# $\theta-$valued Łukasiewicz–Moisil algebras with and without negation

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**Abstract**

Philosophical problems arising from the idea that there are statements which are neither true nor false, led to the formulation of many–valued logics by Łukasiewicz. Since then, plenty of research has been developed in this area. In 1968, G. Moisil found an example which gave him the motivation he had been looking for in order to legitimate the introduction and study of infinitely–valued Łukasiewicz algebras, so he defined $\theta-$valued Łukasiewicz algebras, where $\theta$ is the order type of a chain.

In this article, we determine a topological duality for $\theta-$valued Łukasiewicz algebras (or $\mathrm{Lk}_\theta-$algebras) equivalent to the one given by Filipoiu in 1980. Not only does the duality enable us to obtain a description of the $\mathrm{Lk}_\theta-$congruences on an $\mathrm{Lk}_\theta-$algebra, but also to characterize the subdirectly irreducible $\mathrm{Lk}_\theta-$algebras. Furthermore, we extend the above study to the case of $\mathrm{Lk}_\theta-$algebras with negation arriving through a different method at the results indicated by V. Boisescu et al (Łukasiewicz–Moisil Algebras, Annals of Discrete Mathematics 49, North–Holland, 1991).

## 1 Introduction and preliminaries

In 1940, G. Moisil introduced $3-$valued and $4-$valued Łukasiewicz algebras with the purpose of obtaining the algebraic counterpart of the corresponding Łukasiewicz logics. A year later, he generalized these algebras by defining $n-$valued Łukasiewicz algebras ([14]) and he studied them from the algebraic point of view. It is well-known that these algebras are not the algebraic counterpart of $n-$valued Łukasiewicz propositional calculi for $n \geq 5$ (see [3, 4]). R. Cignoli ([5, 6]) found algebraic counterparts for $n \geq 5$ and he called them proper $n-$valued Łukasiewicz algebras. It is worth mentioning that in 1997, M. Fidel ([10]) described a propositional calculus which has $n-$valued Łukasiewicz algebras as the algebraic counterpart.

On the other hand, in 1968 G. Moisil ([15]) defined $\theta-$valued Łukasiewicz algebras, where $\theta$ is the order type of a chain. The membership to a fuzzy set

in the sense of Zadeh is the example where this author found the motivation in order to legitimate the introduction and study of these algebras.

Let $\theta \geq 2$ be the order type of a totally ordered set $J$ with least element 0 being $J = \{0\} + I$ (ordinal sum). Following Boisescu et al ([3]) recall that:

A $\theta-$valued Łukasiewicz–Moisil algebra (or $\mathrm{Lk}_\theta-$algebra) is an algebra $\langle A, \vee, \wedge, 0, 1, \{\phi_i\}_{i\in I}, \{\overline{\phi}_i\}_{i\in I}\rangle$ of type $(2, 2, 0, 0, \{1\}_{i\in I}, \{1\}_{i\in I})$ where $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and for all $i \in I$, $\phi_i$ and $\overline{\phi}_i$ satisfy the following conditions:

(L1) $\phi_i$ is an endomorphism of bounded distributive lattices,

(L2) $\phi_i x \vee \overline{\phi_i} x = 1,\ \phi_i x \wedge \overline{\phi_i} x = 0,$

(L3) $\phi_i \phi_j x = \phi_j x,$

(L4) $i \leqslant j$ implies $\phi_i x \leqslant \phi_j x.$

(L5) $\phi_i x = \phi_i y$ for all $i \in I$ imply $x = y.$

A $\theta$-valued Łukasiewicz–Moisil pre-algebra (or $\mathrm{Lk}_\theta$-pre-algebra) is an algebra $\langle A, \vee, \wedge, 0, 1, \{\phi_i\}_{i\in I}, \{\overline{\phi}_i\}_{i\in I}\rangle$ of type $(2, 2, 0, 0, \{1\}_{i\in I}, \{1\}_{i\in I})$ where $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and for all $i \in I$, $\phi_i$ and $\overline{\phi}_i$ fullfil (L1)–(L4).

The idea of considering these notions separately is due to Beznea ([1]) in the case $\theta = n$.

We shall denote by $\boldsymbol{\mathfrak{P}\mathfrak{L}k_\theta}$ the category of $\mathrm{Lk}_\theta-$pre-algebras and their corresponding homomorphisms and by $\boldsymbol{\mathfrak{L}k_\theta}$ the category of $\mathrm{Lk}_\theta-$algebras which is a full subcategory of the first one.

Suppose now that the set $J$ defined above has both least element 0 and greatest element 1.

A $\theta-$valued Łukasiewicz–Moisil algebra (pre-algebra) with negation (or $\mathrm{nLk}_\theta-$ algebra (pre-algebra)) is an algebra $\langle A, \vee, \wedge, \sim, 0, 1, \{\phi_i\}_{i\in I}, \{\overline{\phi}_i\}_{i\in I}\rangle$ of type $(2, 2, 1, 0, 0, \{1\}_{i\in I}, \{1\}_{i\in I})$ where

(NL1) $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ is a De Morgan algebra,

(NL2) $\langle A, \vee, \wedge, 0, 1, \{\phi_i\}_{i\in I}, \{\overline{\phi}_i\}_{i\in I}\rangle$ is an $\mathrm{Lk}_\theta-$algebra (pre-algebra),

(NL3) $\overline{\phi}_i x = \sim \phi_i x,$

(NL4) there is a decreasing involution $d : I \to I$ such that $\phi_i \sim x = \sim \phi_{di} x.$

The category of the $\mathrm{Lk}_\theta-$algebras with negation and their corresponding homomorphisms will be denoted by $\boldsymbol{n\mathfrak{L}k_\theta}$.

The notions and results announced here will be used throughout the paper.

It is well known that there are congruences in the classes of $\mathrm{Lk}_\theta-$algebras and $\mathrm{nLk}_\theta-$algebras such that the quotient algebra doesn't satisfy the determination principle. That is the reason why a new notion is defined as follows: a $\theta\mathrm{Lk}_\theta-$congruence on an $\mathrm{Lk}_\theta-$algebra is a bounded distributive lattice congruence $\vartheta$ such that $(x, y) \in \vartheta$ if and only if $(\phi_i x, \phi_i y) \in \vartheta$ for all $i \in I$ ([2, 13]).

On the other hand, the interest in the following example arises from the statement proved in [3], which we will show below in Theorem 2.29 and Corollary 2.17 by a different reasoning:

Let ${B_2}^{[I]}$ be the set of all increasing functions from $I$ to the Boolean algebra $B_2$ with two elements and let $\theta$ be the order type of the totally ordered set $\{0\} + I$ with least element 0. Then, ${B_2}^{[I]}$ can be made into an $\mathrm{Lk}_\theta-$algebra $\langle {B_2}^{[I]}, \wedge, \vee, 0, 1, \{\phi_i\}_{i\in I}, \{\overline{\phi}_i\}_{i\in I}\rangle$ where the operations of the lattice $\langle {B_2}^{[I]}, \wedge, \vee, 0, 1\rangle$ are defined pointwise and for all $i, j \in I$, $(\phi_i f)(j) = f(i)$ and $(\overline{\phi}_i f)(j) = (f(i))'$, being $x'$ the Boolean complement of $x$. If the set $I$ has least and greatest element and there is a decreasing involution $d : I \longrightarrow I$, then the $\mathrm{Lk}_\theta-$algebra ${B_2}^{[I]}$ can be made into an $\mathrm{nLk}_\theta-$algebra $\langle {B_2}^{[I]}, \wedge, \vee, \sim, 0, 1, \{\phi_i\}_{i\in I}, \{\overline{\phi}_i\}_{i\in I}\rangle$, where $(\sim f)(i) = (f(di))'$ for all $i \in I$.

A. Filipoiu ([3, 11, 12]) presented a duality to the case of $\mathrm{Lk}_\theta-$algebras. In order to do this, he considered the category $\boldsymbol{\mathcal{P}r\theta}$ whose objects are $\theta-$valued Priestley spaces (or $\mathrm{P}_\theta-$spaces) and whose morphisms are increasing continuous functions with certain additional conditions. Besides, this author proved that this category is dually equivalent to $\boldsymbol{\mathfrak{L}k_\theta}$. Furthermore, he extended this duality and the one obtained by Cornish and Fowler ([9]), to the case of $\mathrm{nLk}_\theta-$algebras.

For a general account of the origins and the theory of Łukasiewicz many valued logics and Łukasiewicz algebras the reader is referred to [3, 7, 8, 13].

This paper is organized as follows. In Section 1, we summarize the definitions and results necessary for further development. In Section 2, we determine a topological duality for $\theta-$valued Łukasiewicz algebras equivalent to the one given by Filipoiu in [11]. This duality allows us to obtain the $\mathrm{Lk}_\theta-$congruences on these algebras. As a consequence, we characterize the subdirectly irreducible $\mathrm{Lk}_\theta$-algebras. Furthermore, we prove that the associated space of subdirectly irreducible $\mathrm{Lk}_\theta$-algebras is a totally ordered set, and therefore, they are the subalgebras of ${B_2}^{[I]}$ described above. In Section 3, we extend the results obtained in Section 2, to the case of $\mathrm{Lk}_\theta-$algebras with negation arriving through a different way at the results indicated in [3].

## 2 $\theta-$valued Łukasiewicz–Moisil algebras without negation

Here we will obtain a characterization of the dual of $\boldsymbol{\mathfrak{Luk}_\theta}$ in terms of ordered topological spaces.

**Definition 2.1** *A $\theta-$valued Łukasiewicz–Moisil pre-space (or $l_\theta P-$pre-space) is a pair $(X, \{f_i\}_{i\in I})$ provided the following conditions are satisfied:*

(lP1) *$X$ is a Priestley space,*

(lP2) *$f_i : X \to X$ is a continuous function,*

(lP3) *$x \leqslant y$ implies $f_i(x) = f_i(y)$,*

(lP4) *$i \leqslant j$ implies $f_i(x) \leqslant f_j(x)$,*

(lP5) *$f_i \circ f_j = f_i$.*

*A $\theta-$valued Łukasiewicz–Moisil space (or $l_\theta P-$space) is an $l_\theta P-$pre-space satisfying the following condition:*

(lP6) *for all $U, V \in D(X)$, $f_i^{-1}(U) = f_i^{-1}(V)$ for all $i \in I$ imply $U = V$, where $D(X)$ is the lattice of all increasing clopen subsets of $X$.*

**Definition 2.2** *Let $(X, \{f_i\}_{i\in I})$ and $(X', \{f'_i\}_{i\in I})$ be $l_\theta P-$ spaces. An $l_\theta P-$function is an isotone continuous function $f : X \to X'$ satisfying $f'_i \circ f = f \circ f_i$ for all $i \in I$.*

The category $\boldsymbol{l_\theta\mathcal{P}}$ has $l_\theta P-$spaces as objects and $l_\theta P-$functions as morphisms and it is a full subcategory of the category $\boldsymbol{\mathcal{P}l_\theta\mathcal{P}}$ of l$_\theta$P–pre–spaces.

Proposition 2.3 enables us to prove that the notion of $P_\theta-$space, given by Filipoiu, is equivalent to that of $l_\theta P-$space.

**Proposition 2.3** *Let $(X, \{f_i\}_{i\in I})$ be an $l_\theta P-$space. Then for each $U \in D(X)$ the following conditons are fulfilled:*

(lP9) *$X \setminus {f_i}^{-1}(U) \in D(X)$,*

(lP10) *$i \leqslant j$ implies ${f_i}^{-1}(U) \subseteq {f_j}^{-1}(U)$.*

**Theorem 2.4** *Let $X$ be a Priestley space and let $f_i : X \to X$ be a function for each $i \in I$. Then the following conditions are equivalent:*

(i) *$(X, \{f_i\}_{i\in I})$ is an $l_\theta P-$space,*

(ii) *$(X, \{f_i\}_{i\in I})$ is a $P_\theta-$space.*

**Corollary 2.5** *Let $\langle A, \vee, \wedge, 0, 1, \{\phi_i\}_{i\in I}, \{\overline{\phi}_i\}_{i\in I}\rangle$ be an $\mathrm{Lk}_\theta-$algebra and let $X(A)$ be the Priestley space associated with $A$. Then $Ł_\theta(A) = (X(A), \{{f_i}^A\}_{i\in I})$ is an $l_\theta P-$space, where for every $i \in I$, the function ${f_i}^A : X(A) \longrightarrow X(A)$ is defined by ${f_i}^A(P) = {\phi_i}^{-1}(P)$ for all $P \in X(A)$.*

Next, we are going to denote by $Ł_\theta(A)$ or by $X(A)$ the $l_\theta P$-space associated with $A$.

**Theorem 2.6** *The category $\boldsymbol{l_\theta\mathcal{P}}$ is naturally equivalent to the dual of the category $\boldsymbol{\mathfrak{L}k_\theta}$.*

As a direct consequence of Theorem 2.6 we conclude that the categories $\boldsymbol{\mathfrak{P}\mathfrak{L}k_\theta}$ and $\boldsymbol{\mathcal{P}l_\theta\mathcal{P}}$ are dually equivalent.

Next we will determine some properties of $l_\theta P-$spaces which will be quite useful in order to characterize the lattice of all congruences and the lattice of all $\theta-$congruences of $\mathrm{Lk}_\theta-$algebras. These results will be taking into account to obtain the subdirectly irreducible $\mathrm{Lk}_\theta-$algebras and the subdirectly irreducible $\theta\mathrm{Lk}_\theta-$algebras.

**Proposition 2.7** *Let $(X, \{f_i\}_{i\in I})$ be an $l_\theta P-$space. Then for all $x \in X$ it holds*

(lP11) *$x \leqslant f_i(x)$ or $f_i(x) \leqslant x$ for all $i \in I$.*

**Corollary 2.8** *If $(X, \{f_i\}_{i\in I})$ is an $l_\theta P-$space, then $X$ is the cardinal sum of the sets $[\{f_i(x)\}_{i\in I}) \cup (\{f_i(x)\}_{i\in I}]$ for $x \in X$.*

**Proposition 2.9** *Let $(X, \{f_i\}_{i\in I})$ be an $l_\theta P-$space where $I$ has least element $0$ and greatest element $1$. Then the following conditions hold, for each $x \in X$:*

(lP13) *$f_0(x) \leqslant x \leqslant f_1(x)$,*

(lP14) *$f_0(x)$ is the unique minimal element in $X$ that precedes $x$,*

(lP15) *$f_1(x)$ is the unique maximal element in $X$ that follows $x$.*

**Corollary 2.10** *Let $(X, \{f_i\}_{i\in I})$ be an $l_\theta P-$space where $I$ has least element $0$ and greatest element $1$. Then, $X$ is the cardinal sum of the sets $[f_0(x), f_1(x)]$ for $x \in X$.*

**Definition 2.11** *Let $(X, \{f_i\}_{i\in I})$ be an $l_\theta P-$space. A subset $Y$ of $X$ is semimodal if $f_i(Y) \subseteq Y$ for all $i \in I$ or equivalently $Y \subseteq {f_i}^{-1}(Y)$ for all $i \in I$.*

The closed and semimodal subsets of the $l_\theta P-$space associated with an $\mathrm{Lk}_\theta-$ algebra play a fundamental role in the characterization of $\mathrm{Lk}_\theta-$congruences on these algebras, as we shall show next.

**Theorem 2.12** *Let $A \in \boldsymbol{\mathfrak{Lk}_\theta}$ and let $Ł_\theta(A)$ be the $l_\theta P-$space associated with $A$. Then, the lattice $\mathcal{C}_S(Ł_\theta(A))$ of all closed and semimodal subsets of $Ł_\theta(A)$ is isomorphic to the dual lattice $Con_{Lk_\theta}(A)$ of all $Lk_\theta-$congruences on $A$, and the isomorphism is the function $\Theta_S$ defined by the prescription $\Theta_S(Y) = \{(a,b) \in A \times A : \sigma_A(a) \cap Y = \sigma_A(b) \cap Y\}$ (see [16, 17, 18]).*

**Definition 2.13** *Let $(X, \{f_i\}_{i\in I})$ be an $l_\theta P-$space. A $\theta-$subset is a semimodal subset $Y$ of $X$ which verifies that $Y \subseteq \overline{\bigcup\limits_{i\in I} f_i(Y)}$.*

**Proposition 2.14** *Let $(X, \{f_i\}_{i\in I})$ be an $l_\theta P-$space and let $Y$ be a subset of $X$. Then the following conditions are equivalent:*

(i) *$Y$ is a closed $\theta-$subset,*

(ii) *there is a subset $Z$ of $X$ such that $Y = \overline{\bigcup\limits_{i\in I} f_i(Z)}$.*

**Corollary 2.15** *Let $(X, \{f_i\}_{i\in I})$ be an $l_\theta P-$space and let ${C_\theta}^0(X)$ be the set of all non-empty, closed and $\theta-$subsets of $X$. Then, a subset $Y$ of $X$ is a minimal element of ${C_\theta}^0(X)$ if and only if there is $x \in X$ such that $Y = \{\overline{f_i(x)}\}_{i\in I}$.*

**Proposition 2.16** *If $(X, \{f_i\}_{i\in I})$ is an $l_\theta P-$space, then $\bigcup\limits_{i\in I} f_i(X)$ is dense in $X$.*

The closed $\theta-$subsets of the $l_\theta P-$space associated with an $\mathrm{Lk}_\theta-$algebra enable us to characterize the $\theta\mathrm{Lk}_\theta-$congruences on these algebras as Theorem 2.17 shows.

**Theorem 2.17** *Let $A \in \boldsymbol{\mathfrak{Lk}_\theta}$ and let $Ł_\theta(A)$ be the $l_\theta P-$space associated with $A$. Then, the lattice $\mathcal{C}_\theta(Ł_\theta(A))$ of all closed $\theta-$subsets of $Ł_\theta(A)$ is isomorphic to the dual lattice $Con_{\theta Lk_\theta}(A)$ of all $\theta\mathrm{Lk}_\theta-$congruences on $A$, and the isomorphism is the function $\Theta_\theta$ defined as in Theorem 2.12.*

**Corollary 2.18** *Every maximal $\mathrm{Lk}_\theta-$congruence on an $\mathrm{Lk}_\theta-$algebra $A$ is a maximal $\theta\mathrm{Lk}_\theta-$congruence on $A$.*

**Corollary 2.19** *Every $\mathrm{Lk}_\theta-$algebra is semisimple.*

**Theorem 2.20** *Let $(X, \{f_i\}_{i\in I})$ be an $l_\theta P-$space and $\mathbb{L}_\theta(X)$ the $\mathrm{Lk}_\theta-$algebra associated with $X$. Then the following conditions are equivalent:*

(i) *$X$ is a totally ordered set,*

(ii) *$\mathbb{L}_\theta(X)$ is a simple $\mathrm{Lk}_\theta-$algebra,*

(iii) *$\mathbb{L}_\theta(X)$ is a simple $\theta\mathrm{Lk}_\theta-$algebra,*

(iv) *$\mathbb{L}_\theta(X)$ is a subdirectly irreducible $\theta\mathrm{Lk}_\theta-$algebra,*

(v) *$\mathbb{L}_\theta(X)$ is a subdirectly irreducible $\mathrm{Lk}_\theta-$algebra.*

**Corollary 2.21** *The algebra ${B_2}^{[I]}$ is a subdirectly irreducible $\mathrm{Lk}_\theta-$algebra.*

In order to prove that every subdirectly irreducible $\mathrm{Lk}_\theta-$algebra is isomorphic to an $\mathrm{Lk}_\theta-$subalgebra of ${B_2}^{[I]}$, we will study some properties of totally ordered $l_\theta P-$spaces and in particular, of the $l_\theta P-$space associated with ${B_2}^{[I]}$.

Let $(D, \prec)$ be a directed set and $Y$ a topological space. In what follows, for a net $\varphi : D \to Y$, we shall write $\varphi \to y_0$ whenever $\varphi$ converges to $y_0$.

**Proposition 2.22** *Let $(X, \{f_i\}_{i\in I})$ be a totally ordered $l_\theta P-$space and for each $x \in X$, $x \neq f_i(x)$ for all $i \in I$, let $K_x = \{k \in I : \ f_k(x) < x\}$ and $J_x = \{i \in I : \ x < f_j(x)\}$. If the order on $K_x$ is the one induced by the order defined on $I$, and the order on $J_x$ is the one induced by the dual order defined on $I$, then $f_j(x) \to x$ or $f_k(x) \to x$.*

**Proposition 2.23** *Let $P \in X({B_2}^{[I]})$ and let $P \neq {\phi_i}^{-1}(P)$ for all $i \in I$. If $K_P = \{k \in I : \ {\phi_k}^{-1}(P) \subset P\}$, $J_P = \{j \in I : \ P \subset {\phi_j}^{-1}(P)\}$, where the order on $K_P$ is the one induced by the order defined on $I$, and the order on $J_P$ is the order induced by the dual order defined on $I$, then either ${\phi_j}^{-1}(P) \to P$ or ${\phi_k}^{-1}(P) \to P$, but not both.*

**Corollary 2.24** *Let $P \in X({B_2}^{[I]})$ and let $P \neq {\phi_i}^{-1}(P)$ for all $i \in I$. If $\{{\phi_{i_d}}^{-1}(P)\}_{d\in D}$ is a net and ${\phi_{i_d}}^{-1}(P) \to P$, then there is $d_o \in D$ such that either $\{{\phi_{i_d}}^{-1}(P)\}_{d\in T_{d_0}} \subseteq \{{\phi_j}^{-1}(P)\}_{j\in J_P}$ or $\{{\phi_{i_d}}^{-1}(P)\}_{d\in T_{d_0}} \subseteq \{{\phi_k}^{-1}(P)\}_{k\in K_P}$, but not both, where $T_d = \{c \in D : \ d \prec c\}$ for all $d \in D$.*

**Lemma 2.25** *If $P \in X({B_2}^{[I]})$, then ${\phi_i}^{-1}(P) \neq {\phi_j}^{-1}(P)$ for all $i, j \in I$, $i \neq j$.*

**Proposition 2.26** *Let $P \in X({B_2}^{[I]})$. Then, $P = {\phi_{i_0}}^{-1}(P)$ for some $i_0 \in I$ if and only if $P = [f)$, where $f : I \longrightarrow B_2$ is defined by $f(i) = 1$ if $i_0 \leqslant i$ and $f(i) = 0$ otherwise.*

**Proposition 2.27** *Let $P \in X({B_2}^{[I]})$ and let $P = {\phi_{i_0}}^{-1}(P)$ for some $i_0 \in I$. If $\{{\phi_{i_d}}^{-1}(P)\}_{d\in D}$ is a net and ${\phi_{i_d}}^{-1}(P) \to P$, then there is $d_0 \in D$ such that ${\phi_{i_d}}^{-1}(P) = {\phi_{i_0}}^{-1}(P)$ for all $d \in T_{d_0}$.*

**Proposition 2.28** *If $(X, \{f_i\}_{i\in I})$ is a totally ordered $l_\theta P-$space, then there is an $l_\theta P-$function from $X(B_2{}^{[I]})$ onto $X$.*

**Theorem 2.29** *Let $A$ be an $\mathrm{Lk}_\theta-$algebra. Then $A$ is subdirectly irreducible if and only if $A$ is isomorphic to an $\mathrm{Lk}_\theta-$subalgebra of $B_2{}^{[I]}$.*

# 3 $\theta-$valued Łukasiewicz–Moisil algebras with negation

In the sequel, taking into account the results established in [9] we will extend the duality obtained in the Section 2 for $\mathrm{Lk}_\theta-$algebras to the case of $\mathrm{nLk}_\theta-$algebras. This duality leads us determine the congruences and the $\theta-$conguences on $\mathrm{nLk}_\theta-$algebras. Finally, we will describe the subdirectly irreducible $\mathrm{nLk}_\theta-$algebras and the subdirectly irreducible $\theta\mathrm{nLk}_\theta-$algebras. In order to this, we introduce the following notion:

**Definition 3.1** *A $\theta-$valued Łukasiewicz–Moisil space (pre-space) with negation or $Nl_\theta P-$space (pre-space) is a triple $(X, g, \{f_i)\}_{i\in I})$ satisfying the following conditions:*

(nlP1) *$(X, g)$ is a De Morgan space ([9]),*

(nlP2) *$(X, \{f_i)\}_{i\in I})$ is an $l_\theta P-$space (pre-space),*

(nlP3) *$f_i \circ g = f_i$,*

(nlP4) *$g \circ f_i = f_{di}$, where $d : I \to I$ is a decreasing involution.*

**Definition 3.2** *Let $(X, g, \{f_i\}_{i\in I})$ and $(X', g', \{f_i{}'\}_{i\in I})$ be $Nl_\theta P-$spaces (pre-spaces). An $Nl_\theta P-$function from $(X, g, \{f_i\}_{i\in I})$ to $(X', g', \{f'_i\}_{i\in I})$ is an $l_\theta P-$function $f : X \to X'$ satisfying that $f \circ g = g' \circ f$.*

The category $\boldsymbol{\mathcal{N}l_\theta\mathcal{P}}$ of $\mathrm{Nl}_\theta P-$spaces is a full subcategory of $\boldsymbol{\mathcal{P}\mathcal{N}l_\theta\mathcal{P}}$ which has $\mathrm{Nl}_\theta P-$pre-spaces as objects and $Nl_\theta P-$functions as morphisms.

**Theorem 3.3** *The category $\boldsymbol{\mathcal{N}l_\theta\mathcal{P}}$ is dually equivalent to the category $\boldsymbol{n\mathfrak{L}k_\theta}$.*

Theorem 3.3 allows us to assert that the categories $\boldsymbol{n\mathfrak{P}\mathfrak{L}k_\theta}$ and $\boldsymbol{\mathcal{P}\mathcal{N}l_\theta\mathcal{P}}$ are dually equivalent.

The following properties of $Nl_\theta P-$spaces enable us to determine the subdirectly irreducible $\mathrm{nLk}_\theta-$algebras.

First, recall that a subset $Y$ of a De Morgan space $(X, g)$ is involutive if $g(Y) = Y$.

**Proposition 3.4** *Every $Nl_\theta P-$space $X$ is the cardinal sum of the closed, semimodal and involutive sets $[f_0(x), f_1(x)]$, for $x \in X$.*

**Lemma 3.5** *If $X \in \boldsymbol{\mathcal{N}l_\theta\mathcal{P}}$, then every closed $\theta-$subset of $X$ is involutive.*

**Theorem 3.6** *Let $A \in \boldsymbol{n\mathfrak{L}k_\theta}$ and let $N\text{Ł}_\theta(A)$ be the $Nl_\theta P$-space associated with $A$. Then, it holds:*

(i) *the lattice $\mathcal{C}_{IS}(N\text{Ł}_\theta(A))$ of all involutive, semimodal and closed subsets of $N\text{Ł}_\theta(A)$ is isomorphic to the dual lattice $Con_{nLk_\theta}(A)$ of all* $\text{nLk}_\theta-$*congruences on $A$,*

(ii) *the lattice $\mathcal{C}_\theta(N\text{Ł}(A))$ of all closed $\theta-$subsets of $N\text{Ł}(A)$ is isomorphic to the dual lattice $Con_{\theta nLk_\theta}(A)$ of all* $\theta\text{nLk}_\theta-$*congruences on $A$,*

*where in both cases, the isomorphism is defined as in Theorem 2.12.*

**Corollary 3.7** *Let $A \in \boldsymbol{n\mathfrak{L}k_\theta}$ and let $\vartheta$ be a binary relation on $A$. Then $\vartheta$ is a $\theta Lk_\theta-$congruence on $A$ if and only if $\vartheta$ is a $\theta nLk_\theta-$congruence on $A$.*

Theorem 3.6 and Corrollary 3.7 combine, as we have already indicated, to yield our promised characterization of the subdirectly irreducible objects in $\boldsymbol{n\mathfrak{L}k_\theta}$.

**Theorem 3.8** *Let $A \in \boldsymbol{n\mathfrak{L}k_\theta}$. Then the following conditions are equivalent:*

(i) *$A$ is a simple* $\text{nLk}_\theta-$*algebra,*

(ii) *$A$ is a simple* $\theta\text{nLk}_\theta-$*algebra,*

(iii) *$A$ is a subdirectly irreducible* $\theta\text{nLk}_\theta-$*algebra,*

(iv) *$A$ is a subdirectly irreducible* $\text{nLk}_\theta-$*algebra,*

(v) *$X(A)$ is totally ordered,*

(vi) *$A$ is isomorphic to an* $\text{nLk}_\theta-$*subalgebra of $B_2{}^{[I]}$.*

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