A DUALITY FOR 3-VALUED ŁUKASIEWICZ IMPLICATION ALGEBRAS WITH Δ

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In this work we give a topological representation for 3-valued Łukasiewicz implication algebras with Δ. Every 3-valued Łukasiewicz implication algebra with Δ is represented as a union of a unique family of implication filters of a suitable 3-valued Łukasiewicz algebra. Inspired in this representation, we introduce the notion of topological 3-valued implication space, and we prove a dual equivalence. The main tool we use in the proofs of this result is the duality between implication algebras and certain topological spaces, developed in [1]. As an application, we describe the 3-valued implication space of free 3-valued Łukasiewicz implication algebra with Δ.

We asume familiarity with the algebras involved here. The equational class of all 3-valued Łukasiewicz algebras will be denoted by $\mathcal{L}_3$. The 3-element chain L_3, $0 < 1/2 < 1$, with the natural lattice structure and the operations $\sim$ and ∇ defined as $\sim 0 = 1, \sim 1/2 = 1/2, \sim 1 = 0$ and $\nabla 0 = 0, \nabla 1/2 = 1, \nabla 1 = 1$, will be denoted by $\mathbf{L}_3$. It is well known that $\mathbf{L}_3$ and its subalgebra $\mathbf{L}_2 = \langle L_2 = \{0,1\}, \vee, \wedge, \sim, \nabla, 0, 1\rangle$ are the subdirectly irreducible algebras of $\mathcal{L}_3$. Given $\mathbf{L} \in \mathcal{L}_3$ and $X \subseteq L$, we will denote by $F(X)$ the implication filter generated by X in $\mathbf{L}$.

An algebra $\mathbf{A} = \langle A, \to, \Delta, 1\rangle$ of type $(2,1,0)$ is a *3-valued Łukasiewicz implication algebra with* Δ if

(1) $x \to (y \to x) = 1$,
(2) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(3) $(x \to y) \to y = (y \to x) \to x$,
(4) $((x \to y) \to (y \to x)) \to (y \to x) = 1$,
(5) $((x \to (x \to y)) \to x) \to x = 1$,
(6) $1 \to x = x$,
(7) $\Delta x \to y = x \to (x \to y)$,
(8) $\Delta(\Delta x \to y) = \Delta x \to \Delta y$.

These algebras were introduced and studied by Figallo in [4]. The variety of all 3-valued Łukasiewicz implication algebras with Δ will be denote by $\mathcal{L}_3^{\{\to,\Delta\}}$. Note that $\langle A, \to, 1\rangle$ is a 3-valued Łukasiewicz implication algebra ([5]). It is known that they are the class of all $\{\to, \Delta, 1\}$-subreducts of 3-valued Łukasiewicz algebras. The $\{\to, \Delta, 1\}$-reduct of $\mathbf{L}_3$ will be denoted by $\mathbf{L}_3^{\to,\Delta}$. The subalgebra of $\mathbf{L}_3^{\to,\Delta}$ with subuniverse consisting of the 2-element chain will be denoted by $\mathbf{L}_2^{\to,\Delta}$. These algebras are the subdirectly irreducible algebras of

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$\mathcal{L}_3^{\{\to,\Delta\}}$, and they are simple. It is also known that the congruences are determined by the implication filters.

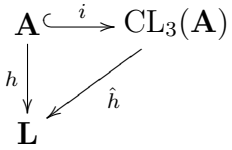
Let $\mathbf{A} \in \mathcal{L}_3^{\{\to,\Delta\}}$, and let $\mathcal{M}$ the set of all maximal implication filters of $\mathbf{A}$. Then $\mathbf{A}$ is isomorphic to a subalgebra $\mathbf{A}'$ of $\mathbf{P} = \prod_{D\in\mathcal{M}} \mathbf{A}/D$. For each $D \in \mathcal{M}$, $\mathbf{A}/D$ is isomorphic to $\mathbf{L}_3^{\to,\Delta}$ or $\mathbf{L}_2^{\to,\Delta}$. Then, $\mathbf{P} \in \mathcal{L}_3$. We identify $\mathbf{A}$ with $\mathbf{A}'$. Consider now the 3-valued Łukasiewicz algebra generated by $\mathbf{A}$ in $\mathbf{P}$, denote by $\mathbf{L}_3[\mathbf{A}]$. We prove that A is an increasing subset of $\mathbf{L}_3[\mathbf{A}]$, and so, $\mathbf{A}$ is isomorphic to a union of implication filters of $\mathbf{L}_3[\mathbf{A}]$. Also, if $F(\mathbf{A})$ is proper then $\mathbf{L}_3[\mathbf{A}]/F(\mathbf{A}) \cong \mathbf{L}_2$. We define the closure of $\mathbf{A}$ by

$$\mathrm{CL}_3(\mathbf{A}) = \begin{cases} \mathbf{L}_3[\mathbf{A}] & \text{if } F(\mathbf{A}) \neq \mathbf{L}_3[\mathbf{A}] \\ \mathbf{L}_3[\mathbf{A}] \times \mathbf{L}_2 & \text{if } F(\mathbf{A}) = \mathbf{L}_3[\mathbf{A}] \end{cases}.$$

The next theorem establishes that $\mathrm{CL}_3(\mathbf{A})$ is the least, up to isomorphism, 3-valued Łukasiewicz algebra in which the implication filter $F(\mathbf{A})$ is maximal and $\mathrm{CL}_3(\mathbf{A})/F(\mathbf{A})$ is the 2-element chain.

Theorem 1. *Let $\mathbf{A} \in \mathcal{L}_3^{\{\to,\Delta\}}$, then:*

1. *A is increasing in $\mathrm{CL}_3(\mathbf{A})$, and A satisfies the finite meet property (*fmp*). Moreover, $\mathrm{CL}_3(\mathbf{A})/F(\mathbf{A}) \cong \mathbf{L}_2$,*
2. *if $\mathbf{L} \in \mathcal{L}_3$ and $h\colon \mathbf{A} \to \mathbf{L}$ is a $\{\to,\Delta\}$-homomorphism, such that $h[\mathbf{A}]$ has the* fmp *in $\mathbf{L}$, then there is a homomorphism $\hat{h}\colon \mathrm{CL}_3(\mathbf{A}) \to \mathbf{L}$ such that $\hat{h} \restriction A = h$, i.e., the diagram*



commutes.

Two 3-valued Łukasiewicz implication algebras with Δ may have the same closure, but the can be distinguished by means of the implication filters contained in them. If $\mathsf{M}(\mathbf{A})$ is the family of all maximal elements in the set of all implication filters of $\mathrm{CL}_3(\mathbf{A})$ contained in $\mathbf{A}$, then $\mathsf{M}(\mathbf{A})$ satisfies:

1. $A = \bigcup_{D\in\mathsf{M}(\mathbf{A})} D$, and $\mathsf{M}(\mathbf{A})$ is an antichain,
2. if M is an implication filter of $\mathrm{CL}_3(\mathbf{A})$ contained in A, then $M \subseteq D$ for some $D \in \mathsf{M}(\mathbf{A})$.

A 3-valued Boolean space is a pair $\langle \mathbf{X}, V\rangle$ such that $\mathbf{X}$ is a Boolean space and $V \subseteq X$ is closed. If the set L_3 is equipped with the discrete topology, and $\langle \mathbf{X}, V\rangle$ is a 3-valued Boolean space, then $\mathbf{C}_3(\mathbf{X}, V)$ denotes the 3-valued Łukasiewicz algebra of all continuous functions $f\colon \mathbf{X} \to \mathbf{L}_3$ such that $f(V) \subseteq L_2$, with the algebraic operations defined pointwise. In [3], it have been proved that for each $\mathbf{A} \in \mathcal{L}_3$, there is a 3-valued Boolean space $\langle X(\mathbf{A}), V_{\mathbf{A}}\rangle$ such that $\mathbf{A} \cong \mathbf{C}_3(X(\mathbf{A}), V_{\mathbf{A}})$. Moreover, $X(\mathbf{A})$ is homeomorphic to the Stone space of the Boolean algebra $B(\mathbf{A})$.

Definition 2. *A space $\langle \mathbf{X}, V, u, \mathcal{C}\rangle$ is a* 3-valued implication space *if*

(1) $\langle \mathbf{X}, V\rangle$ *is a 3-valued Boolean space, and* u *is a fix element of* $\mathbf{X}$ *such that* $u \in V$,
(2) $\mathcal{C}$ *is an antichain, respect to inclusion, of closed set of* $\mathbf{X}$ *such that* $\bigcap \mathcal{C} = \{u\}$,
(3) *if* C *is a closed set of* $\mathbf{X}$ *such that for every clopen* N *of* $\mathbf{X}$, $C \subseteq N$ *implies there is* $D \in \mathcal{C}$ *such that* $D \subseteq N$, *then there is* $D' \in \mathcal{C}$ *such that* $D' \subseteq C$.

Definition 3. *A function* $f\colon \langle \mathbf{X}_1, V_1, u_1, \mathcal{C}_1\rangle \to \langle \mathbf{X}_2, V_2, u_2, \mathcal{C}_2\rangle$ *is* $i3$-continuous *if* f *is continuous,* $f(V_1) \subseteq V_2$, $f(u_1) = u_2$, *and for all* $C \in \mathcal{C}_2$ *there is* $D \in \mathcal{C}_1$ *such that* $D \subseteq f^{-1}[C]$.

Let $\mathbf{A} \in \mathcal{L}_3^{\{\to,\Delta\}}$, and consider $\mathrm{CL}_3(\mathbf{A})$, $F(\mathbf{A})$ and $\mathsf{M}(\mathbf{A})$ as before. Recall that the lattice of implication filters of $\mathrm{CL}_3(\mathbf{A})$ is isomorphic to the lattice of filters of $B(\mathrm{CL}_3(\mathbf{A}))$, and that there is a correspondence between filters and closed sets of $\mathrm{St}(B(\mathrm{CL}_3(\mathbf{A})))$. So, we have a correspondence between implication filters D of $\mathrm{CL}_3(\mathbf{A})$ and closed sets of $\mathrm{St}(B(\mathrm{CL}_3(\mathbf{A})))$ given by $D \mapsto C_{\Delta D} = \{U \in \mathrm{St}(B(\mathrm{CL}_3(\mathbf{A}))) : \Delta D \subseteq U\}$. Consider the space

$$\mathbb{F}(\mathbf{A}) = \langle \mathbf{X}(\mathrm{CL}_3(\mathbf{A})), V_{\mathrm{CL}_3(\mathbf{A})}, \Delta F(\mathbf{A}), \mathcal{C}(\mathbf{A})\rangle,$$

where $\mathbf{X}(\mathrm{CL}_3(\mathbf{A})) \cong \mathrm{St}(B(\mathrm{CL}_3(\mathbf{A})))$, $V_{\mathrm{CL}_3(\mathbf{A})} = \{U \in \mathrm{St}(B(\mathrm{CL}_3(\mathbf{A}))) : \mathrm{CL}_3(\mathbf{A})/F(U) \cong \mathbf{L}_2\}$, and $\mathcal{C}(\mathbf{A}) = \{C_{\Delta D} : D \in \mathsf{M}(\mathbf{A})\}$. From Theorem 1 and conditions of $\mathsf{M}(\mathbf{A})$, we have that $\mathbb{F}(\mathbf{A})$ is a 3-valued implication space.
We have proved that there exists a dual equivalence between the category whose objects are the 3-valued Łukasiewicz implication algebras with Δ and whose arrows are homomorphisms, and the category of 3-valued implication spaces with morphisms as $i3$-continuous functions.

The $i3$-implication space of free algebras. We will denote by $\mathbf{F}_{\mathcal{K}}(X)$ the free algebra in a variety $\mathcal{K}$ with a set X of generators. Let X and X^* be sets such that $\mid X \mid = \mid X^* \mid$. Since $\mathbf{F}_{\mathcal{L}_3}(X^*) \in \mathcal{L}_3^{\{\to,\Delta\}}$, we can consider the $\mathcal{L}_3^{\{\to,\Delta\}}$-subalgebra generated by X in $\mathbf{F}_{\mathcal{L}_3}(X^*)$. It have been proved that $\mathbf{F}_{\mathcal{L}_3^{\{\to,\Delta\}}}(X)$ is isomorphic to this subalgebra. Moreover, if we identify X with X^*, we have that $\mathbf{F}_{\mathcal{L}_3^{\{\to,\Delta\}}}(X) = \bigcup_{x\in X}[\Delta x)$, where $[\Delta x)$ is the implication filter generated by Δx in $\mathbf{F}_{\mathcal{L}_3}(X)$.

It is known that $B(\mathbf{F}_{\mathcal{L}_3}(X))$ is $\mathbf{F}_{\mathcal{B}}(Y)$, the free Boolean algebra over the poset $Y = \{\Delta x, \nabla x : x \in X\}$ ([2]). For each $x \in X$, $\{\Delta x, \nabla x\}$ is a chain called *principal chain*. Moreover, there is a bijection from the set of upwards closed subsets $S \subseteq Y$ onto the ultrafilters of $\mathbf{F}_{\mathcal{B}}(Y)$, given by $S \to F(S \cup \{\neg y : y \in Y \backslash S\}) = U_S$. Therefore, $\mathbf{F}_{\mathcal{L}_3}(X)$ is isomorphic to the algebra $\mathbf{C}_3(\mathrm{St}(\mathbf{F}_{\mathcal{B}}(Y)), V_{\mathbf{F}_{\mathcal{L}_3}(X)})$, where $U_S \in V_{\mathbf{F}_{\mathcal{L}_3}(X)}$ if and only if S is a union of principal chains [2, Thm. 2.1]. From this, we have that $\mathbf{F}_{\mathcal{L}_3}(X)/F(X) \cong \mathbf{L}_2$, since $F(X) = F(U_Y)$. Then, $\mathrm{CL}_3(\mathbf{F}_{\mathcal{L}_3^{\{\to,\Delta\}}}(X)) = \mathbf{F}_{\mathcal{L}_3}(X)$. Also, we have proved that $\mathsf{M}(\mathbf{F}_{\mathcal{L}_3^{\{\to,\Delta\}}}(X)) = \{[\Delta x) : x \in X\}$.

Then $\mathbf{X}(\mathrm{CL}_3(\mathbf{F}_{\mathcal{L}_3^{\{\to,\Delta\}}}(X))) \cong \mathbf{X}(\mathbf{F}_{\mathcal{L}_3}(X)) \cong \mathrm{St}(B(\mathbf{F}_{\mathcal{L}_3}(X))) \cong \mathrm{St}(\mathbf{F}_{\mathcal{B}}(Y))$. For each $x \in X$, $C_{[\Delta x)} = \{U_S \in \mathrm{St}(\mathbf{F}_{\mathcal{B}}(Y) : \Delta x \in U_S\}$. Observe that $\bigcap C_{[\Delta x)} = \{U_Y\}$. Then, $\langle \mathrm{St}(\mathbf{F}_{\mathcal{B}}(Y)), V, \{C_{[\Delta x)} : x \in X\}, U_Y\rangle$ is the $i3$-implication space of $\mathbf{F}_{\mathcal{L}_3^{\{\to,\Delta\}}}(X)$, where $U_S \in V$ is and only if S is union of principal chains.

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