Monadic distributive lattices and monadic augmented
Kripke frames

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Abstract
In this article, we continue the study of monadic distributive lattices
(or m-lattices) which are a natural generalization of monadic Heyting
algebras, introduced by Monteiro and Varsavsky and developed ex-
austively by Bezhanishvili. First, we extended the duality obtained
by Cignoli for $Q$-distributive lattices to m-lattices. This new duality
allows us to describe in a simple way the subdirectly irreducible alge-
bras in this variety. Next, we introduce the category $mK\mathcal{F}$ whose ob-
jects are monadic augmented Kripke frames and whose morphisms are
increasing continuous functions verifying certain additional conditions
and we prove that it is equivalent to the one obtained above. Finally,
we show that the category of perfect augmented Kripke frames given
by Bezhanishvili for monadic Heyting algebras is a proper subcategory
of $mK\mathcal{F}$.

1 Preliminaries
In this section, in order to simplify reading, we summarize the fundamental
concepts we use.

Let $X, Y$ be sets. Given a relation $R \subseteq X \times Y$, for each $Z \subseteq X$, $R(Z)$
will denote the image of $Z$ by $R$. If $Z = \{x\}$, we shall write $R(x)$ instead of
$R(\{x\})$. Moreover, for each $V \subseteq Y$, $R^{-1}(V)$ will denote the inverse image of
$V$ by $R$, i.e. $R^{-1}(V) = \{x \in X : R(x) \cap V \neq \emptyset\}$. If $V = \{y\}$, we shall write
$R^{-1}(y)$ instead of $R^{-1}(\{y\})$. Besides, if $R, T \subseteq X \times X$ then the relation
$R \circ T$ is defined by putting $(x, y) \in R \circ T$ if and only if there is $z \in X$ such
that $(x, z) \in T$ and $(z, y) \in R$.

If $X$ is a poset (i.e. partially ordered set) and $Y \subseteq X$, then we shall
denote by $(Y \uparrow = \{y\})$ the set of all $x \in X$ such that $x \leq y$ ($y \leq x$) for
some $y \in Y$, and we shall say that $Y$ is increasing (decreasing) if $Y = \{y\}$
($Y = \{y\}$). Furthermore, $\max Y$ (min $Y$) will denote the set of maximal
(minimal) elements of $Y$. 

Recall that R. Cignoli [4] introduced the category $q\mathcal{L}$ of $Q$-distributive lattices and $Q$-homomorphisms, where a $Q$-distributive lattice is an algebra $(L, \vee, \wedge, \neg, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $\neg$ is a quantifier on $L$, that is a unary operator on $L$ which satisfies the following equalities:

$(M1) \neg 0 = 0, \quad (M2) x \wedge \neg x = x,$
$(M3) \neg(x \wedge y) = \neg x \wedge \neg y, \quad (M4) \neg(x \vee y) = \neg x \vee \neg y.$

We shall denote the objects in $q\mathcal{L}$ by $L$ or $(L, \neg)$.

In addition, this author extended Priestley duality [9, 10, 11] to the category $q\mathcal{L}$. To this aim, he considered the category $q\mathcal{P}$ whose objects are $q$-spaces and whose morphisms are $q$-functions. Specifically, a $q$-space is a pair $(X, E)$ such that $X$ is a Priestley space and $E$ is an equivalence relation on $X$ which satisfies the following conditions:

$(E1)$ $EU \in D(X)$ for each $U \in D(X)$ where $EU = \{y \in X : (x, y) \in E$ for some $x \in U\}$ and $D(X)$ is the set of all clopen (i.e. simultaneously closed and open) increasing subsets of $X$,
$(E2)$ the equivalence classes for $E$ are closed in $X$,

and a $q$-function from a $q$-space $(X, E)$ into another one $(X', E')$ is an order-preserving continuous function $f : X \rightarrow X'$ such that $E f^{-1}(U) = f^{-1}(E'U)$ for all $U \in D(X')$. Besides, he proved that $q\mathcal{L}$ is dually equivalent to $q\mathcal{P}$.

G. Bezhanishvili [2] developed the duality theory for monadic Heyting algebras. In order to determine one of these dualities, this author introduced the category $p\mathcal{AKF}$ of perfect augmented Kripke frames (or paK-frames) and their corresponding morphisms (or paK-functions), which we shall describe below.

A quadruple $(X, \Omega, R, E)$ is a perfect augmented Kripke frame if the following conditions are satisfied:

$(k1)$ $(X, R, E)$ is an augmented Kripke frame or equivalently

(i) $(X, R)$ is a non-empty partially ordered set,
(ii) $E$ is an equivalence relation on $X$,
(iii) $R \circ E \subseteq E \circ R$.

$(k2)$ $(X, \Omega, R)$ and $(X, \Omega, E \circ R)$ are perfect Kripke frames which means that if $T = R$ or $T = E \circ R$ then

(i) $(X, T)$ is a non-empty quasi-ordered set,
(ii) $(X, \Omega)$ is a Stone (i.e. 0-dimensional, compact and Hausdorff) space,
(iii) for all $x \in X$, $T(x)$ is closed in $X$,
(iv) for all clopen $A$ of $X$, $T^{-1}(A)$ is a clopen of $X$. 

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(k3) for all increasing clopen $A$ of $X$, $E(A)$ is a clopen of $X$.

Let $(X_1, \Omega_1, R_1, E_1)$ and $(X_2, \Omega_2, R_2, E_2)$ be paK-frames. A paK-function from $(X_1, \Omega_1, R_1, E_1)$ into $(X_2, \Omega_2, R_2, E_2)$ is a continuous function $f : X_1 \to X_2$ which verifies the following conditions:

(kf1) $f$ is strongly isotone with respect to $R_1$ and $E_1 \circ R_1$,
(kf2) $f$ is almost strongly isotone with respect to $E_1$.

Hence, the category of monadic Heyting algebras is dually equivalent to $pAKF$ (see [2]).

2 Subdirectly irreducible algebras in $\mathcal{M}$

Recall that [5] a monadic distributive lattice (or m–lattice) is a triple $(L, \Delta, \nabla)$ where $L$ is a bounded distributive lattice and $\Delta$, $\nabla$ are unary operations on $L$ satisfying the above mentioned identities M1–M4 and the following ones:

(M5) $\nabla \nabla x = \nabla x,$
(M6) $\Delta 1 = 1,$
(M7) $x \land \Delta x = \Delta x,$
(M8) $\Delta (x \land y) = \Delta x \land \Delta y,$
(M9) $\Delta \Delta x = \Delta x,$
(M10) $\nabla \Delta x = \Delta x,$
(M11) $\Delta \nabla x = \nabla x.$

The category of m–lattices and their corresponding homomorphisms will be denoted by $\mathcal{M}$.

In what follows firstly, we will determine a topological duality for these algebras which extends the results obtained in [4].

Definition 2.1 A $q$-space $(X, E)$ is an mq-space if the following conditions are fulfilled:

(mq1) $(x, y) \in E$ and $y \leq z$ imply that there is $w \in X$ such that $x \leq w$ and $(w, z) \in E$,
(mq2) for every $V \in D(X)$, $(E(X \setminus V))$ is a clopen subset of $X$.

Definition 2.2 Let $(X_1, E_1)$ and $(X_2, E_2)$ be $q$-spaces. An $mq$-function $f : X_1 \to X_2$ is a $q$-function which verifies

(mqf1) $(E_1(f^{-1}(X_2 \setminus V))) = f^{-1}((E_2(X_2 \setminus V)))$ for all $V \in D(X_2)$.

We will denote by $\mathcal{M}$ the category whose objects are $mq$-spaces and whose morphisms are $mq$-functions.

The properties of $mq$-spaces and $mq$-functions that follow are necessary to prove that $\mathcal{M}$ and $\mathcal{M}$ are dually equivalent.

Lemma 2.3 Let $(X, E)$ be a $q$-space. Then $E(A)$ is a closed set for every closed subset $A$ of $X$, i.e. $E$ is a closed relation.
**Lemma 2.4** Let $(X, E)$ be an $mq$-space. Then the following conditions are equivalent:

(i) $x \in X \setminus (E(X \setminus U)]$ for each $U \in D(X)$,

(ii) $E([x]) \subseteq U$.

**Proposition 2.5** (i) If $(X, E)$ is an $mq$-space, then $(D(X), \triangle_E, \nabla_E)$ is an $m$-lattice, where $\nabla_E(V) = E(V)$ and $\triangle_E V = X \setminus (E(X \setminus V)]$ for each $V \in D(X)$.

(ii) If $(I, \triangle, \nabla)$ is an $m$-lattice, then $(X(I), \sigma_I)$ is an $mq$-space, where $E_I = \{(P, T) \in X(I) \times X(I) : P \cap \nabla(L) = T \cap \nabla(L)\}$. Besides, $\sigma_I$ is an $m$-isomorphism.

Next, our attention is focused on studying some properties of order-preserving continuous functions between $mq$-spaces, in order to obtain another description of $mq$-functions.

**Lemma 2.6** Let $(X_1, E_1)$ and $(X_2, E_2)$ be $q$-spaces. If $f : X_1 \rightarrow X_2$ is a $q$-function then the following condition holds:

$$(qf_1) \ (E_1(f^{-1}(X_2 \setminus V)]) \subseteq f^{-1}((E_2(X_2 \setminus V)])$$

for each $V \in D(X_2)$.

**Proposition 2.7** Let $(X_1, E_1)$ and $(X_2, E_2)$ be $mq$-spaces. Then for each order-preserving continuous function $f : X_1 \rightarrow X_2$ the following conditions are equivalent:

$qf_1$) $(E_1(f^{-1}(X_2 \setminus V)]) \subseteq f^{-1}((E_2(X_2 \setminus V)])$ for each $V \in D(X_2)$,

$qf_2$) $f(E_1([x])) \subseteq E_2([f(x)])$ for each $x \in X_1$,

$qf_3$) $[f(E_1([x]))] \subseteq E_2([f(x)])$ for each $x \in X_1$.

**Proposition 2.8** Let $(X_1, E_1)$ and $(X_2, E_2)$ be $mq$-spaces. Then for each order-preserving continuous function $f : X_1 \rightarrow X_2$ the following conditions are equivalent:

$mqf_2$) $f^{-1}((E_2(X_2 \setminus V)]) \subseteq (E_1(f^{-1}(X_2 \setminus V)])$ for each $V \in D(X_2)$,

$mqf_3$) $E_2([f(x)]) \subseteq [f(E_1([x]))]$ for each $x \in X_1$.

The above results allow us to obtain the description of $mq$-functions we were looking for.

**Corollary 2.9** Let $(X_1, E_1)$ and $(X_2, E_2)$ be $mq$-spaces. Then for each order-preserving continuous function $f : X_1 \rightarrow X_2$ the following conditions are equivalent:

(i) $f$ is an $mq$-function,

(ii) $f$ is a $q$-function which verifies (mqf3),

(iii) $f$ is a $q$-function which verifies (mqf4): $E_2([f(x)]) = [f(E_1([x]))]$.
Next, we are going to characterize the isomorphisms in the category $\mathcal{mQ}$ for which Lemma 2.10 is fundamental.

**Lemma 2.10** Let $(X_1, E_1)$ and $(X_2, E_2)$ be $mq$-spaces. If $f : X_1 \rightarrow X_2$ is an isomorphism in $q\mathcal{P}$, then $E_2([f(x)]) = f(E_1([x]))$ for each $x \in X_1$.

**Proposition 2.11** Let $(X_1, E_1), (X_2, E_2)$ be $mq$-spaces and $f : X_1 \rightarrow X_2$ a function. Then the following conditions are equivalent:

(i) $f$ is an isomorphism in $q\mathcal{P}$,
(ii) $f$ is an isomorphism in $\mathcal{mQ}$.

From the above results and using the usual procedures we conclude

**Theorem 2.12** The category $\mathcal{mQ}$ is naturally equivalent to the dual of the category $\mathcal{M}$.

Next, we obtain a characterization of the congruence lattice on monadic distributive lattices by means of certain closed subsets of its associated $mq$-space. This fact allows us to describe the congruence lattice on $Q$-distributive lattices completing the results obtained in [4].

**Definition 2.13** Let $(X, E)$ be an $mq$-space. A subset $Y$ of $X$ is id-saturated provided that $\min E([y]) \cup \max (E(y)) \subseteq Y$ for all $y \in Y$.

**Theorem 2.14** Let $L \in \mathcal{M}$ and let $X(L)$ be the $mq$-space associated with $L$. Then, the lattice $C_{\text{idS}}(X(L))$ of closed and id-saturated subsets of $X(L)$ is isomorphic to the dual lattice $\text{ConM}(L)$ of $m$-congruences on $L$.

Then, as a direct consequence of the proof of Theorem 2.14, we obtain Corollary 2.16 which determines the congruences on $Q$-distributive lattices.

**Definition 2.15** Let $(X, E)$ be a $q$-space. A subset $Y$ of $X$ is i-saturated provided that $\max E(y) \subseteq Y$ for all $y \in Y$.

**Corollary 2.16** Let $L$ be a $Q$-distributive lattice and $X(L)$ the $q$-space associated with $L$. Then, the lattice $C_{\text{idS}}(X(L))$ of closed and $i$-saturated subsets of $X(L)$ is isomorphic to the dual lattice $\text{ConQ}(L)$ of $Q$-congruences on $L$.

Theorem 2.14 will be taken into account to describe the subdirectly irreducible members of $\mathcal{M}$.

**Corollary 2.17** Let $L \in \mathcal{M}$ and let $X(L)$ be the $mq$-space associated with $L$. If $Y$ is an id-saturated subset of $X(L)$, then $\overline{Y}$ is also id-saturated (where $\overline{Y}$ denotes the closure of $Y$).
Proposition 2.18 Let \((X, E)\) be an mq-space. Then the following conditions are equivalent:

(i) \(\nabla_{E}\) is the simple quantifier,
(ii) \(E = X \times X\).

Proposition 2.19 Let \((X, E)\) be an mq-space such that \(\nabla_{E}\) is the simple quantifier. Then for all non-empty subset \(Y\) of \(X\) the following conditions are equivalent:

(i) \(Y\) is id-saturated,
(ii) \(\min X \cup \max X \subseteq Y\).

The tools we shall employ to prove Theorem 2.22 are Corollary 2.20 and Lemma 2.21.

Corollary 2.20 Let \((X, E)\) be an mq-space such that \(\nabla_{E}\) is the simple quantifier. Then \(\min X \cup \max X\) is the lowest non-empty closed and id-saturated subset of \(X\).

Lemma 2.21 Let \((X, E)\) be an mq-space. Then the following conditions hold:

(i) for each \(x \in X\), \(E(|x|)\) is a closed and id-saturated subset of \(X\),
(ii) \(\nabla_{E}(U)\) is an id-saturated subset of \(X\) for all \(U \in D(X)\).

Theorem 2.22 Let \((X, E)\) be an mq-space. Then the following conditions are equivalent:

(i) \((D(X), \Delta_{E}, \nabla_{E})\) is a simple monadic distributive lattice,
(ii) \((\Delta_{E}, \nabla_{E})\) is simple and \(X = \min X \cup \max X\).

Lemma 2.23 Let \((X, E)\) be an mq-space and \(\tau_{S} = \{X \setminus F : F \in C_{idS}(X)\}\). Then, \(\tau_{S}\) defines a topology on \(X\) whose closed sets are exactly the members of \(C_{idS}(X)\). Besides, the Priestley topology is finer than \(\tau_{S}\).

Let \(X\) be an mq-space and \(Y \subseteq X\). We shall denote by \(Y^{S}\) the closure of \(Y\) when \(X\) is endowed with the topology \(\tau_{S}\).

Corollary 2.24 Let \((X, E)\) be an mq-space. Then \(Y = Y^{S}\) for all id-saturated subset \(Y\) of \(X\).

In order to prove Proposition 2.26 we have

Lemma 2.25 Let \((X, E)\) be an mq-space. Then, \(\max E(x) \cup \min E(|x|)^{S} = \max E(x)^{S}\) for each \(x \in X\).
Proposition 2.26 Let $(X, E)$ be an mq–space. If $Y$ is a closed subset of $X$, then the following conditions are equivalent:

(i) $Y$ is id–saturated,

(ii) for each $y \in Y$, $\max E(y)^S \subseteq Y$.

Theorem 2.27 Let $(X, E)$ be an mq–space. Then the following conditions are equivalent:

(i) $(D(X), \Delta_E, \nabla_E)$ is a subdirectly irreducible monadic distributive lattice but not simple,

(ii) one and only one of these conditions hold:

(a) $\{x \in X : \max E(x)^S = X\}$ is a proper non-empty open subset of $X$,

(b) there is $x \in X$ such that $x \notin \max E(x)^S$ and $X = \max E(x)^S \cup \{x\}$.

Proposition 2.28 Let $L$ be an $m$–lattice and $X(L)$ the Priestley space associated with $L$. Then the following conditions are equivalent:

(i) $\max X \cup \min X = X$,

(ii) for each $a, b \in L$ if $a \not\leq b$, there is $M \in \max X(L)$ such that $a \in M$ and $b \not\in M$ or there is $P \in \min X(L)$ such that $a \in P$ and $b \not\in P$.

Theorem 2.29 Let $(L, \nabla, \Delta)$ be a monadic distributive lattice. Then, the following conditions are equivalent:

(i) $(L, \nabla, \Delta)$ is simple,

(ii) $(\nabla, \Delta)$ is simple and for each $a, b \in L$ if $a \not\leq b$ there is $c \in L$ such that $a \land c \neq 0$ and $b \land c = 0$, or there is $d \in L$ such that $b \lor d \neq 1$ and $a \lor d = 1$.

3 Monadic augmented Kripke frames

Our next task is to show the relationship between the categories $\mathbf{pAKF}$ and $\mathbf{mQ}$. To this end, we determine in the first place a new topological duality for monadic distributive lattices by considering the category whose objects are augmented Kripke frames which verify certain additional conditions. More precisely,

Definition 3.1 A monadic augmented Kripke frame (or mk–frame) is a quadruple $(X, \Omega, \leq, E)$ where $(X, \leq)$ is a non-empty partially ordered set, $E$ is an equivalence relation on $X$ and the following conditions are verified:

(mk1) $(X, \leq, E)$ is an augmented Kripke frame,

(mk2) $(X, \Omega, \leq)$ is a Priestley space,

(mk3) $E$ is a closed relation,
(mk4) \( E(U) \) is an open subset of \( X \) for all \( U \in D(X) \),

(\( m k 5 \) \( ( E(X \setminus U) \) is an open subset of \( X \) for all \( U \in D(X) \).

In what follows, we will denote monadic augmented Kripke frames by \((X, \leq, E)\).

**Definition 3.2** Let \((X_1, \leq, E_1)\) and \((X_2, \leq, E_2)\) be \( mk \)-frames. An \( mk \)-function is a function \( f : X_1 \rightarrow X_2 \) which verifies the following conditions:

- \( mkf1 \) \( f \) is continuous,
- \( mkf2 \) \( x \leq y \) implies \( f(x) \leq f(y) \),
- \( mkf3 \) \( (x, y) \in E_1 \) implies \( (f(x), f(y)) \in E_2 \),
- \( mkf4 \) \( E_2(f(x)) \subseteq (f[E_1(x)]) \) for all \( x \in X_1 \),
- \( mkf5 \) \( E_2([f(x)]) \subseteq [f(E_1(x))] \) for all \( x \in X_1 \).

We will denote by \( mK\mathcal{F} \) the category whose objects are all \( mk \)-frames and whose morphisms are all \( mk \)-functions.

**Proposition 3.3** Let \( X \) be a non-empty set. Then the following conditions are equivalent:

- \( i \) \( (X, \Omega, \leq, E) \) is an mq-space,
- \( ii \) \( (X, \Omega, \leq, E) \) is an \( mk \)-frame.

Next, we are going to show that the notions of \( mq \)-function and \( mk \)-function are also equivalent. To this end, first we will indicate a characterization of \( q \)-functions proved in [6], from which we obtain a new description of \( mq \)-functions.

**Proposition 3.4** ([6, Proposition 2.1]) Let \((X_1, E_1)\) and \((X_2, E_2)\) be \( q \)-spaces and \( f \) an order-preserving continuous function from \( X_1 \) into \( X_2 \). Then the following conditions are equivalent:

- \( i \) \( f \) is a \( q \)-function,
- \( ii \) \( f \) satisfies
  
  \( f1 \) \( (x, y) \in E_1 \) implies \( (f(x), f(y)) \in E_2 \),
  
  \( f2 \) \( E_2(f(x)) \subseteq (f[E_1(x)]) \) for all \( x \in X_1 \).

**Proposition 3.5** Let \((X_1, E_1)\), \((X_2, E_2)\) be \( mq \)-spaces and \( f : X_1 \rightarrow X_2 \) a function. Then the following conditions are equivalent:

- \( i \) \( f \) is an \( mq \)-function,
- \( ii \) \( f \) is continuous and increasing function satisfying
  
  \( f1 \) \( (x, y) \in E_1 \) implies \( (f(x), f(y)) \in E_2 \),
(f2) \( E_2(f(x)) \subseteq (f(E_1(x)) \) for all \( x \in X_1, \)

(mq3) \( E_2([f(x)]) \subseteq [f(E_1([x])]) \) for all \( x \in X_1. \)

**Corollary 3.6** Let \((X_1, E_1), (X_2, E_2)\) be mq-spaces and \(f : X_1 \to X_2\) a function. Then the following conditions are equivalent:

(i) \( f \) is an mq-function,

(ii) \( f \) is an mk-function.

**Theorem 3.7** The categories \( \mathcal{M} \) and \( \mathcal{MKF} \) are dually equivalent.

Below, we will indicate the relationship between \( \mathcal{PAKF} \) and \( \mathcal{MKF} \).

**Proposition 3.8** Every perfect augmented Kripke frame \((X, \Omega, R, E)\) is a monadic augmented Kripke frame.

In general, the converse of Proposition 3.8 is not true as the following example shows:

**Example 3.9** Let \( \mathbb{R} \) be the set of real numbers endowed by the Euclidean topology and \( \mathcal{F} \) the set of all closed subsets of \( \mathbb{R} \). It is well known that \((\mathcal{F}, \cap, \cup, 0, \mathbb{R})\) is a bounded distributive lattice. Besides, \((\mathcal{F}, \nabla, \triangle)\) is a monadic distributive lattice where the operators \( \nabla, \triangle \) are defining by the prescriptions \( \nabla 0 = 0 \) and \( \nabla F = \mathbb{R} \) for each \( F \neq 0 \); \( \triangle \mathbb{R} = \mathbb{R} \) and \( \triangle F = 0 \) for each \( F \neq \mathbb{R} \). Then, the monadic augmented Kripke frame \((X(\mathcal{F}), \subseteq, E_{\nabla})\) associated with \( \mathcal{F} \) is not perfect since for all \( U \in D(X(\mathcal{F})) \) we have that \( U \) is not an open subset of \( X(\mathcal{F}) \). Indeed, if it were, it follows that \( X(\mathcal{F}) \setminus \langle U \rangle \in D(X(\mathcal{F})) \) for all \( U \in D(X(\mathcal{F})) \) and therefore, \( F \to \emptyset \) would be defined, which is a contradiction.

Our next task will be to show that the morphisms between perfect Kripke frames are also morphisms between monadic augmented Kripke frames. First, we will determine properties of mk-frames which will be useful to this aim.

**Lemma 3.10** If \((X, \leq, E)\) is an mk-frame, then

(i) for each \( x \in X \), \( \max E_{E \leq}(x) \neq \emptyset, \)

(ii) \( E = E_{E \leq}. \)

**Lemma 3.11** Let \((X_1, \leq, E_1), (X_2, \leq, E_2)\) be mk-frames and \(f : X_1 \to X_2\) strongly isotope with respect to \( E_1 \circ \leq \). Then,

(i) \( f \) is isotope with respect to \( E_1, \)

(ii) \( f \) is almost strongly isotope with respect to \( E_1 \circ \leq \).

**Proposition 3.12** Every morphism in \( \mathcal{PAKF} \) is a morphism in \( \mathcal{MKF} \).

From Proposition 3.8, Example 3.9 and Proposition 3.12 we conclude

**Theorem 3.13** The category \( \mathcal{PAKF} \) is a proper subcategory of \( \mathcal{MKF} \).
References


