verify this query, we must check if the matrix of the formula

$$F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5 \wedge N_t \wedge O_t \rightarrow (K_t \wedge L_t)$$

is valid.

Following the same procedure employed in the first case, it is easy to see that the matrix for this formula is valid.

Notice that we concluded that Paul has both diseases K and L, in contradiction to what DOC_1 said in the formulas F_1 and F_2. The reason for our conclusion was that the information given by the pathologist is in contradiction to that one provided by the doctor. In a classical logic setting, the above contradiction would made the knowledge base trivial (all formulas could be derived from the knowledge base). In our case, the inconsistency is limited to the literals K and L; any atomic formula containing these literals can be proved.

Case 3: The previous example shows an inconsistent knowledge base, used to derive an inconsistent formula. However, the system can handle inconsistencies in a non trivial way. Let us exemplify this condition by using the same set of symptoms as in case 2 to see if Paul *has not* the disease M. This condition can be written as M_f, so that the new situation is described by the following formula:

$$F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5 \wedge N_t \wedge O_t \rightarrow M_f$$

Following the evaluation procedure once more, we verify that the matrix for this formula is *not valid*.

The last two queries show that our method can deal with inconsistences in the knowledge base without every formula becoming derivable.

Implementation Issues In [161], an implementation using the Standard ML Language [181] for our method was described, but this process are not included here for reasons of space.

The ideas presented here can also be used, for instance, in a reasoning given by an intelligent agent that admits contradictory information. We believe that a system with this characteristic would be more robust than traditional ones, since it would keep the inconsistency bound to a subset of its formulas, without affecting other parts of the knowledge base.

7.2.4 Inductive paraconsistent logic

Of course science does not use only deductions. So, a natural question is whether we can build 'non-classical inductive logics' on a pair with non-classical deductive logics, in particular aiming at developing systems in Artificial Intelligence (AI). John Pollock, who has been investigating several aspects of defeasible forms of reasoning also in this context, says that

> "A common misconception about reasoning is that reasoning is deducing, and in good reasoning the conclusions follow logically from the premisses. It is now generally recognized both in philosophy and in AI that non-deductive reasoning is at least as common as deductive reasoning, and a reasonable epistemology must accommodate both." [202]

The literature on non deductive ways of reasoning presents various systems of computational tools devoted to non-monotonic and defeasible reasoning ([198], [202], to mention two of them). A most difficult task for AI experts is to further improve the AI systems to 'reason' and 'make inferences' also from vague propositions, as we humans usually do, to which we in general cannot attribute with certainty one of the two truth-values *true* or *false*. Frequently, our (human) reasoning is performed by attributing only some 'degree of confidence' as either the involved propositions are true or not. This is what may happen, for instance, when we are visiting a foreign country and someone gives us a (vague) information about the location of a certain place (perhaps because the native is also not sure about the right geography of the city), and we 'believe' in the information with a certain degree of confidence and 'decide' the way to be taken.

Here, we shall present a sketch of a way of dealing with such 'degrees of confidence' making use of paraconsistent logics. The result is that a different process involving vagueness is achieved, which we believe should also be seriously considered by all AI researchers. Our main motivation is of course to handle vague information mechanically, but in this section we shall limit ourselves to the description of a vague inductive logic we hope can be useful in the mechanical treatment of inductive information. Further developments should provide a way of elaborating, say, expert systems based on our scheme. The main ideas are from [96].

In talking of 'inductive reasoning', we of course need to say in what sense we are using the word 'induction'. Here, we take as 'inductive' whatever reasoning so that the truthfulness of the premises does not entail necessarily that the conclusion is true, but that it should be regarded as 'plausible'. The way of measuring this 'plausibility' can be discussed, but we shall not do it here (but see [77]). Anyway, the question, to put it in short, is: is there an 'inductive' paraconsistent logic? The answer was put in the positive form firstly in [168], and them also in [96] from different perspectives. In this last work, the developed 'inductive annotated system' expresses that a possible expert system elaborated for dealing with vagueness and with degrees of confidence, when faced with situations like the just mentioned one, should opt for the more prudent situation, provided by a 'warning rule' which tells us to be cautious in attributing degrees of confidence. As shown in the paper, this is in accordance with a rational stance, for we may say that 'rationality' means also the tentative of optimizing our rational degrees of confidence in the propositions we are concerned with, but with the caution of not taking conclusions with degrees of confidence greater than those attributed to the premisses.

Our point is to extend the common ways of using arguments by accepting that a proposition may have a certain 'degree of vagueness'. For instance, 'Peter is smart' is a vague proposition. Furthermore, we still aim at to attribute a degree of confidence in the truthfulness of these propositions, as suggested in the previous sections. In other words, we intend to suppose that we believe that Peter is smart with some degree of confidence. Peter's mother has a great confidence in such a proposition, but his teacher may be not so confident. This degree of confidence can be interpreted as an amount of confidence someone accepts in relation to a proposition.

In order to deal with these two concepts related to propositions, namely its vagueness and its degree of confidence, we make use of annotated logics.

Let us call $\mathcal{I}_\tau$ a propositional logic whose language has the following categories of primitive symbols: a countable set of propositional letters, which stay for propositions (we use $P, Q, \ldots$ as syntactical variables for propositions); the elements $\mu, \ldots, \mu_1, \ldots$ of a complete lattice τ ordered by $\leq$, termed *the values of vagueness* and the usual logical connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$), as well as auxiliary symbols (parentheses).

The concept of *formula* of $\mathcal{I}_\tau$ is introduced in the following way:

(i) If P is a propositional letter and $\mu \in \tau$, then $P : \mu$ is a formula of $\mathcal{I}_\tau$ (atomic formula).

(ii) If α and β are formulas, then $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$ are formulas.[18]

(iii) Every formula is obtained from just one of the two above clauses.

Furthermore, we employ a standard way of eliminating parentheses, and Greek capital letters for denoting collections of formulas. Intuitively speaking, $P : \mu$ means that P is true with degree of vagueness μ. Let us remark that we are attaching degrees of vagueness to atomic formulas only, and not to formulas in general; so, expressions like

$$((P : \mu_1) \vee (Q : \mu_2)) : \mu \tag{12}$$

are not well formed in our system.

Definition 7.2.11

(i) *If P is a propositional letter and $\mu \in \tau$, then:*

(ii) $\neg^0 P : \mu$ *means* $P : \mu$

(iii) $\neg^1 P : \mu$ *means* $\neg(P : \mu)$

(iv) $\neg^k P : \mu$ *means* $\neg(\neg^{k-1}(P : \mu))$, *with k a natural number, $k \neq 0$.*

(v) *Let* $\sim: \tau \longrightarrow \tau$ *be a fixed mapping.*[19] *We shall write* $\sim \mu$ *instead of* $\sim (\mu)$ *from now on. If $\mu \in \tau$, then:*

[18] $\alpha \leftrightarrow \beta$ is introduced in the standard way.

[19] The specific definition of this mapping depends on the particular application. For instance, by taking τ to be the unit interval $[0, 1] \subseteq \Re$ and $\sim (x) =_{\mathsf{def}} 1 - x$, the introduction of 'fuzzy' ways of reasoning can be performed within the scope of annotated logics (see [113]).

(a) $\sim^0 \mu$ *means* μ

(b) $\sim^1 \mu$ *means* $\sim \mu$

(c) $\sim^k \mu$ *means* $\sim (\sim^{k-1} \mu)$, *for* $k \neq 0$ *being a natural number.*

Expressions like $P : \mu$ are called *annotated atoms*, while $\neg^k(\alpha : \mu)$ are *hyper-literals* of order k ($k \geq 0$); the other formulas are called *complex*.

Semantics Let τ be the complete lattice above with least element $\perp$ and greatest element $\top$; let $h : \mathcal{P} \longrightarrow \tau$ be a mapping, called an *interpretation* of $\mathcal{I}_\tau$, where $\mathcal{P}$ is the collection of propositional letters of $\mathcal{I}_\tau$. The image of the proposition P by the mapping h shall be denoted $P : \mu$, where $\mu \in \tau$. Informally speaking, as we have said, $P : \mu$ means that P is true with degree of vagueness μ. To each interpretation h we associate a *valuation* $v_h : \mathcal{F} \longrightarrow \{0, 1\}$, where $\mathcal{F}$ is the above defined collection of formulas of $\mathcal{I}_\tau$. Intuitively speaking, 1 and 0 stand for 'true' and 'false' respectively.

Particular applications may demand appropriate choices of the complete lattice, as the papers in our References show. Here, to cope with the above mentioned case studies, we shall be concerned with a particular finite linearly ordered set $\tau = \{\mu_1, \ldots, \mu_4\}$ (with $\mu_1 \leq \cdots \leq \mu_4$) for expressing the distinct degrees of vagueness of a proposition, but of course the scheme here presented is quite general.

Definition 7.2.12 *If h and v_h are as above and P is a propositional letter and α and β denote formulas, then:*

(i) $v_h(P : \mu) = 1$ *iff* $\mu \leq h(P)$.

(ii) $v_h(\neg^k(P : \mu)) = v_h(\neg^{k-1}(P :\sim \mu))$, *where* $k \neq 0$.

(iii) $v_h(\alpha \wedge \beta) = 1$ *iff* $v_h(\alpha) = v_h(\beta) = 1$.

(iv) $v_h(\alpha \vee \beta) = 1$ *iff* $v_h(\alpha) = 1$ *or* $v_h(\beta) = 1$.

(v) $v_h(\alpha \rightarrow \beta) = 1$ *iff either* $v_h(\alpha) = 0$ *or* $v_h(\beta) = 1$.

(vi) *if α is a complex formula, then* $v_h(\neg\alpha) = 1$ *iff* $v_h(\alpha) = 0$.

If $v_h(\alpha) = 1$, we say that v_h *satisfies* α, and that it does not satisfy α otherwise (that is, when $v_h(\alpha) = 0$). If Γ is a set of formulas, then we say that a formula α is a *semantic consequence* of (the formulas of) Γ, and write $\Gamma \models \alpha$, iff for every valuation v_h such that $v_h(\beta) = 1$ for each $\beta \in \Gamma$, then $v_h(\alpha) = 1$. A formula α is *valid* iff $\emptyset \models \alpha$, and in this case we write $\models \alpha$.

As usual, we say that a valuation v_h is a *model* for a set Γ of formulas iff $v_h(\beta) = 1$ for every $\beta \in \Gamma$. In particular, v_h is a model of α iff $v_h(\alpha) = 1$. The other concepts like maximal non-trivial sets of formulas and so on are defined like the standard ones.

The postulates of $\mathcal{I}_\tau$ If α, β and γ are formulas and P is a propositional letter, then the postulates (axioms plus inference rules) of $\mathcal{I}_\tau$ are the following (adapted from [113], [106]):

(I1) All the postulates of classical positive logic.

(I2) If α and β are complex formulas, then the following is an axiom: $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha)$.

(I3) If α is complex, then $\alpha \vee \neg\alpha$ is an axiom.

(I4) If α is complex and β is a formula whatsoever, then $\alpha \rightarrow (\neg\alpha \rightarrow \beta)$ is an axiom.

Then, classical logic holds for complex formulas. The presence of inconsistencies will be allowed at the level of atomic formulas only [113].

(I5) $P :\perp$ is an axiom. The technical motive for using this axiom is that $v_h(P :\perp) = 1$ iff $h(\alpha) \geq 0$, which is always true.

(I6) If $\lambda \leq \mu$, then $P : \mu \to P : \lambda$

(I7) $\neg^k(P : \mu) \leftrightarrow \neg^{k-1}(P :\sim\mu)$, if $k \neq 0$.

(I8) If α is a formula whatsoever, then if $\alpha \to (P : \mu_i)$, $i \in I$, then $\alpha \to (P : \bigsqcup_{i\in I} \mu_i)$. If τ is a finite lattice, then this axiom may be replaced by the following one (cf. [113]):

$$P : \mu_1 \wedge \ldots \wedge P : \mu_n \to P : \bigsqcup_{i=1}^{n} \mu_i \tag{13}$$

The syntactical concepts of $\mathcal{I}_\tau$ are introduced in the standard way, so as in particular, the symbol of deduction $\vdash$ (see [113]).

We can prove the soundness and completeness of the logic $\mathcal{I}_\tau$ with respect to the semantic described in the previous section, as we shall sketch below. Let us first introduce a definition:

Definition 7.2.13 (Strong Negation)

$$\neg^*\alpha =_{\mathsf{def}} \alpha \to ((\alpha \to \alpha) \wedge \neg(\alpha \to \alpha)) \tag{14}$$

It is easy to prove that $\neg^*$ has all the properties of classical negation, hence the classical laws hold when $\neg^*$ is used instead our $\neg$ in the formulas of our system. For instance, the reductio ad absurdum $(\alpha \to \beta) \to ((\alpha \to \neg^*\beta) \to \neg^*\alpha)$ is a theorem of $\mathcal{I}_\tau$, so as is the excluded middle law $\alpha \vee \neg^*\alpha$. Despite we can show that if α is a complex formula, then $\neg\alpha \leftrightarrow \neg^*\alpha$ is valid (see the theorem below), this does not hold for formulas in general; for instance, if Q is a hyper-literal, then in general $\neg Q \leftrightarrow \neg^*Q$ is not valid.[20]

Other results are the following:

Theorem 7.2.2

(i) *If* $\Gamma, \alpha \vdash \beta$*, then* $\Gamma \vdash \alpha \to \beta$ *(the Deduction Theorem).*

(ii) *If* $\Gamma \vdash \alpha$ *and* $\Gamma \vdash \alpha \to \beta$*, then* $\Gamma \vdash \beta$

(iii) $\alpha \wedge \beta \vdash \alpha$, $\alpha \wedge \beta \vdash \beta$, $\alpha, \beta \vdash \alpha \wedge \beta$

(iv) $\alpha \vdash \alpha \vee \beta$, $\beta \vdash \alpha \vee \beta$

(v) $\Gamma, \alpha \vdash \gamma$ *and* $\Gamma, \beta \vdash \gamma$*, then* $\Gamma, \alpha \vee \beta \vdash \gamma$ *(Proof by Cases)*

(vi) $\Gamma, \alpha \vdash \beta$ *and* $\Gamma, \alpha \vdash \neg^*\beta$*, then* $\Gamma \vdash \neg^*\alpha$ *(Reductio ad Absurdum)*

(vii) $\alpha, \neg^*\alpha \vdash \beta$, $\neg^*\neg^*\alpha \vdash \alpha$, $\alpha \vdash \neg^*\neg^*\alpha$

(viii) *If* α *is complex, then* $\neg^*\alpha \leftrightarrow \neg\alpha$

(ix) $(\alpha : \mu_i)_{i\in I} \vdash \alpha : \bigsqcup_{i\in I} \mu_i$

(x) *If* $\Gamma \vdash \alpha$*, then* $\Gamma \models \alpha$ *(Soundness Theorem).*

In order to prove the completeness theorem, we need a few definitions and results.

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[20] See [113].

Definition 7.2.14

(i) $\overline{\Gamma} =_{\mathsf{def}} \{\alpha : \Gamma \vdash \alpha\}$

(ii) Γ *is trivial iff* $\overline{\Gamma} = \mathcal{F}$, *where* $\mathcal{F}$ *is the set of formulas of* $\mathcal{I}_\tau$; *otherwise,* Γ *is non-trivial.*

(iii) Γ *is inconsistent iff there exists* α *such that both* α *and* $\neg\alpha$ *belong to* $\overline{\Gamma}$. *Otherwise,* Γ *is consistent.*

(iv) Γ *is strongly inconsistent iff there exists* α *such that both* α *and* $\neg^*\alpha$ *belong to* $\overline{\Gamma}$. *Otherwise,* Γ *is strongly consistent.*

It is easy to see that Γ is strongly inconsistent iff it is trivial and that Γ is strongly consistent iff it is non-trivial. Furthermore, by an adequate choice of τ, we may prove that there exist inconsistent but non-trivial sets of formulas, which are still not strongly inconsistent [102]. So, the logic $\mathcal{I}_\tau$ is a paraconsistent logic. This means that there exist interpretations h and formulas α such that $v_h(\alpha) = v_h(\neg\alpha) = 1$. But we may also prove that for certain τ, there are formulas α and interpretations h such that $v_h(\alpha) = v_h(\neg\alpha) = 0$. So, $\mathcal{I}_\tau$ is also a paracomplete logic. All these results are treated in details in the papers listed in the References.

Lemma 7.2.1 *Every non-trivial set of formulas is a subset of some maximal non-trivial set of formulas.*

Proof: See [113], [102].▮

The completeness theorem results from the following Lemma:

Lemma 7.2.2 *If* Γ *is a maximal non-trivial set of formulas, then its characteristic function* $\chi_\Gamma : \mathcal{F} \longrightarrow \{0,1\}$ *is a model of* Γ, *that is, such a mapping is a valuation such that* $\chi_\Gamma(\beta) = 1$ *for every* $\beta \in \Gamma$.

Proof: The trick is to define a valuation v_h, for a given interpretation h, in such a way so that the rules of $\mathcal{I}_\tau$ are 'preserved'. This means that, given Γ, we may define $h : \mathcal{P} \longrightarrow \tau$ such that for every proposition $P \in \mathcal{P}$,

$$h(P) =_{\mathsf{def}} \bigsqcup_i \{\mu_i : \mu_i \in \Gamma\} \tag{15}$$

It is now a not difficult to prove that the valuation generated by such an interpretation coincides with the characteristic function χ_Γ.▮

As a consequence, we have the completeness theorem:

Theorem 7.2.3 *If* $\Gamma \models \alpha$, *then* $\Gamma \vdash \alpha$.

Proof: See [102], [113].▮

Degrees of confidence Now, we shall sketch a *theory of confidence*, which enables us to attribute degrees of confidence to propositions, even to vague ones.

Our degrees of confidence are in general only qualitative, characterized by the elements of an appropriate lattice with least and greatest elements. Abstractly speaking, degrees of confidence are elements of a lattice σ, and are attributed to the formulas of the language $\mathcal{I}_\tau$, when the propositional variables are interpreted as denoting specific vague statements as described above. In order to do so, let σ be a lattice with least and greatest elements denoted respectively by $\perp\prime$ and $\top\prime$.

The algebraic lattice operations are represented by $\sqcap$ and $\sqcup$, and the corresponding partial order by $\leq$. If $\mathcal{F}$ is the set of formulas of the logic $\mathcal{I}_\tau$, let $\mathsf{C} : \mathcal{F} \longrightarrow \sigma$ be a mapping satisfying the postulates below, were α and β denote annotated propositions whatever:

(C1) $\mathsf{C}(\alpha \wedge \neg^*\alpha) = \bot'$

(C2) $\mathsf{C}(\alpha \vee \neg^*\alpha) = \top'$

(C3) $\mathsf{C}(\bigvee_{i\in I} \alpha_i) \geq \bigsqcup_{i\in I} \mathsf{C}(\alpha_i)$, for I finite.

(C4) $\mathsf{C}(\bigwedge_{i\in I} \alpha_i) \leq \sqcap_{i\in I} \mathsf{C}(\alpha_i)$, for I finite.

(C5) If $\vdash \alpha \leftrightarrow \beta$, then $\mathsf{C}(\alpha) = \mathsf{C}(\beta)$.

Any such a mapping C is called a *confidence function*. Depending on the applications, the above postulates can be extended by other convenient ones, for instance those expressing that if $\vdash \alpha \to \beta$, then $\mathsf{C}(\alpha) \leq \mathsf{C}(\beta)$, and that $\mathsf{C}(\alpha) \sqcup \mathsf{C}(\neg^*\alpha) = \top'$.

Our proposal is to combine the concept of vagueness with that of confidence, that is, the logic $\mathcal{I}_\tau$ with the operator C; this is similar to the introduction of probability measures in a classical system of probability calculus. Obviously, an "algebra of confidence", as we suggest, extends the classical case of subjective measure of probability, defined on Boolean algebras of propositions. On the other hand, our procedure also encompasses Zadeh's theory of possibility (cf. [232], [123]).

When the degrees of confidence reduce to strict degrees of belief, *i.e.*, subjective probability, they can be handled in the standard way (following the rules of Bayesian probability calculus). Be such hypothesis satisfied or not, it is convenient to strength our system with extra rules as the following Warning Rule (we write $P : \mu : \lambda$ for $\mathsf{C}(P : \mu) = \lambda$):[21]

[The Warning Rule]

$$\frac{P : \mu_i : \lambda_i \,,\ P : \mu_j : \lambda_j}{P : \mu_i \sqcap \mu_j : \lambda_i \sqcup \lambda_j} \quad \Gamma \tag{16}$$

where Γ stands for the set of side conditions which are the underlying base for the application of the rule, conditions that are accepted as true.

The above rule has an intuitive appeal and constitutes a tool for dealing simultaneously with degrees of confidence and of vagueness. Some inductive rules, such as those of induction by simple enumeration and analogy (cf. [79]), complemented by the degrees of vagueness and of confidence, are also useful to reinforce the logic we are trying to built; in the same vein, the methods of the theory of possibility may help here.

We designate the resulting system of inductive logic by $\mathbf{I}_\tau$. Our principal objective is to employ an appropriate $\mathbf{I}_\tau$ to the mechanization of inductive inference, say in robotics and in the theory of expert systems. Let us, then, outline how this can be done by considering a simple case-study. Suppose we are working in an insurance company and need to provide a way of classifying people into disjointed classes by age, since the company needs to distinguish young people from old people in order to differentiate among several prices for the insurance premiums. For example, the company could guide its interviewers that they should classify people in (say) four different categories: (C_1) "young" (less or equal than 25 years), (C_2) "not so young" (less or equal than 35 years and more than 25), (C_3) "not so old" (between 35 and 45 years) and (C_4) "old" (more than 45 years). Of course this is just an example

[21] We remark that, in writing the rule, we have used $\sqcap$ and $\sqcup$ to denote the algebraic operations in both lattices τ and σ by simplicity, but of course this will not cause any confusion.

and more detailed and accurate descriptions could be required. But suppose the Government would like to extend (and to pay for) the insurance services also for the people who don't have certificates of birth, and who sometimes don't know precisely their age, and their hard and poor conditions of life may confound the interviewers in what concerns attributing them precise ages.[22]

So, the information provided by the interviewers may be sometimes considered as not completely precise. For instance, interviewer A may account of to the company an information which represents his belief that John looks to be not so old (that is, it seems to the interviewer that John has an age between 35 and 45 years) and similarly he may account that "Paul looks not so young" (that is, John and Paul were classified as satisfying respectively the predicates C_3 and C_2 above, but the interviewer is not completely sure about the correctness of such a classification). If John and Paul have their certificates of birth, then apparently there is no problem regarding the adequate manipulation of the information. But suppose they don't have their certificates. How to deal with this situation? Since the attribution of ages cannot be done with precision, the interviewer, in the case of doubt, may adopt one of the following alternatives, depending on several involved factors: (1) to classify John as "old" and Paul as "not so young" and (2) to classify John as "not so old" and Paul as "young". These mentioned 'factors' may be for instance the expertise of the interviewer (that is, his experience in the job), or his interest in the defense of some kind of policy, or some guidance, as for instance an hypothetical company's interest in classifying people in classes of ages so that the due premium prices are as greater as possible, so getting more money from the Government.[23]

Humans in general go around situations of this kind by fixing some *ad hoc* criterion in a more or less arbitrary way: they may "decide" what to do by fixing some rule, or by deciding case by case. But, what happens if the case is to be handled by an expert system? In other words, is it possible to keep the system with the capacity of dealing with propositions to which different "degrees of confidence" are attached to, or to follow, for instance, an insurance policy where it is necessary to express a "confidence" in the information provided by the interviewers?

Let τ be a linearly ordered set $\tau = \{\mu_1, \ldots, \mu_4\}$, which is a complete lattice (were $\mu_1(=\perp) \leq \ldots \leq \mu_4(= \top)$). Suppose that two interviewers (this example can be generalized) give as inputs $P : \mu_2$ and $P : \mu_3$ and suppose that P stands for 'John is old'. Then, according to that example, $P : \mu_2$ means 'John is not so young', while $P : \mu_3$ says that 'John is not so old'. More precisely, the first interviewer has classified John as having an age between 25 and 35 years, while the second interviewer admitted that he is between 35 and 45 years old.

An expert system, in order to attach a certain value to John's due taxes, may follows the following rule, being $C_1, \ldots, C_4$ the classes of ages in the above considered example: *in the case of doubt if someone belongs to either class C_i or C_{i+1}, classify him/her as belonging to the class C_i*. So, in the exemplified case, John would be considered as 'not so young', and then his due taxes would be supposed to be smaller than if he were classified as being 'not so old'.

The choice of $P : \mu_2$ may be interpreted as resulting from the application of the Warning Rule, since it was preferred to put the greater degree of confidence in the vaguer proposition. In other words, according to the convention made in writing the Warning Rule, if we write $P : \mu_2 : \lambda_2$ for $\mathsf{C}(P : \mu_2) = \lambda_2$ and $P : \mu_3 : \lambda_3$ for $\mathsf{C}(P : \mu_3) = \lambda_3$, then we may say that from the 'premisses' $P : \mu_2$ and $P : \mu_3$ we have arrived at a 'conclusion' $P : \mu_2$ with greater degree of confidence. This can be expressed by saying that the 'conclusion' is $P : \mu_2 \sqcap \mu_3 : \lambda_2 \sqcup \lambda_3$ (we remark that τ is a linear lattice in which $\mu_2 \sqcap \mu_3 = \mu_2$).

[22]This situation is common in developing and in poor countries.
[23]Of course the situation could be precisely the opposite.

Roughly speaking, the Warning Rule expresses that a possible expert system elaborated for dealing with vagueness and with degrees of confidence, when faced with situations like the just mentioned one, should opt for the more prudent situation. This is in accordance with a rational stance, for we may say that 'rationality' means also the tentative of optimizing our rational degrees of confidence in the propositions we are concerned with, but with the caution of not taking conclusions with degrees of confidence greater than those attributed to the premisses.

7.3 Foundations of physics

7.3.1 Logic and physics

During the International Congress of Mathematicians, held in Paris in 1900, David Hilbert presented a list of 23 Problems of Mathematics which in his opinion should occupy the efforts of mathematicians in the century to come. To solve one of the problems become a way of achieving something really important in mathematics, and several Fields medals were awarded for this kind of endeavour. The sixth problem of his celebrated list dealt with the axiomatization of the theories of physics; Hilbert proposed "to treat in the same manner [as Hilbert himself had done with geometry], by means of axioms, those physical sciences in which mathematics plays an important part" [142].

In the XXth century, much was done in this direction, in continuation to those efforts already developed in the previous period, as remarked by Hilbert himself in his mentioned paper. The parallel development of logic in the last century and the development of non-classical systems, linked to some philosophical views about science and the presentation of scientific theories, such as that one of the neo-positivists, with emphasis in logic and language, forced philosopher to acknowledge that behind (or underlying) the axiomatic version of a scientific theory there are also the postulates of a higher-order logic kind, which provide the grounds for its deductive and mathematical counterparts. Generally, we can think as 'logical' those postulates of first-order logic (with equality), while the 'mathematical' postulates can be those of a set theory, like Zermelo-Fraenkel (of course alternative approaches can be found, but we will not discuss this point here). So, in such a framework, logic plays an important role, and in particular if there are reasons to suspect that some logical system, other than classical logic, is to be used in axiomatizing a certain domain, its explicit details are to be made explicit.

For instance, when we hear something about the relationships between logic and quantum physics, we usually tend to relate the subject with the so called 'quantum logics', a field that has its 'official' birth in Birkhoff and von Neumann's well known paper from 1936. This is completely justified, for their fundamental work caused the development of a wide field of research in logic. Today there are various 'quantum logical systems', including some paraconsistent quantum logics (see [115], [116], [117]), although they have been studied specially as pure mathematical systems, far from applications to the axiomatization of the microphysical world and also far from the insights of the forerunners of quantum mechanics.

Of course an axiomatization of a given empirical theory is not always totally determinate, and the need for a logic distinct from the classical as the underlying logic of quantum theory is still an open question. The axiomatic basis of a scientific theory depends on the several aspects of the theory, explicitly or implicitly, appropriate to take into account its structure. So, for example, Ludwig [170] studies an axiomatization of quantum mechanics based on classical logic. All stances, that of employing a logic like paraconsistent logic (or other kind of system), and that of Ludwig, are in principle acceptable, since they treat different perspectives of the same domain of discourse, and different 'perspectives' of a domain of science may demand for distinct logical apparatuses; this is a philosophical point of view radically different from the classical, and it is against the idea that there is one logic, *vis.*, classical logic. As we have said in the Introduction, the possibility of using non-standard systems in the foundations of physics (and in general of science) does not necessarily entail that classical logic is wrong, or that (in particular) quantum theory *needs* another logic. Physicists probably will continue to use classical (informal) logic in the near future. But we should realize that other forms of logic may help us in the better understanding of certain features of the quantum world as well, not easily treated by classical devices. One of the interesting examples is that of involving the concept of complementarity, which can be treated without deep and detailed prolegomena. Concerning the use of non-classical systems in science, in particular in physics, only the future of this discipline will decide what is the better solution in respect to the use of alternative formalisms or keeping the theories within a 'classical' domain. Such a decision involves pragmatic factors, as it seems to be obvious.

To summarize our point, we believe that there is no just one 'true logic', and distinct logical (so as mathematical and perhaps even physical) systems, like paraconsistent logic, may be useful to approach different aspects of a wide field of knowledge like quantum theory. The important point is that the scientist should be open to the justifiable revision of concepts, a point very lucidly emphasized by Niels Bohr, who wrote:

> "For describing our mental activity, we require, on one hand, an objectively given content to be placed in opposition to a perceiving subject, while, on the other hand, as is already implied in such an assertion, no sharp separation between object and subject can be maintained, since the perceiving subject also belongs to our mental content. From these circumstances follows not only the relative meaning of every concept, or rather of every word, the meaning depending upon our arbitrary choice of view point, but also we must, in general, be prepared to accept the fact that a complete elucidation of one and the same object may require diverse points of view which defy a unique description. Indeed, strictly speaking, the conscious analysis of any concept stands in a relation of exclusion to its immediate application. The necessity of taking recourse to a complementarity, or reciprocal, mode of description is perhaps most familiar to us from psychological problems. In opposition to this, the feature which characterizes the so-called exact sciences is, in general, the attempt to attain to uniqueness by avoiding all reference to the perceiving subject. This endeavour is found most consciously, perhaps, in the mathematical symbolism which sets up for our contemplation an ideal of objectivity to the attainment of which scarcely any limits are set, so long as we remain within a self-contained field of applied logic. In the natural sciences proper, however, there can be no question of a strictly self-contained field of application of the logical principles, since we must continually count on the appearance of new facts, the inclusion of which within the compass of our earlier experience may require a revision of our fundamental concepts." (cf. ibid.)

7.3.2 A case involving paraconsistency

In this section, as an example, we shall provide a general look on the use of paraconsistent logic to approach what we call 'complementarity theories'. These are based on Bohr's view on complementarity, a concept introduced by him in his famous 'Como Lecture', in 1927, although the basic ideas go back to 1925. The consequences of his view were fundamental, particularly for the development of the Copenhagen interpretation of quantum mechanics and constitutes, as it is largely recognized in the literature, as one of the most fundamental contributions to the development of quantum theory. But we would like to remark that we are not making exegesis of Bohr's ideas, for, as it is well known, they are quite controversial.[24] Anyway, we will use the basic motivation underlying a possible reading on the very idea of complementarity to provide the grounds for defining a general class of theories ($\mathcal{C}$-theories). As we shall see, the logic of such theories is a particular paraconsistent logic termed *paraclassical*.

Roughly speaking, we say that a theory T admits a complementarity interpretation, or that T is a $\mathcal{C}$-theory, if T encompasses 'true' formulas α and β which are 'mutually exclusive' in the sense that their conjunction yields to a strict contradiction if classical logic is applied. More precisely, if $\vdash$ is the symbol of deduction of classical logic, then, there exists γ such that, being α and β complementary, we have (in T) $\alpha, \beta \vdash \gamma \wedge \neg\gamma$. Of course if the underlying logic of T is classical logic, then T, once involving complementary propositions, is contradictory, or inconsistent.

So, let us call C an axiomatized system for the classical propositional calculus (the developments made here can be extended to quantification). The concept of deduction in C is taken to be the standard one; we use the symbol $\vdash$ to represent deductions in C. Furthermore, the formulas of C are denoted by Greek lowercase letters, while Greek uppercase letters stand for sets of formulas. The symbols $\neg$, $\rightarrow$, $\wedge$, $\vee$ and $\leftrightarrow$ have their usual meanings, and standard conventions in writing formulas will be also assumed without further comments.

Definition 7.3.1 *Let Γ be a set of formulas of C and let α be a formula (of the language of C). Then we say that α is a (syntactical)* P-consequence *of Γ, and write $\Gamma \vdash_{\mathsf{P}} \alpha$, if and only if*

(P1) $\alpha \in \Gamma$, *or*

(P2) α *is a classical tautology, or*

(P3) *There exists a consistent (according to classical logic) subset $\Delta \subseteq \Gamma$ such that $\Delta \vdash \alpha$ (in classical logic).*

We call $\vdash_{\mathsf{P}}$ the relation of P-*consequence*.

Definition 7.3.2 P *is the logic whose language is that of* C *and whose relation of consequence is that of* P*-consequence. Such a logic will be called* paraclassical.

It is immediate that, among others, the following results can be proved:

Theorem 7.3.1

1. *If α is a theorem of the classical propositional calculus* C *and if Γ is a set of formulas, then $\Gamma \vdash_{\mathsf{P}} \alpha$; in particular, $\vdash_{\mathsf{P}} \alpha$.*

2. *If Γ is consistent (according to* C*), then $\Gamma \vdash \alpha$ (in* C*) iff $\Gamma \vdash_{\mathsf{P}} \alpha$ (in* P*).*

3. *If $\Gamma \vdash_{\mathsf{P}} \alpha$ and if $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathsf{P}} \alpha$ (The defined notion of* P*-consequence is monotonic.)*

4. *The notion of* P*-consequence is recursive.*

5. *Since the theses of* P *(valid formulas of* P*) are those of* C*,* P *is decidable.*

[24] For references and further discussion on this particular point, see [43], [97].

Definition 7.3.3 *A set of formulas Γ is* P*-trivial iff* $\Gamma \vdash_{\mathsf{P}} \alpha$ *for every formula* α*. Otherwise,* Γ *is* P*-non-trivial. (Similarly we define the concept of a set of formulas being trivial in* C*).*

Definition 7.3.4 *A set of formulas* Γ *is* P*-inconsistent if there exists a formula* α *such that* $\Gamma \vdash_{\mathsf{P}} \alpha$ *and* $\Gamma \vdash_{\mathsf{P}} \neg\alpha$*. Otherwise,* Γ *is* P*-consistent.*

Theorem 7.3.2

1. *If* α *is an atomic formula, then* $\Gamma = \{\alpha, \neg\alpha\}$ *is* P*-inconsistent, but* P*-non-trivial.*

2. *If the set of formulas* Γ *is* P*-trivial, then it is trivial (according to classical logic). If* Γ *is non-trivial, then it is* P*-nontrivial.*

3. *If* Γ *is* P*-inconsistent, then it is inconsistent according to classical logic. If* Γ *is consistent according to classical logic, then* Γ *is* P*-consistent.*

A semantical analysis of P, for instance a completeness theorem, can be obtained without difficulty, as indicated in [103]. We remark that the set $\{\alpha \wedge \neg\alpha\}$, where α is a propositional variable, is trivial according to classical logic, but it is not P-trivial. Notwithstanding, we are not suggesting that complementary propositions should be understood necessarily as pairs of contradictory sentences. This is made clear by the following definition:

Definition 7.3.5 (Complementarity Theories or $\mathcal{C}_{mp}$-theories) *A* $\mathcal{C}$*-theory is a set of formulas* T *of the language of* C *(the classical propositional calculus) closed by the relation of* P*-consequence, that is,* $\alpha \in T$ *for any* α *such that* $T \vdash_{\mathsf{P}} \alpha$*. In other words,* T *is a theory whose underlying logic is* P*. A* $\mathcal{C}_{mp}$*-theory is a* $\mathcal{C}$*-theory subjected to meaning principles.*

Of course the definition of a $\mathcal{C}_{mp}$-theory is a little bit vague. However, for instance in the case of a meaning principle that introduces restrictions in the acceptable statements of the theory, the hypothesis and axioms used in deductions have to satisfy such restrictive conditions. For instance, if a meaning principle of a theory T is formulated as Heisenberg Uncertainty Principle, this circumstance will impose obvious restrictions to certain statements of T.

Theorem 7.3.3 *There exist* $\mathcal{C}$*-theories and* $\mathcal{C}_{mp}$*-theories that are inconsistent, although are* P*-non-trivial.*

Proof: Immediate consequence of Theorem 5.1.2. ∎

Finally, we state a result (Theorem 7.3.4), whose proof is an immediate consequence of the definition of P-consequence. However, before stating the theorem, let us introduce a definition:

Definition 7.3.6 (Complementary Propositions) *Let* T *be a* $\mathcal{C}_{mp}$*-theory (in particular, a* $\mathcal{C}$*-theory) and let* α *and* β *be formulas of the language of* T*. We say that* α *and* β *are* T*-complementary (or simply complementary) if there exists a formula* γ *of the language of* T *such that:*

1. $T \vdash_{\mathsf{P}} \alpha$ and $T \vdash_{\mathsf{P}} \beta$

2. $T, \alpha \vdash_{\mathsf{P}} \gamma$ and $T, \beta \vdash_{\mathsf{P}} \neg\gamma$ (in particular, $\alpha \vdash_{\mathsf{P}} \gamma$ and $\beta \vdash_{\mathsf{P}} \neg\gamma$).

Theorem 7.3.4 *If* α *and* β *are complementary theorems of a* $\mathcal{C}_{mp}$*-theory* T *and* $\alpha \vdash_{\mathsf{P}} \gamma$ *and* $\beta \vdash_{\mathsf{P}} \neg\gamma$*, then in general* $\gamma \wedge \neg\gamma$ *is not a theorem of* T*.*

Proof: Immediate, as a consequence of Theorem 5.1.2. ∎

In other words, T is inconsistent from the point of view of classical logic, but it is P-non-trivial.

It should be emphasized that our way of characterizing complementarity does not mean that complementary propositions are always contradictory, for α and β above are not necessarily one the negation of the other. However, as complementary propositions, we may derive from them (in classical logic) a contradiction; to exemplify, we remark that 'x is a particle' is not the direct negation of 'x is a wave', but 'x is a particle' *entails* that x is not a wave. This reading of complementarity as not indicating strict contradiction, as we have already made clear, is in accordance with Bohr himself; let us quote him once more to reinforce this idea. Bohr says (cf. [97]):

> "In considering the well-known paradoxes which are encountered in the application of the quantum theory to atomic structure, it is essential to remember, in this connection, that the properties of atoms are always obtained by observing their reactions under collisions or under the influence of radiation, and that the (...) limitation on the possibilities of measurement is directly related to the apparent contradictions which have been revealed in the discussion of the nature of light and of the material particles. In order to emphasize that we *are not concerned here with real contradictions*, the author [Bohr himself] suggested in an earlier article the term 'complementarity'." (italics ours).

Let us give a simple example of a situation involving a $\mathcal{C}_{mp}$-theory. Suppose that our theory T is a fragment of quantum mechanics admitting Heisenberg relations as a meaning principle and having as its underlying logic paraclassical logic. If α and β are two incompatible propositions according to Heisenberg's principle, we can interpret this principle as implying that α entails $\neg\beta$ (or that β entails $\neg\alpha$). So, even if we add α and β to T, we will be unable to derive, in T, $\alpha \wedge \beta$. Analogously, Pauli's Exclusion Principle has also an interpretation as that of Heisenberg's.

As we said before, the basic characteristic of $\mathcal{C}_{mp}$-theories is that, in making P-inferences, we suppose that some sets of statements we handle are consistent. In other words, $\mathcal{C}_{mp}$-theories are closer to those theories scientists *actually* use in their everyday activity than those theories with the classical concept of deduction. In other words, paraclassical logic (and paraconsistent logics in general) seems to fit more accurately the way scientists reason when stating their theories.

7.3.3 Generalization: the paralogic associated to a given logic

The technique used above to define the paraclassical logic associated to classical logic can be generalized to other logics $\mathcal{L}$ (including logics having no negation symbol, but we will not deal with this case here), as well as the concept of a $\mathcal{C}_{mp}$-theory. More precisely, starting with a logic $\mathcal{L}$, which can be seen as a pair $\mathcal{L} = \langle \mathcal{F}, \vdash \rangle$, where $\mathcal{F}$ is an abstract set called the set of formulas of $\mathcal{L}$ and $\vdash \subseteq \mathcal{P}(\mathcal{F}) \times \mathcal{F}$ is the deduction relation of $\mathcal{L}$ (which is subjected to certain postulates depending on the particular logic $\mathcal{L}$), we can define the $\mathsf{P}_{\mathcal{L}}$-logic associated to $\mathcal{L}$ (the 'paralogic' associated to $\mathcal{L}$) as follows.

Let $\mathcal{L}$ be a logic, which may be classical logic, intuitionistic logic, some paraconsistent logic or, in principle, any other logical system. By simplicity, we suppose that the language of $\mathcal{L}$ has a symbol for negation, $\neg$. Then,

Definition 7.3.7 *A theory based on $\mathcal{L}$ (an $\mathcal{L}$-theory) is a set of formulas Γ of the language of $\mathcal{L}$ closed under $\vdash_{\mathcal{L}}$ (the symbol of deduction in $\mathcal{L}$). In other words, $\alpha \in \Gamma$ for every formula α such that $\Gamma \vdash_{\mathcal{L}} \alpha$.*

Definition 7.3.8 *An $\mathcal{L}$-theory Γ is $\mathcal{L}$-inconsistent if there exists a formula α of the language of $\mathcal{L}$ such that $\Gamma \vdash_{\mathcal{L}} \alpha$ and $\Gamma \vdash_{\mathcal{L}} \neg\alpha$, where $\neg\alpha$ is the negation of α. Otherwise, Γ is $\mathcal{L}$-consistent.*

Definition 7.3.9 *An $\mathcal{L}$-theory Γ is $\mathcal{L}$-trivial if $\Gamma \vdash_{\mathcal{L}} \alpha$ for any formula α of the language of $\mathcal{L}$. Otherwise, Γ is $\mathcal{L}$-non-trivial.*

Then, we define the $\mathsf{P}_{\mathcal{L}}$-logic associated with $\mathcal{L}$ whose language and syntactical concepts are those of $\mathcal{L}$, except the concept of deduction, which is introduced as follows: we say that α is a $\mathsf{P}_{\mathcal{L}}$-*syntactical consequence* of a set Γ of formulas, and write $\Gamma \vdash_{\mathsf{P}_{\mathcal{L}}} \alpha$ if and only if:

(1) $\alpha \in \Gamma$, or

(2) α is a provable formula of $\mathcal{L}$ (that is, $\vdash_{\mathcal{L}} \alpha$), or

(3) There exists $\Delta \subseteq \Gamma$ such that Δ is $\mathcal{L}$-non-trivial, and $\Delta \vdash_{\mathcal{L}} \alpha$.

For instance, we may consider the paraconsistent calculus $\mathcal{C}_1$ as our logic $\mathcal{L}$. Then the paralogic associated with $\mathcal{C}_1$ is a kind of 'para-paraconsistent' logic.

It seems worthwhile to note the following in connection with the paraclassical treatment of theories. Sometimes, when one has a paraclassical theory T such that $T \vdash_{\mathsf{P}} \alpha$ and $T \vdash_{\mathsf{P}} \neg\alpha$, there exist *appropriate* propositions β and γ such that T can be replaced by a classical consistent theory T' in which $\beta \to \alpha$ and $\gamma \to \neg\alpha$ are theorems. If this happens, the logical difficulty is in principle eliminable and classical logic maintained.

7.4 Morality and Law

In its standard form, deontic logic can be taken as a kind of classical modal logic through the introduction of deontic operators, O for obligatory, P for permitted, V for prohibited and F for indifferent. One of the interesting problems involving deontic logic is that the standard systems do not enable the existence of 'true' moral dilemmas, expressed by formulas of the form $O\alpha \wedge O\neg\alpha$, which says that both α and its negation are obligatory. This is due to the fact that in these systems $O\alpha \wedge O\neg\alpha \to O\beta$ is a theorem. Hence, a moral dilemma trivializes the whole system. Other problems related to these logics are created by paradoxes like Ross' paradox and the Good Samaritan paradox [210].

One of the standard solutions to problems like the existence of moral dilemmas is simply to sustain that they really cannot exist, but for some philosophers they are genuine possibilities (see [86] for references). So, it should be interesting to tolerate them, but obviously without deontic triviality; paraconsistent logics are good candidates for providing the necessary tools.

The obligatory operator (so as the other mentioned above) has various interpretations, the most common being in the sense of *moral* obligations (and moral prohibitions etc.) and *legal* obligations (legal prohibitions etc.). If we write O_l and O_m for the legal and the moral obligations respectively (the same notation applies to the other operators), it is interesting, among other things, to study the principles $O_l\alpha \to P_m\alpha$ and $O_m\alpha \to P_l\alpha$, which seen quite 'natural'. To handle these situations, some deontic paraconsistent logics were introduced; see also [208].

8 Concluding remarks

We said in several parts of this paper that it is impossible for someone to follow all the literature on paraconsistent logics. We have leaving aside various important authors and topics not because we consider them as not important or that the subjects are not relevant, but simply because it is out of our capacity even to mention them all here. But we would like to say that nowadays 'paraconsistency' became a field of knowledge so wide, of which the World Congresses on Paraconsistency do really justice.

To emphasize what we just said, it is enough to ask for 'paraconsistent logic' at `www.google.com`; in January 31, 2004, 17:15h, Brazil time, we have found 7,190 entries! So, we think we should stop here.

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