A DUALITY FOR 3-VALUED ŁUKASIEWICZ IMPLICATION ALGEBRAS WITH $\Delta$

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In this work we give a topological representation for 3-valued Łukasiewicz implication algebras with $\Delta$. Every 3-valued Łukasiewicz implication algebra with $\Delta$ is represented as a union of a unique family of implication filters of a suitable 3-valued Łukasiewicz algebra. Inspired in this representation, we introduce the notion of topological 3-valued implication space, and we prove a dual equivalence. The main tool we use in the proofs of this result is the duality between implication algebras and certain topological spaces, developed in [1]. As an application, we describe the 3-valued implication space of free 3-valued Łukasiewicz implication algebra with $\Delta$.

We assume familiarity with the algebras involved here. The equational class of all 3-valued Łukasiewicz algebras will be denoted by $L_3$. The 3-element chain $L_3 = \{0, 1/2, 1\}$, with the natural lattice structure and the operations $\sim$ and $\triangledown$ defined as $\sim 0 = 1$, $\sim 1/2 = 1/2$, $\sim 1 = 0$ and $\triangledown 0 = 0, \triangledown 1/2 = 1, \triangledown 1 = 1$, will be denoted by $L_3$. It is well known that $L_3$ and its subalgebra $L_2 = \langle L_2 = \{0, 1\}, \lor, \land, \sim, \triangledown, 0, 1\rangle$ are the subdirectly irreducible algebras of $L_3$. Given $L \in L_3$ and $X \subseteq L$, we will denote by $F(X)$ the implication filter generated by $X$ in $L$.

An algebra $A = \langle A, \rightarrow, \Delta, 1 \rangle$ of type $(2, 1, 0)$ is a 3-valued Łukasiewicz implication algebra with $\Delta$ if

1. $x \rightarrow (y \rightarrow x) = 1$,
2. $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,
3. $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
4. $((x \rightarrow y) \rightarrow (y \rightarrow x)) \rightarrow (y \rightarrow x) = 1$,
5. $((x \rightarrow (x \rightarrow y)) \rightarrow x) \rightarrow x = 1$,
6. $1 \rightarrow x = x$,
7. $\Delta x \rightarrow y = x \rightarrow (x \rightarrow y)$,
8. $\Delta(\Delta x \rightarrow y) = \Delta x \rightarrow \Delta y$.

These algebras were introduced and studied by Figallo in [4]. The variety of all 3-valued Łukasiewicz implication algebras with $\Delta$ will be denote by $L_3^{(\rightarrow, \Delta)}$. Note that $\langle A, \rightarrow, 1 \rangle$ is a 3-valued Łukasiewicz implication algebra ([5]). It is known that they are the class of all $\{\rightarrow, \Delta, 1\}$-subreducts of 3-valued Łukasiewicz algebras. The $\{\rightarrow, \Delta, 1\}$-reduct of $L_3$ will be denoted by $L_3^{\rightarrow, \Delta}$. The subalgebra of $L_3^{\rightarrow, \Delta}$ with subuniverse consisting of the 2-element chain will be denoted by $L_2^{\rightarrow, \Delta}$. These algebras are the subdirectly irreducible algebras of


Key words and phrases. Łukasiewicz implication algebras, 3-valued Łukasiewicz algebras, implication spaces, dual categorical equivalence.

This paper is partially supported by Universidad Nacional del Sur and CONICET.
$\mathcal{L}_3^{(-,\Delta)}$, and they are simple. It is also known that the congruences are determined by the implication filters. Let $A \in \mathcal{L}_3^{(-,\Delta)}$, and let $\mathcal{M}$ the set of all maximal implication filters of $A$. Then $A$ is isomorphic to a subalgebra $A'$ of $P = \prod_{D \in \mathcal{M}} A/D$. For each $D \in \mathcal{M}$, $A/D$ is isomorphic to $L_3^{(-,\Delta)}$ or $L_2^{(-,\Delta)}$. Then, $P \in \mathcal{L}_3$. We identify $A$ with $A'$. Consider now the 3-valued Lukasiewicz algebra generated by $A$ in $P$, denote by $L_3[A]$. We prove that $A$ is an increasing subset of $L_3[A]$, and so, $A$ is isomorphic to a union of implication filters of $L_3[A]$. Also, if $F(A)$ is proper then $L_3[A]/F(A) \cong L_2$. We define the closure of $A$ by

$$\text{CL}_3(A) = \begin{cases} L_3[A] & \text{if } F(A) \neq L_3[A] \\ L_3[A] \times L_2 & \text{if } F(A) = L_3[A] \end{cases}.$$ 

The next theorem establishes that $\text{CL}_3(A)$ is the least, up to isomorphism, 3-valued Lukasiewicz algebra in which the implication filter $F(A)$ is maximal and $\text{CL}_3(A)/F(A)$ is the 2-element chain.

**Theorem 1.** Let $A \in \mathcal{L}_3^{(-,\Delta)}$, then:

1. $A$ is increasing in $\text{CL}_3(A)$, and $A$ satisfies the finite meet property (fmp). Moreover, $\text{CL}_3(A)/F(A) \cong L_2$.

2. If $L \in \mathcal{L}_3$ and $h: A \to L$ is a $\{-,\Delta\}$-homomorphism, such that $h[A]$ has the fmp in $L$, then there is a homomorphism $\hat{h}: \text{CL}_3(A) \to L$ such that $\hat{h} \upharpoonright A = h$, i.e., the diagram

$$\begin{array}{ccc}
A & \xleftarrow{i} & \text{CL}_3(A) \\
\downarrow h & & \downarrow \hat{h} \\
L & & L
\end{array}$$

commutes.

Two 3-valued Lukasiewicz implication algebras with $\Delta$ may have the same closure, but they can be distinguished by means of the implication filters contained in them. If $M(A)$ is the family of all maximal elements in the set of all implication filters of $\text{CL}_3(A)$ contained in $A$, then $M(A)$ satisfies:

1. $A = \bigcup_{D \in M(A)} D$, and $M(A)$ is an antichain,

2. if $M$ is an implication filter of $\text{CL}_3(A)$ contained in $A$, then $M \subseteq D$ for some $D \in M(A)$.

A 3-valued Boolean space is a pair $\langle X, V \rangle$ such that $X$ is a Boolean space and $V \subseteq X$ is closed. If the set $L_3$ is equipped with the discrete topology, and $\langle X, V \rangle$ is a 3-valued Boolean space, then $C_3(X, V)$ denotes the 3-valued Lukasiewicz algebra of all continuous functions $f: X \to L_3$ such that $f(V) \subseteq L_2$, with the algebraic operations defined pointwise. In [3], it has been proved that for each $A \in \mathcal{L}_3$, there is a 3-valued Boolean space $\langle X(A), V_A \rangle$ such that $A \cong C_3(X(A), V_A)$. Moreover, $X(A)$ is homeomorphic to the Stone space of the Boolean algebra $B(A)$.

**Definition 2.** A space $\langle X, V, u, C \rangle$ is a 3-valued implication space if
(1) \(\langle X, V \rangle\) is a 3-valued Boolean space, and \(u\) is a fixed element of \(X\) such that \(u \in V\).

(2) \(C\) is an antichain, respect to inclusion, of closed set of \(X\) such that \(\bigcap C = \{u\}\).

(3) if \(C\) is a closed set of \(X\) such that for every clopen \(N\) of \(X\), \(C \subseteq N\) implies there is \(D \in C\) such that \(D \subseteq N\), then there is \(D' \in C\) such that \(D' \subseteq C\).

**Definition 3.** A function \(f: \langle X_1, V_1, u_1, C_1 \rangle \rightarrow \langle X_2, V_2, u_2, C_2 \rangle\) is \(i3\)-continuous if \(f\) is continuous, \(f(V_1) \subseteq V_2\), \(f(u_1) = u_2\), and for all \(C \in C_2\) there is \(D \in C_1\) such that \(D \subseteq f^{-1}[C]\).

Let \(A \in \mathcal{L}_3^{\langle \rightarrow, \Delta \rangle}\), and consider \(CL_3(A)\), \(F(A)\) and \(M(A)\) as before. Recall that the lattice of implication filters of \(CL_3(A)\) is isomorphic to the lattice of filters of \(B(CL_3(A))\), and that there is a correspondence between filters and closed sets of \(St(B(CL_3(A)))\). So, we have a correspondence between implication filters \(D\) of \(CL_3(A)\) and closed sets of \(St(B(CL_3(A)))\) given by \(D \mapsto C_{\Delta D} = \{U \in St(B(CL_3(A))) : \Delta D \subseteq U\}\). Consider the space

\[
\mathbb{F}(A) = \langle X(CL_3(A)), V_{CL_3(A)}, \Delta F(A), C(A) \rangle,
\]

where \(X(CL_3(A)) \cong St(B(CL_3(A)))\), \(V_{CL_3(A)} = \{U \in St(B(CL_3(A))) : CL_3(A)/F(U) \cong L_2\}\), and \(C(A) = \{C_{\Delta D} : D \in M(A)\}\). From Theorem 1 and conditions of \(M(A)\), we have that \(\mathbb{F}(A)\) is a \(3\)-valued implication space.

We have proved that there exists a dual equivalence between the category whose objects are the \(3\)-valued Lukasiewicz implication algebras with \(\Delta\) and whose arrows are homomorphisms, and the category of \(3\)-valued implication spaces with morphisms as \(i3\)-continuous functions.

**The \(i3\)-implication space of free algebras.** We will denote by \(F_K(X)\) the free algebra in a variety \(K\) with a set \(X\) of generators. Let \(X\) and \(X^*\) be sets such that \(|X| = |X^*|\). Since \(F_L_3(X^*) \in \mathcal{L}_3^{\langle \rightarrow, \Delta \rangle}\), we can consider the \(\mathcal{L}_3^{\langle \rightarrow, \Delta \rangle}\)-subalgebra generated by \(X\) in \(F_L_3(X^*)\). It has been proved that \(F_L_3(X^*)\) is isomorphic to this subalgebra. Moreover, if we identify \(X\) with \(X^*\), we have that \(F_L_3(X^*) = \bigcup_{x \in X} \Delta x\), where \(\Delta x\) is the implication filter generated by \(\Delta x\) in \(F_L_3(X)\).

It is known that \(B(F_L_3(X))\) is \(B(Y)\), the free Boolean algebra over the poset \(Y = \{\Delta x, \nabla x : x \in X\}\) ([2]). For each \(x \in X\), \(\Delta x, \nabla x\) is a chain called principal chain. Moreover, there is a bijection from the set of upwards closed subsets \(S \subseteq Y\) onto the ultrafilters of \(F_B(Y)\), given by \(S \rightarrow F(S \cup \{-y : y \in Y\}) = U_S\). Therefore, \(F_L_3(X)\) is isomorphic to the algebra \(C_3(St(F_B(Y)), V_{F_L_3(X)})\), where \(U_S \in V_{F_L_3(X)}\) if and only if \(S\) is a union of principal chains [2, Thm. 2.1]. From this, we have that \(F_L_3(X)/F(X) \cong L_2\), since \(F(X) = F(U_Y)\). Then, \(CL_3(F_L_3(X)) = F_L_3(X)\). Also, we have proved that

\[
M(F_L_3(X)) = \{[\Delta x] : x \in X\}.
\]

Then \(X(CL_3(F_L_3(X))) \cong X(F_L_3(X)) \cong St(B(F_L_3(X))) \cong St(F_B(Y))\). For each \(x \in X\), \(C_{\Delta x} = \{U_S \in St(F_B(Y)) : \Delta x \subseteq U_S\}\). Observe that \(\bigcap C_{\Delta x} = \{U_Y\}\). Then, \(\langle St(F_B(Y)), V, \{C_{\Delta x} : x \in X\} \cup \{U_Y\} \rangle\) is the \(i3\)-implication space of \(F_L_3(X)\), where \(U_S \in V\) is and only if \(S\) is union of principal chains.
References


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