Quantifying in Extensive Games

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Abstract. Recently, there have been a lot of approaches that uses a modal logic to model and reason about games. It seems natural to study quantification whenever games are modeled by means of the modal approach. In this work, we use a first-order modal logic, namely Game Analysis Logic (GAL), in order to study the alternatives of quantification for the most used solution concepts, namely Nash Equilibrium (NE) and subgame perfect equilibrium (SPE), for extensive games. We also characterize these concepts by means of the structure of an extensive game according to the different ways of quantification. Despite the fact that quantifying in a modal context might be troublesome, we show that for NE and SPE the alternatives are equivalent each other.

1 Introduction

Games are abstract models of decision-making in which decision-makers (players) interact in a shared environment to accomplish their goals. Several models have been proposed to analyze a wide variety of applications in many disciplines such as Mathematics, Computer Science and even political and social sciences among others.

Game Theory \cite{vonNeumannMorgenstern} has its roots in the work of von Neumann and Morgenstern \cite{vonNeumannMorgenstern} and uses mathematics in order to model and analyze games in which the decision-makers pursue rational behavior in the sense that they choose their actions after some process of optimization and take into account their knowledge or expectations of the other players’ behavior. Game Theory provides general game definitions as well as reasonable solution concepts for many kinds of situations in games. Typical examples of this kind of research come from phenomena emerging from Markets, Auctions and Elections.

Although historically Game Theory has been considered more suitable to perform quantitative analysis than qualitative ones, there have been a lot of approaches that emphasizes Game Analysis on a qualitative basis, by using an adequate logic in order to express games as well as their solution concepts. Some of the most representatives of these logics are: Coalitional Logic \cite{Conboy2010}; Game Logic \cite{Boutilier2000}; Game Logic with Preferences \cite{VanGelder2004}; Alternating-time Temporal Logic (ATL) \cite{Alur1998} and its variation Counter-factual ATL (CATL) \cite{Burch2002}, a good progress report of the use of ATL and its extensions is provided in \cite{Alur}; Coalitional Game Logic (CGL) \cite{Boutilier2004} that reasons about coalitional games. To see more details about the connections and open problems between logic and games, we point out \cite{Hendricks2010}.

It is well-known that quantifying in modal logics is troublesome \cite{Gabbay2000}. It seems natural to study quantification whenever games are modeled by means of the modal approach. In order to observe how this problem may happen in games as simple as extensive games with perfect information \cite{vonNeumannMorgenstern}, we use a first-order modal logic which is based on the standard logic CTL \cite{Alur1993}, namely Game Analysis Logic (GAL) \cite{Haeusler2009}. In \cite{Haeusler2010}, we represent these games by means of models of GAL and their main solution concepts that are Nash equilibrium (NE) and subgame perfect equilibrium (SPE) by means of modal formulas; however, we do not take into account the alternatives of quantification in the context of games.

The main difference to the logics mentioned above is that GAL has a first-order apparatus. As a consequence, we are able to define many concepts, such as utility, in an easier way; moreover, we can study quantification in the context of games that in the main guideline of this work. It is worth mentioning that the ATL logic, in which the operators of CTL are parameterized by sets of players, can be seen as a fragment of GAL using the first-order feature of GAL; thus, there is no need for such a parameterization in GAL.
The solution concept of NE requires that each player’s strategy be optimal, given the other players’ strategies. And, the solution concept of SPE requires that the action prescribed by each player’s strategy be optimal, given the other players’ strategies, after every history. In SPE concept, the structure of the extensive game is taken into account explicitly, while, in the solution concept of NE, the structure is taken into account only implicitly in the definition of the strategies. Thus, the usual definitions of NE for extensive games does not regard to the sequential structure of the game. We can see this clearly in this quotation:

“The first solution concept [Nash Equilibrium] we define for an extensive game ignores the sequential structure of the games; it treats the strategies as choices that are made once and for all before play begins.” [8, pages 93]

Other authors discuss that NE is related to the structure of the extensive game; however, their formal definitions are presented in the usual way. See the quotation below:

“We also saw that some of these Nash Equilibria may rely on “empty threats” of suboptimal play at histories that are not expected to occur - that is, at histories off the path of the equilibrium.” [5, pages 72]

In this article, we characterize Nash Equilibrium by means of the rationales of the players on the equilibrium’s path. Moreover, we aim to study the different ways of quantification for NE and SPE solution concepts in a first-order modal logic (GAL) regarding to the paths of the game.

Consider a GAL model of an extensive game as a kind of first-order Kripke frame (each world is a first-order structure), such that the relationships between worlds are defined according to the actions of the game. As an example, Figure 1.b below represents part of the GAL model associated to the extensive game in Figure 1.a.

![GAL model](image_url)

(a) - Extensive form representation  (b) - A GAL representation

**Fig. 1.** Mapping an extensive game into a GAL model.

We can consider the quantification alternatives of *de dicto* and *de re* in the GAL logic. Definitions 1 and 2 as well as definitions 3 and 4 seem to be adequate, respectively, to express SPE and NE. The quantifications of the strategies take place in different contexts in these definitions. In this article, we show how they might have different and inadequate meanings according to the alternative chosen for GAL quantification. Note the emphasis (by means of boldface) on the role played by the quantification in each definition.

**Definition 1.** A subgame perfect equilibrium of an extensive game \( \Gamma = \langle N, H, P, (u_i) \rangle \) is a strategy profile \( s^* = (s^*_1, \ldots, s^*_n) \) such that for every player \( i \in N \) and every history \( h \in H \) for which \( P(h) = i \) we have

\[
u_i(O_h(h, s^*_1, \ldots, s^*_n)) \geq u_i(O_h(h, s^*_1, \ldots, s_i, \ldots, s^*_n)),
\]

for every strategy \( s_i \in S_i \).

**Definition 2.** A subgame perfect equilibrium of an extensive game \( \Gamma = \langle N, H, P, (u_i) \rangle \) is a strategy profile \( s^* = (s^*_1, \ldots, s^*_n) \) such that for every player \( i \in N \), every strategy \( s_i \in S_i \) and every history \( h \in H \) for which \( P(h) = i \) we have

\[
u_i(O_h(h, s^*_1, \ldots, s^*_n)) \geq u_i(O_h(h, s^*_1, \ldots, s_i, \ldots, s^*_n))
\]
Definition 3. A Nash equilibrium of an extensive game \(\Gamma = (N, H, P, (u_i))\) is a strategy profile \(s^* = (s_1^*, \ldots, s_n^*)\) such that for every player \(i \in N\) and every history on the path of the strategy profile \(s^*\) (i.e. \(h \in O(s^*)\)) for which \(P(h) = i\) we have
\[
u_i(O_h(h, s_1^*, \ldots, s_n^*)) \geq u_i(O_h(h, s_1^*, \ldots, s_i^*, \ldots, s_n^*))\]
for every strategy \(s_i \in S_i\).

Definition 4. A Nash equilibrium of an extensive game \(\Gamma = (N, H, P, (u_i))\) is a strategy profile \(s^* = (s_1^*, \ldots, s_n^*)\) such that for every player \(i \in N\), every strategy \(s_i \in S_i\) and every history on the path of the strategy profile \(s^*\) (i.e. \(h \in O(s^*)\)) for which \(P(h) = i\) we have
\[
u_i(O_h(h, s_1^*, \ldots, s_n^*)) \geq u_i(O_h(h, s_1^*, \ldots, s_i^*, \ldots, s_n^*))\]

This work is divided into 4 parts: Section 2 introduces Game Analysis Logic; Section 3 presents extensive games as well their correspondence in the GAL logic; and, finally, Section 4 concludes this work.

2 Game Analysis Logic (GAL)

We model and analyze games using a many-sorted modal first-order logic language, called Game Analysis Logic (GAL), that is a logic based on the standard Computation Tree Logic (CTL) [3]. A game is a model of GAL, called game analysis logic structure, and an analysis is a formula of GAL.

The games that we model are represented by a set of states \(SE\) and a set of actions \(CA\).

A state \(e \in SE\) is defined by both a first-order interpretation and a set of players, where: 1- The first-order interpretation is used to represent the choices and the consequences of the players’ decisions. For example, we can use a list to represent the history of the players’ choices until a certain state; 2- The set of players represents the players that have to decide simultaneously at a state. This set must be a subset of the players’ set of the game. The other players cannot make a choice at this state. For instance, we can model games such as auction games, where all players are in all states, or even games as Chess or turn-based synchronous game structure, where only a single player has to make a choice at each state. Notice that we may even have some states where none of the players can make a decision that can be seen as states of the nature.

An action is a relation between two states \(e_1\) and \(e_2\), where all players in the state \(e_1\) have commit themselves to move to the state \(e_2\). Note that this is an extensional view of how the players commit themselves to take a joint action. Of course, this can have an intentional view and be expressed in a formal language.

We refer to \((A_k)_{k \in K}\) as a sequence of \(A_k\)'s with the index \(k \in K\). Sometimes we will use more than one index as in the example \((A_k, l)_{k \in K \times L}\). We can also use \((A_k, B_l)_{k \in K, l \in L}\) to denote the sequence of \((A_k)_{k \in K}\) followed by the sequence \((B_l)_{l \in L}\). Throughout of this article, when the sets of indexes are clear in the context, we will omit them.

A path \(\pi(e)\) is a sequence of states (finite or infinite) that could be reached through the set of actions from a given state \(e\) that has the following properties: 1- The first element of the sequence is \(e\); 2- If the sequence is infinite \(\pi(e) = (e_k)_{k \in \mathbb{N}}\), then \(\forall k \geq 0\ we have \langle e_k, e_{k+1} \rangle \in CA\); 3- If the sequence is finite \(\pi(e) = (e_0, \ldots, e_i)\), then \(\forall k\ such that \ 0 \leq k < l \ we have \langle e_k, e_{k+1} \rangle \in CA\) and there is no \(e'\) such that \(\langle e_l, e' \rangle \in CA\). The game behavior is characterized by its paths that can be finite or infinite. Finite paths end in a state where the game is over, while infinite ones represent a game that will never end.

Below we present the formal syntax and semantics of GAL. As usual, we call the sets of sorts \(S_i\), predicate symbols \(P\), function symbols \(F\) and players \(N\) as a non-logic language in contrast to the logic language that contains the quantifiers and the connectives. We use the notation \(T_s\), where \(s\) is a sort, to denote the set of sorts of term \(s\) defined in a standard way. The modalities can be read as follows.

- \([EX]\alpha\) - ‘exists a path \(\alpha\) in the next state’
- \([EF]\alpha\) - ‘exists a path \(\alpha\) in the future’
- \([EG]\alpha\) - ‘exists a path \(\alpha\) globally’
- \([E\alpha\beta]\) - ‘exists a path \(\alpha\) until \(\beta\)’
- \([AX]\alpha\) - ‘for all paths \(\alpha\) in the next state’
- \([AF]\alpha\) - ‘for all paths \(\alpha\) in the future’
- \([AG]\alpha\) - ‘for all paths \(\alpha\) globally’
- \([A\alpha\beta]\) - ‘for all paths \(\alpha\) until \(\beta\)’
Definition 5 (Syntax of GAL). Let \( \langle S, F, P, N \rangle \) be a non-logic language, and \( t_1, \ldots, t_n \) be terms of sorts \( s_1, \ldots, s_n \), and \( P : s_1 \ldots s_n \) be a predicate symbol, and \( i \) be a player, and \( x_s \) be a variable of sort \( s \). The logic language of GAL is generated by the following BNF definition:

\[
\Phi ::= \top \mid i \mid P(t_1, \ldots, t_n) \mid (t_1 \approx t'_1) \mid (\neg \Phi) \mid (\Phi \land \Phi) \mid (\Phi \lor \Phi) \mid (\Phi \rightarrow \Phi) \mid \exists x_\Phi \mid \forall x_\Phi \\
\mid [EX]\Phi \mid [AX]\Phi \mid [EF]\Phi \mid [AF]\Phi \mid [AG]\Phi \mid E(\Phi \cup \Phi) \mid A(\Phi \cup \Phi)
\]

It is well-known that the operators \( \lor, \land, \top, \bot, [AF], [EF], [AG], [EG] \) and \([EX]\) can be given by the following usual abbreviations.

\[
\begin{align*}
\alpha \land \beta & \iff (\alpha \rightarrow \beta) \\
[EX]\alpha & \iff \neg[AG]\neg \alpha \\
[AF]\alpha & \iff A(\top \cup \alpha) \\
[EF]\alpha & \iff E(\top \cup \alpha) \\
[AG]\alpha & \iff \neg E(\top \cup \neg \alpha) \\
[EG]\alpha & \iff \forall x(\alpha(x) \iff \neg \exists x \neg \alpha(x))
\end{align*}
\]

Definition 6 (Structure of GAL). Let \( \langle S, F, P, N \rangle \) be a non-logic language of GAL. A Game Analysis Logic Structure for this non-logic language is a tuple \( G = \langle SE, SE_0, CA, (D_s), (F_{f,e}), (P_{p,e}), (N_e) \rangle \) such that:

- \( SE \) is a non-empty set, called the set of states.
- \( SE_0 \) is a set of initial states, where \( SE_0 \subseteq SE \).
- For each state \( e \in SE, N_e \) is a subset of \( N \).
- \( CA \subseteq SE \times SE \), the set of actions of the game\(^3\), in which if there is at least one player in the state \( e_1 \), then exists a state \( e_2 \) such that \( \langle e_1, e_2 \rangle \in CA \).
- For each sort \( s \in S, D_s \) is a non-empty set, called the domain of sort \( s \)\(^4\).
- For each function symbol \( f : w \rightarrow s \) of \( F \) and each state \( e \in SE \), \( F_{f,e} \) is a function such that

\[
F_{f,e} : \left( \prod_{s_e \in w} D_{s_e} \right) \rightarrow D_s.
\]
- For each predicate symbol \( p : w \rightarrow P \) and each state \( e \in SE \), \( P_{p,e} \) is a relation such that

\[
P_{p,e} \subseteq \left( \prod_{s_e \in w} D_{s_e} \right).
\]

A GAL-structure is of finite model if the set of states \( SE \) and each set of domains \( D_s \) are finite. Otherwise, it is of infinite model. Note that even when a GAL-structure is finite we might have infinite paths.

In order to provide the semantics of GAL, we define a valuation function as a mapping \( \sigma_s \) that assigns to each free variable \( x_s \) of sort \( s \) some member \( \sigma_s(x_s) \) of domain \( D_s \). As we use terms, we extend every function \( \sigma_s \) to a function \( \bar{\sigma}_s \) from state and term to element of sort \( s \) that is done in a standard way. When the valuation functions are not necessary, we will omit them.

Definition 7 (Semantics of GAL). Let \( G = \langle SE, SE_0, CA, (D_s), (F_{f,e}), (P_{p,e}), (N_e) \rangle \) be a GAL-structure, and \( (\sigma_s) \) be valuation functions, and \( \alpha \) be a GAL-formula, where \( s \in S, f \in F, p \in P \) and \( e \in SE \). We write \( G, (\sigma_s) \models_e \alpha \) to indicate that the state \( e \) satisfies the formula \( \alpha \) in the structure \( G \) with valuation functions \( (\sigma_s) \). The formal definition of satisfaction \( \models_e \) proceeds as follows:

\[
\begin{align*}
G, (\sigma_s) & \models_e \top \\
G, (\sigma_s) & \models_e i \iff i \in N_e \\
G, (\sigma_s) & \models_e p(t_1^{s_1}, \ldots, t_n^{s_n}) \iff (\bar{\sigma}_s(e, t_1^{s_1}), \ldots, \bar{\sigma}_s(e, t_n^{s_n})) \in P_{p,e} \\
G, (\sigma_s) & \models_e (t_1 \approx t'_1) \iff \sigma_s(e, t_1) = \sigma_s(e, t'_1) \\
G, (\sigma_s) & \models_e \neg \alpha \iff \neg G, (\sigma_s) \models_e \alpha \\
G, (\sigma_s) & \models_e (\alpha \rightarrow \beta) \iff IF G, (\sigma_s) \models_e \alpha \THEN G, (\sigma_s) \models_e \beta \\
G, (\sigma_s) & \models_e [AX] \alpha \iff \forall e' \in SE \text{ such that } (e, e') \in CA \text{ we have } G, (\sigma_s) \models_e \alpha \text{ (see Figure 2.a).} \\
G, (\sigma_s) & \models_e [E \alpha \cup \beta] \iff \exists \text{ a finite (or infinite) path } \pi(e) = (e_0 e_1 e_2 \ldots e_i), \text{ such that exists a } k \text{ where } k \geq 0, \text{ and } G, (\sigma_s) \models_{e_k} \beta, \text{ and for all } j \text{ where } 0 \leq j < k, \text{ and } G, (\sigma_s) \models_{e_j} \alpha \text{ (see Figure 2.b).}
\end{align*}
\]

\(^3\) This relation is not required to be total as in the CTL case. The idea is because we have finite games.

\(^4\) In algebraic terminology \( D_s \) is a carrier for the sort \( s \).
\[ G, (\sigma_s) \models_e A(\alpha U \beta) \iff \text{for all finite (and infinite) paths such that } \pi(e) = (e_0e_1e_2...e_i), \text{ exists a } k \text{ where } k \geq 0, \text{ and } G, (\sigma_s) \models_{e_k} \beta, \text{ and for all } j \text{ where } 0 \leq j < k, \text{ and } G, (\sigma_s) \models_{e_j} \alpha \text{ (see Figure 2.c).} \]

\[ G, (\sigma_s, \sigma_s k) \models_e \exists \exists x \models_e \alpha \iff \exists \exists \exists d \in D_{s_k} \text{ such that } G, (\sigma_s, \sigma_s k(x_{s_k}|d)) \models_e \alpha, \text{ where } \sigma_s k(x_{s_k}|d) \text{ is the function which is exactly like } \sigma_s k \text{ except for one thing: At the variable } x_{s_k} \text{ it assumes the value } d. \text{ This can be expressed by the equation:} \]

\[
\sigma_s(x_{s_k}|d)(y) = \begin{cases} 
\sigma_s(y), & \text{if } y \neq x_{s_k} \\
\sigma_s(x_{s_k}), & \text{if } y = x_{s_k}
\end{cases}
\]

Fig. 2. Modal Connectives of GAL.

It is well-known that there is no complete and sound Deductive System for a first-order CTL [11]. Thus, GAL is also non-axiomatizable. However, we argue that we can reason about games using a model checking approach for GAL. In [16], we present a prototype of a model-checker for GAL, namely GALV, that has been developed according to the main intentions of the approach advocated here. GALV is available for download at www.tecmf.inf.puc-rio.br/DaviRomero.

One of the main problematic issues in the first-order modal context is the interaction between the modal operators and the quantifiers. In order to see that, consider the following two formulas.

\[ EF \forall x \text{Free}(x) \] (1)
\[ \forall x[EF]\text{Free}(x) \] (2)

Formula 1 asserts that at some day everybody will simultaneously be free. On the other hand, formula 2 asserts that everybody will be free at some day, but it does not imply that everybody will simultaneously be free. It should be clear that formula 1 implies formula 2, but the converse does not hold.

When the quantifier appears before the modal operator, we say that the quantification is \textit{de re}. On the other hand, when the quantifier appears after the modal operator, we say that the quantification is \textit{de dicto}.

Figure 3 sums up the relationship between the \textit{de re} and \textit{de dicto} in GAL. Note that in the existential quantification always holds that \textit{de re} implies \textit{de dicto}, and the universal quantification always holds that \textit{de dicto} implies \textit{de re}. For the purpose of this article we are concerned with the relationship between the universal quantification and the operators \([EF]x\) and \([AG]x\). In the first case, \textit{de dicto} implies \textit{de re}, but it does not hold the converse. On the other hand, for the operator \([AG]x\) we have an equivalence between the two alternatives of quantification.

3 Game Theory in Game Analysis Logic

In this section we present the correspondence between the extensive games and the GAL-structures as well as the solution concepts of NE and SPE and the formulas of GAL.

An extensive game is a model in which each player can consider his plan of action at every time of the game at which he or she to make a choice. There are two kinds of models: game with perfect information; and games with imperfect information. For the sake of simplicity we restrict the games to models of perfect information. A general model that allows imperfect information is straightforward. Below we present the formal definition and the example shown in Figure 1.a.
Definition 8. An extensive game with perfect information is a tuple \( \langle N, H, P, (u_i) \rangle \), where

- \( N \) is a set, called the set of players.
- \( H \) is a set of sequences of actions (finite or infinite), called the set of histories, that satisfies the following properties
  - the empty sequence is a history, i.e., \( \emptyset \in H \).
  - if \( (a_k)_{k \in K} \in H \) where \( K \subseteq \mathbb{N} \) and for all \( l \leq |K| \), then \( (a_k)_{k=0,...,l} \in H \).
  - if \((a_0...a_k) \in H \) for all \( k \in \mathbb{N} \), then the infinite sequence \((aa_1...) \in H \).

A history \( h \) is terminal if it is infinite or it has no action such that \((h,a) \in H \). We refer to \( T \) as the set of terminals.
- \( P \) is a function that assigns to each non-terminal history a player.
- For each player \( i \in N \), a utility function \( u_i \) on \( T \).

Example 1. An example of a two-player extensive game \( \langle N, H, P, (u_i) \rangle \), where:
\[
N = \{1,2\}; \quad H = \{\emptyset, (A), (B), (A,L), (A,R)\}; \quad P(\emptyset) = 1 \text{ and } P((A)) = 2; \quad u_1((B)) = 1, \quad u_2((A,L)) = 0, \quad u_2((A,R)) = 0
\]

A strategy of player \( i \) is a function that assigns an action for each non-terminal history for each \( P(h) = i \). For the purpose of this article, we represent a strategy as a tuple. In order to avoid confusing when we refer to the strategies or the histories, we use ‘(‘ and ‘)’ to the strategies and ‘(‘ and ‘)’ to the histories. In Example 1, Player 1 has to make a decision only at the initial state and he or she has two strategies \( (A) \) and \( (B) \). Player 2 has to make a decision after the history \( (A) \) and he or she has two strategies \( (L) \) and \( (R) \). We denote \( S_i \) as the set of player \( i \)’s strategies. We denote \( s = (s_i) \) as a strategy profile. We refer to \( O(s_1,...,s_n) \) as an outcome that is the terminal history when each player follows his or her strategy \( s_i \). In Example 1, \( \langle (B), (L) \rangle \) is a strategy profile in which Player 1 chooses \( B \) after the initial state and Player 2 chooses \( L \) after the history \( (A) \), and \( O((B), (L)) \) is the outcome \( (B) \). In a similar way, we refer to \( O_h(h,s_1,...,s_n) \) as the outcome when each player follows his or her strategy \( s_i \) from history \( h \). In Example 1, \( O_h((A), (B), (L)) \) is the outcome \( (A, L) \) and \( u_1(O_h((A), (B), (L))) = u_1((A,L)) = 0 \).

We invite the reader to verify that the outcomes \( \langle (A), (R) \rangle \) and \( \langle (B), (L) \rangle \) are the Nash equilibria in Example 1. Game theorists can argue that the solution \( \langle (B), (L) \rangle \) is not reasonable when the players regard to the sequence of the actions. To see that the reader must observe that after the history \( (A) \) there is no way for Player 2 commit himself or herself to choose \( L \) instead of \( R \) since he or she will be better off choosing \( R \) (his or her utility is 1 instead of 0). Thus, Player 2 has an incentive to deviate from the equilibrium, so this solution is not a subgame perfect equilibrium. On the other hand, we invite the reader to verify that the solution \( \langle (A), (R) \rangle \) is the only subgame perfect equilibrium.

We can model an extensive game \( \Gamma = \langle N, H, P, (u_i) \rangle \) as a GAL-structure in the following way. Each history \( h \in H \) (from the extensive game) is represented by a state, in which a 0-ary symbol \( h \) designates a history of \( \Gamma \) (the one that the state is coming from), so \( h \) is a non-rigid designator. The set of the actions of the GAL-structure is determined by the set of actions of each history, i.e., given a history \( h \in H \) and an action \( a \) such that \((h,a) \in H \), then the states namely \( h \) and \((h,a) \) are in the set of actions of the GAL-structure, i.e. \( \langle h, (h,a) \rangle \in CA \). Function \( P \) determines the player that has to make a choice at every state, i.e. \( N_h = \{ P(h) \} \). The utilities functions are rigidly defined as in the extensive game. The initial state is the state represented.
The GAL-structure of Example 1 is 
\[ \langle H, O, H, (\sigma_i) \rangle \]
where each \( \sigma_i \) is a rigidly defined as in the extensive game. 

Consider the following formulas as expressing subgame perfect equilibrium definitions 1 and 2, respectively. A strategy profile \( s^* = (s_1^*, \ldots, s_n^*) \) is a SPE if and only if formula 3 (or formula 4) holds at the initial state \( 0 \), where each \( \sigma_S(v^*_{s_i}) = s_i^* \). This is in fact the case, if one verifies by the mapping from extensive games into GAL models. In fact, taking de dicto as well de re into account for \([AG]\) the formulas are equivalent each other.

\[
[AG] \left( \bigwedge_{i \in N} i \rightarrow \forall v_{s_i} \left( u_i(O(h, v_{s_1}^*, \ldots, v_{s_n}^*)) \geq u_i(O(h, v_{s_1}, \ldots, v_{s_n}, \ldots, v_{s_n}^*)) \right) \right) 
\]

\[ \forall v_{s_1}, \ldots, v_{s_n} [AG] \left( \bigwedge_{i \in N} i \rightarrow \left( u_i(O(h, v_{s_1}, \ldots, v_{s_n}^*)) \geq u_i(O(h, v_{s_1}^*, \ldots, v_{s_n}, \ldots, v_{s_n}^*)) \right) \right) \]

On the other hand, formulas 5 and 6 expressing Nash equilibrium according to definitions 3 and 4, respectively, are not equivalent under both interpretation for quantification for \([EG]\). As this relationship might suggest, formula 5 represents NE, while formula 6 does not represent. Although, both formulas represent NE because both formulas take always the same path (the equilibrium’s path). Thus, a strategy profile \( s^* = (s_1^*, \ldots, s_n^*) \) is a NE if and only if formula 5 (or formula 6) holds at the initial state \( 0 \), where each \( \sigma_S(v^*_{s_i}) = s_i^* \).

\[
[EG] \left( \bigwedge_{i \in N} i \rightarrow \forall v_{s_i} \left( u_i(O(h, v_{s_1}^*, \ldots, v_{s_n}^*)) \geq u_i(O(h, v_{s_1}, \ldots, v_{s_n}, \ldots, v_{s_n}^*)) \right) \right) 
\]

\[ \forall v_{s_1}, \ldots, v_{s_n} [EG] \left( \bigwedge_{i \in N} i \rightarrow \left( u_i(O(h, v_{s_1}, \ldots, v_{s_n}^*)) \geq u_i(O(h, v_{s_1}^*, \ldots, v_{s_n}, \ldots, v_{s_n}^*)) \right) \right) \]

In order to guarantee the correctness of the representation of both subgame perfect equilibrium and Nash equilibrium, we state the theorem below. Proof is provided in Appendix A.

**Theorem 1.** Let \( \Gamma \) be an extensive game, and \( \mathcal{G}_T \) be a GAL-structure for \( \Gamma \), and \( \alpha \) be a subgame perfect equilibrium formula for \( \mathcal{G} \) as defined in equation 3 (or in equation 4), and \( \beta \) be a Nash equilibrium formula as defined in equation 5 (or in equation 6), and \( (s_1^*) \) be a strategy profile,

\[ \text{Note that this set is finite if the game is finite.} \]
and \((\sigma_{S_i})\) be valuations functions for sorts \((S_i)\).

- A strategy profile \((s_i^*)\) is a SPE if \(\mathcal{G}_{\mathcal{R}_i}(\sigma_{S_i}) \models \alpha\), where each \(\sigma_{S_i}(v_i^*) = s_i^*\)
- A strategy profile \((s_i^*)\) is a NE if \(\mathcal{G}_{\mathcal{R}_i}(\sigma_{S_i}) \models \beta\), where each \(\sigma_{S_i}(v_i^*) = s_i^*\)

4 Conclusion

In this work, we have used a first-order modal logic (GAL) to model and reason about extensive games, in which a game is a model of GAL and a solution concept is a formula. As one can expect, quantifying in the context of extensive games might be troublesome. However, for the most used solution concepts, namely Nash equilibrium (NE) and subgame perfect equilibrium (SPE), the alternatives of quantification \emph{de re} and \emph{de dicto} are equivalent each other. For SPE, the equivalence is given by the equivalence for \([AG]\). On the other hand, the formulas for NE are not equivalent under both interpretation for quantification for \([EG]\); however, the equivalence for NE is true, since the quantification is taken by a specific path (the equilibrium’s path). As a future work, we intend to characterize the solution concept of iterated elimination of weakly dominated strategies (forward induction) by means of the structure of an extensive game, and, in addition, study the alternatives of quantification.

References

Appendix A - Proof of Theorem 1

Here we prove the correctness of the definitions 3 and 4 of Nash equilibrium (see lemma below) as well as the correspondence in the GAL logic (see theorem below).

Lemma 1. The following assertions are equivalent each other

1. For every player \( i \in N \) we have \( u_i(O(s_1^i, \ldots, s_n^i)) \geq u_i(O(s_1^i, \ldots, s_i, \ldots, s_n^i)) \) for every \( s_i \in S_i \).
2. For every \( h \in O(s_1^* \ldots, s_n^*) \) in which \( P(h) = i \) we have \( u_i(O_h(h, s_1^i, \ldots, s_n^i)) \geq u_i(O_h(h, s_1^i, \ldots, s_i, s_1^i, \ldots, s_n^i)) \) for every \( s_i \in S_i \).
3. For every \( s_i \in S_i \) and every \( h \in O(s_1^*, \ldots, s_n^*) \) in which \( P(h) = i \) we have \( u_i(O_h(h, s_1^i, \ldots, s_n^i)) \geq u_i(O_h(h, s_1^i, \ldots, s_i, s_1^i, \ldots, s_n^i)) \).

Proof. - (1 \( \implies \) 2) Suppose by contradiction that NOT for every \( h \in O(s_1^i, \ldots, s_n^i) \) in which \( P(h) = i \) we have \( u_i(O_h(h, s_1^i, \ldots, s_n^i)) \geq u_i(O_h(h, s_1^i, \ldots, s_i, s_1^i, \ldots, s_n^i)) \) for every \( s_i \in S_i \). Thus, there is a \( h \in O(s_1^i, \ldots, s_n^i) \) in which \( P(h) = i \) we have that there is a strategy \( s_i \in S_i \) such that \( u_i(O_h(h, s_1^i, \ldots, s_n^i)) < u_i(O_h(h, s_1^i, \ldots, s_i, s_1^i, \ldots, s_n^i)) \). On the other hand, we have by hypothesis that \( u_i(O(s_1^i, \ldots, s_n^i)) \geq u_i(O(s_1^i, \ldots, s_i, s_1^i, \ldots, s_n^i)) \) for every \( s_i \in S_i \), which means that \( u_i(O_h(h, s_1^i, \ldots, s_n^i)) \geq u_i(O_h(h, s_1^i, \ldots, s_i, s_1^i, \ldots, s_n^i)) \).

- (2 \( \implies \) 1) Suppose by contradiction that NOT for every \( i \in N \) we have \( u_i(O(s_1^i, \ldots, s_n^i)) \geq u_i(O(s_1^i, \ldots, s_i, s_1^i, \ldots, s_n^i)) \) for every \( s_i \in S_i \). Then, there is a player \( i \in N \) such that \( u_i(O(s_1^i, \ldots, s_n^i)) < u_i(O(s_1^i, \ldots, s_i, s_1^i, \ldots, s_n^i)) \) for some \( s_i \in S_i \). Now consider the following cases: Player \( i \) does not make a move at a history \( h \in O(s_1^i, \ldots, s_n^i) \), and hence he or she cannot change his or her utility, i.e. \( u_i(O(s_1^i, \ldots, s_n^i)) = u_i(O(s_1^i, \ldots, s_i, s_1^i, \ldots, s_n^i)) \). Player \( i \) makes a move at a history \( h \in O(s_1^i, \ldots, s_n^i) \), which means that there is a history \( h \in O(s_1^i, \ldots, s_n^i) \) in which \( P(h) = i \) such that \( u_i(O_h(h, s_1^i, \ldots, s_n^i)) < u_i(O_h(h, s_1^i, \ldots, s_i, s_1^i, \ldots, s_n^i)) \) (contradiction with the hypothesis of 1).

Proof 1 \( \implies \) 3 is similar to 1 \( \implies \) 2.
Proof 3 \( \implies \) 1 is similar to 2 \( \implies \) 1.

Theorem 2. Let \( \Gamma \) be an extensive game, and \( G_{\Gamma} \) be a GAL-structure for \( \Gamma \), and \( \alpha \) be a subgame perfect equilibrium formula for \( G \) as defined in equation 3 (or in equation 4), and \( \beta \) be a Nash equilibrium formula as defined in equation 5, and \( (s_1^*) \) be a strategy profile, and \( (\sigma_S) \) be valuations functions for sorts \( (S) \).

1. A strategy profile \( (s_1^*) \) is a SPE \( \iff \) \( G_{\Gamma},(\sigma_S) \models \alpha \), where each \( \sigma_S(v_1^*) = s_1^* \)
2. A strategy profile \( (s_1^*) \) is a NE \( \iff \) \( G_{\Gamma},(\sigma_S) \models \beta \), where each \( \sigma_S(v_1^*) = s_1^* \)

Proof. 1. A strategy profile \( (s_1^*) \) is a SPE \( \iff \) \( G_{\Gamma},(\sigma_S) \models \alpha \), where each \( \sigma_S(v_1^*) = s_1^* \)

Due to the equivalence between de re and de dicto of the universal quantification of the operator \( [AG] \), we prove de dicto only.

A strategy profile \( (s_1^*) \) is a SPE of \( \Gamma \):
\[ \iff \] for every player \( i \) and every history \( h \in H \) for which \( P(h) = i \) we have \( u_i(O_h(h, s_1^i, \ldots, s_n^i)) \geq u_i(O_h(h, s_1^i, \ldots, s_i, s_1^i, \ldots, s_n^i)) \), for every strategy \( s_i \in S_i \).

By the definition of \( G_{\Gamma} \) from \( \Gamma \), we have that every state of \( G_{\Gamma} \), which represents a history of \( \Gamma \), is reached by a path from the initial state \( \emptyset \); moreover, we have that each domain of player \( i \)'s strategy \( D_S \) is interpreted by the set of strategies \( S_i \) (i.e. \( D_S = S_i \)), and the player that has to take a move in a state \( e_k \), which represents the history \( h_k \), is defined by the function \( P \) (i.e. \( N_e = \{ P(h_k) \} \)), and, finally, the symbol \( h \) is interpreted in \( e_k \) by the history \( h_k \) (i.e. \( \sigma_H(e_k, h) = h_k \)). As a consequence of this definition, we have
\[ \iff \] for all paths \( \pi(\emptyset) = e_0, e_1, \ldots \) and for all \( k \geq 0 \), for every player \( i \in N \) such that IF \( i \in N_e \) THEN we have for all \( d_i \in D_S \)
\( (u_i(O_h(\sigma_H(e_k, h), s_1^i, \ldots, s_n^i)) \geq u_i(O_h(\sigma_H(e_k, h), s_1^i, \ldots, d_i, \ldots, s_n^i))) \).
As function $O_h$ and utility functions $(u_i)$ are rigidly interpreted as in the extensive game $\Gamma$, we have

\[ \iff \text{ for all paths } \pi(\emptyset) = e_0, e_1, \ldots \text{ and for all } k \geq 0, \text{ for every player } i \in N \text{ such that } \]

\[ \text{if } G_r, (\sigma_i) \models e_k i \text{ then for all } d_i \in \mathcal{D}_i, \text{ we have } \]

\[ G_r, (\sigma_i(v_s, d_i)) \models e_k \left( u_i(O_h(h, v_{S_1}^*, \ldots, v_{S_n}^*)) \geq u_i(O_h(h, v_{S_1}^*, \ldots, v_{S_n}^*)) \right), \]

\[ \text{where each } \sigma_i(v_{S_i}^*) = s_i^* . \]

\[ \iff \text{ for all paths } \pi(\emptyset) = e_0, e_1, \ldots \text{ and for all } k \geq 0, \text{ we have } \]

\[ G_r, (\sigma_i) \models e_k \left( \bigwedge_{i \in N} i \rightarrow \forall v_{S_i} \left( u_i(O_h(h, v_{S_1}, \ldots, v_{S_n}^*)) \geq u_i(O_h(h, v_{S_1}^*, \ldots, v_{S_n}^*)) \right) \right), \]

\[ \text{where each } \sigma_i(v_{S_i}^*) = s_i^* . \]

2. A strategy profile $(s_1^*, \ldots, s_n^*)$ is a NE $\iff G_r, (\sigma_i) \models \emptyset \beta$, where each $\sigma_i(v_{S_i}^*) = s_i^* . \]

Despite the fact that the alternatives of quantification for $[EG]$ are not equivalent each other, formula 5 and 6 express NE. The proofs are similar each other and use the lemma above, which guarantees the correctness of the definitions 3 and 4. Thus, we present de dicto alternative only.

(a)

A strategy profile $(s_1^*, \ldots, s_n^*)$ is a NE of $\Gamma$.

\[ \iff \text{ for every player } i \text{ and every history } h \in O(s_i^*) \text{ in which } P(h) = i \text{ we have } \]

\[ u_i(O(s_1^*, \ldots, s_n^*)) \geq u_i(O(s_1^*, \ldots, s_i, s_i^*)), \text{ for every strategy } s_i \in S_i. \]

We take the path $\pi(\emptyset) = e_0, e_1, \ldots$ in $G_r$ that is defined by histories $h_0, h_1, \ldots$ on the equilibrium’s path $O(s_1^*, \ldots, s_n^*)$ according to definition of $G_r$ from $\Gamma$. Thus, we have

\[ \iff \text{ there is a path } \pi(\emptyset) = e_0, e_1, \ldots \text{ such that for all } k \geq 0 \text{ we have } \sigma_H(e_k, h) \in O(s_1^*, \ldots, s_n^*) \]

\[ \text{AND for every player } i \in N \text{ IF } i \in N_{e_k} \text{ THEN for all } s_i \in S_i \text{ we have } \]

\[ u_i(O_h(\bar{h}, v_{S_1}^*, \ldots, v_{S_n}^*)) \geq u_i(O_h(\bar{h}, v_{S_1}^*, \ldots, v_{S_n}^*)) \],

\[ \text{where each } \sigma_i(v_{S_i}^*) = s_i^* . \]

As function $O$, $O_h$ and utility functions $(u_i)$ are rigidly interpreted as in the extensive game $\Gamma$, we have

\[ \iff \text{ def } \text{ there is a path } \pi(\emptyset) = e_0, e_1, \ldots \text{ such that for all } k \geq 0 \text{ we have } \]

\[ G_r, (\sigma_i) \models e_k h \in O(v_{S_1}, \ldots, v_{S_n}) \text{ AND } \]

\[ G_r, (\sigma_i) \models e_k \bigwedge_{i \in N} i \rightarrow \forall v_{S_i} \left( u_i(O_h(h, v_{S_1}, \ldots, v_{S_n}^*)) \geq u_i(O_h(h, v_{S_1}^*, \ldots, v_{S_n}^*)) \right), \]

\[ \text{where each } \sigma_i(v_{S_i}^*) = s_i^* . \]

\[ \iff \]

\[ G_r, (\sigma_i) \models [\emptyset] \left( h \in O(v_{S_1}, \ldots, v_{S_n}) \right. \wedge \]

\[ \left. \bigwedge_{i \in N} i \rightarrow \forall v_{S_i} \left( u_i(O_h(h, v_{S_1}, \ldots, v_{S_n}^*)) \geq u_i(O_h(h, v_{S_1}^*, \ldots, v_{S_n}^*)) \right) \right) \]

\[ \text{where each } \sigma_i(v_{S_i}^*) = s_i^* . \]