A Sequent System for LP

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Abstract. This paper presents a Gentzen-type sequent system for Priest's three-valued paraconsistent logic LP. This sequent system is not canonical because it introduces non-standard axioms. Furthermore, the rules for the conditional and negation connectives are not the classical ones. Some philosophical consequences of this type of sequent presentation for many-valued logics are discussed.

1 Introduction

This paper introduces a sequent system for Priest's many-valued and paraconsistent logic LP (Logic of Paradox)[6]. Priest presents this logic to deal with paradoxes, vague contexts and the alleged existence of true contradictions or dialetheias. A logic is paraconsistent if and only if its consequence relation is paraconsistent, and a consequence relation is paraconsistent if and only if it is not explosive. A consequence relation ⊢ is explosive if and only if, for any formulas A and B, {A, ¬A} ⊢ B (ECQ). In addition, Priest's logic LP is a three-valued system in which both a formula as its negation can receive a designated truth value. In his 1979 paper and in [7], [8] and [9] Priest offers a semantic characterization of LP. He also offers a natural deduction formulation of LP in [9]. Anthony Bloesch provides in [3] a formulation of LP as a system of signed tableaux and Tony Roy offers in [10] another presentation of LP as a natural deduction system.

In this paper a sequent system for LP (SLP) will be presented. Nevertheless, this sequent formulation has some differential properties with respect to standard Gentzen systems [4]. This is not surprising taking into account that A. Avron [2] has proven that a sequent system for a correct and complete many-valued with a finite characteristic matrix, must have either non-standard structural rules, or non-standard axioms, or non-standard operational rules. The SLP system has no structural rules. This calculus does not the classical rules for negation, in order to block the derivation of the ECQ rule. It is not possible, either, to preserve the classical rules for the conditional connective, because the modus ponens rule is not a valid one in LP. Furthermore, the SLP rules for the negation and the conditional connectives differ from standard operational rules which introduce only one occurrence of a connective in its conclusion and do not display other occurrences of connectives in it. The last section of this paper is devoted to the discussion of some philosophical questions...
posed by this kind of formulation of many-valued logics in general and of LP in particular.

2 Priest’s Paraconsistent Logic LP

The usual characterization of paraconsistent logics defines them as those that allow us to draw sensible conclusions from inconsistent but non-trivial information. A logic satisfying this characterization can differentiate inconsistent theories from trivial ones. From a logical point of view, a theory is a set \( \Gamma \) of formulas closed under a logical consequence relation. A theory \( \Gamma \) is inconsistent if and only if, for some formula \( A \), both \( A \) and \( \neg A \) belong to \( \Gamma \). A theory \( \Gamma \) is trivial if and only if, for every formula \( A \), \( A \in \Gamma \). The consequence relation of classical logic does not differentiate between inconsistency and triviality: a set of formulas \( \Gamma \) closed under classical consequence is inconsistent if and only if it is trivial.

Nevertheless, the formal specification of this intuitive characterization is a complicated question for which we do not find a satisfactory answer in the literature. The most commonly accepted formal characterization of paraconsistent logics takes it as a property of the consequence relations of those logics: a logic is paraconsistent if and only if its consequence relation is paraconsistent, i.e. if it does not validate the ex contradictione quodlibet rule. This characterization of paraconsistent logics is based, thus, in a purely negative criterion.

A method to achieve paraconsistency is to use a many-valued truth-functional logic. A many-valued logic in which it is possible for both a formula and its negation to receive a designated value and in which validity is defined in terms of preservation of the designated values, will be able to avoid the explosive character of the classic consequence relation.

Graham Priest’s three-valued logic LP (Logic of Paradox) exemplifies this strategy. Priest maintains that there are propositions that can be both true and false, such as self-reference paradoxes and set theory paradoxes. An LP-valuation is a function which assigns to each well-formed formula one and only one of the truth values of the set \( \{t, p, f\} \) according to the following truth tables:

<table>
<thead>
<tr>
<th>( \neg )</th>
<th>( \wedge )</th>
<th>( \vee )</th>
<th>( \supset )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>( t )</td>
<td>( t )</td>
<td>( t )</td>
</tr>
<tr>
<td>( f )</td>
<td>( f )</td>
<td>( p )</td>
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<td>( p )</td>
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The third truth value \( p \) (paradoxical) can be read as both true and false and, unlike what happens in the three-valued Lukasiewicz’ and Kleene’s logics, is a designated value together with the truth value \( t \) (true). A formula \( A \) is a semantic consequence in LP of a set of premisses \( \Sigma \), in symbols \( \Sigma \vdash_A \), if and only if there is LP-valuation \( v \) such that for every \( B \in \Sigma \), \( v(B) = t \) or \( v(B) = p \), and \( v(A) = f \). A formula \( A \) is an LP
logical truth, in symbols $\vDash_A$, if and only if for every LP-valuation $\nu$, $\nu(A) = t$ or $\nu(A) = p$.

All the theorems of classical propositional logic are valid in LP, but the following classical inferences —among others— are not valid in this system:

- $(A \land \neg A) \vdash B$ (ECQ)
- $(A \supset B), A \vdash B$ (Modus ponens)
- $(A \supset B), \neg B \vdash \neg A$ (Modus tollens)
- $(A \lor B), \neg A \vdash B$ (Disjunctive Syllogism)
- $(A \supset B), (B \supset C) \vdash (A \supset C)$ (Transitivity)
- $(A \supset (B \land \neg B)) \vdash \neg A$ (Reductio ad Absurdum)

3 The Sequent System SLP

In this section a sequent system SLP for Priest’s logic LP will be introduced. The antecedent $\Gamma$ and the consequent $\Delta$ of a sequent $\Gamma \Rightarrow \Delta$ of this calculus are finite sets of propositional formulas, rather than sequences or lists of formulas as in Gentzen’s original system. This eliminates the need in SLP for the structural rules of contraction and interchange. Furthermore, the form of the axioms —for example, $\Gamma, A \Rightarrow \Delta, A$ instead of $A \Rightarrow A$— compensates for the absence of the structural rule of thinning. SLP is a system without structural rules, like Kleene’s G4 calculus [5] for classical logic.

The axioms for this system are:

- Ax.1 $\Gamma, A \Rightarrow \Delta, A$
- Ax.2 $\Gamma, \neg A \Rightarrow \Delta, \neg A$
- Ax.3 $\Rightarrow \Delta, A, \neg A$

The third axiom is needed to compensate for the absence of the standard negation rules that allow us to move a formula from the antecedent (consequent) of a sequent to its consequent (antecedent) and to introduce an occurrence of negation. Thus, the addition of this axiom makes it possible to prove the excluded middle law using the rule of disjunction introduction in the consequent.

The operational rules are:

Conjunction

$\land \Rightarrow$

<table>
<thead>
<tr>
<th>$\Gamma, A, B \Rightarrow \Delta$</th>
<th>$\Rightarrow \land$</th>
<th>$\Gamma \Rightarrow \Delta, A$</th>
<th>$\Gamma \Rightarrow \Delta, B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma, A \land B \Rightarrow \Delta$</td>
<td>$\Gamma \Rightarrow \Delta, A \land B$</td>
<td>$\Gamma \Rightarrow \Delta, A \land B$</td>
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Disjunction
The conjunction and disjunction rules are classical ones, those of Kleene’s system G4, whereas the conditional and negation ones are not. Standard operational rules introduce only one occurrence of a connective in its conclusion and do not display any other occurrences of other connectives in its formulation. It is not the case, either, that the formulas that occur in the premisses of these rules are subformulas of the formula that occurs in the conclusion, although in all the cases they are the affirmation or the negation of subformulas of the formula that occurs in the conclusion.

It must be pointed out that none of the rules allow the passage of a formula from the antecedent (consequent) of a sequent to its consequent (antecedent). This fact
blocks, for example, the derivation of the classically valid sequent $\neg A \Rightarrow B$ in the SLP calculus.

The rules of this system are invertible, i.e. all the sequents in the premises are derivable whenever the conclusion is. Therefore, a derivation can be built beginning with the formula to be proved and using the rules backwards until we arrive a sequent that is not derivable, or until we check that every branch of the derivation ends in an axiom.

The following two trees, the proofs of the contraposition rule and the *modus ponens* law, are examples of derivations in this system:

On the other hand, the following tree shows that the *modus ponens* rule is invalid in SLP:

The standard proofs of soundness and completeness for classical sequent systems can be straightforwardly adapted to provide their counterparts for SLP.
4 The Meaning of Connectives in SLP

This presentation of LP as a sequent calculus raises some philosophical questions concerning the meaning of logical connectives in this system. From an inferentialist point of view—which can be traced back to Gentzen—the meaning of logical connectives is determined by their introduction rules, in conjunction with the structural rules of the system. Nevertheless, it has been maintained that not any rule of introduction of a logical operation can be considered a definition of that operation. A bona fide definition must fulfill certain requirements. Here we wish to point out that in this presentation of LP—and the same can be maintained of the presentation of J3 in [1]—some of the rules do not fulfill two of those requirements: the purity and the simplicity properties. A rule that displays occurrences of just one connective is pure; if, in addition, that connective occurs just once in the formulation of the rule, then the rule is simple. The absence of purity in the rule of introduction of a connective suggests that the meaning of this connective cannot be characterized independently of the meaning of the other connectives which occur in the rule. The purity of the rules is, in turn, related to another property that is considered important for a formulation of a logic as a sequent system: the separation property. A system has this property if its rules are pure and any derivable sequent in the system has a proof that requires, in addition to the structural rules, logical rules only for the connectives that occur in that sequent. The separation property is usually considered as a sign that the inferential behaviour of each of the connectives of the system can be isolated, so that the proof of a derivable sequent reflects (only) the interaction of the inferential behaviours of connectives that occur in that sequent.

These considerations leave as an open question whether this presentation of LP as a sequent system is the most adequate one to reflect the meaning of its connectives and of the notion of logical consequence that it characterizes.

References

