The concept of axiom in Hilbert's thought

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Introduction

In this talk I would like to present a study on Hilbert's foundational papers. I propose a categorization into two periods that can be distinguished by the role and the content of the axioms. The role of axiomatization has always obscured the role played by the axioms in Hilbert's thought, and the latter is very instructive in order to understand the philosophical ideas that aimed his proposals.

The first period (1891-1918)

The first period starts with Hilbert's lessons on projective geometry held in Könisberg in 1891 and centers around the works in geometry and the sketched attempt to prove the consistency of a weak form of arithmetic: *Die Grundlagen der Mathematik* (1899) and *Über die Grundlagen der Logik und der Arithmetik* (1904). In the works of this period, Hilbert's concept of axiom is linked to a "*deepening of the foundations* of the individual domains of knowledge". Indeed the axiomatization of a theory is gained finding the more important ideas of a field and by making explicit the logical structure of the corresponding domain of knowledge. In this first period Hilbert has a precise idea of what axioms are: they are *implicit definitions*. But what kind of entities are defined by the implicit definitions of the axioms? This problem was explicitly raised by Frege in the a latter to Hilbert:

> The characteristic marks you give in your axioms are apparently all higher than first-level; i.e., they do not answer to the question "What properties must an object have in order to be a point (a line, a plane, etc.)?", but they contain, e.g., second-order relations, e.g., between the concept *point* and the concept *line*. It seems to me that you really want to define second-level concepts but do not clearly distinguish them first-level ones[1].

Indeed Hilbert is not precise in saying what axioms define, sometimes they seem to define mathematical objects, but sometimes he says that axioms define the relations between them. This concern is important, since it reveals the problem that Hilbert tries to solve: the relationship between a

[1]Letter from Frege to Hilbert January 6th, 1900; in [Frege 1980], p.46.

modern axiomatic attitude and the old conception of axioms as expression of spatial intuitions. Indeed, while maintaining the possibility of a multiple interpretation of the axiom system for geometry, Hilbert says that:

> These axioms may be arranged in five groups. Each of these groups expresses, by itself, certain related fundamental facts of our intuition[2].

Then it may seem that the role of axioms manifests a cluster of ideas that are not sharpened yet: axioms derive from a new notion of intuition that is not spatial, as Hilbert explicitly say, but they perfectly fit with our spatial intuition. The problem that arises here is that of the adequacy of a formal notion to an informal one.

In the next section, the analysis of the notion of intuition that lays behind this conception of axiom will also clear Hilbert's solution to the problem of adequacy. What is important to stress is that the conception of axiom depends on the role played by intuition in the choice of a good axiomatization.

The second period (1919-1930)

For more then a decade Hilbert abandons the foundational studies and this is mostly due to the lack of a good formalization of logic. Then, after an initial enthusiasm in Russell's and Whitehead's *Principia Mathematica*, Hilbert comes back to the foundation of mathematics. While the work *Axiomatishes Denken* (1918) still shows a conception of the axioms very similar to the first period, Hilbert's second period begins, publicly, in the early Twenties[3].

In the lectures *Neubergründung der Mathematik. Erste Mittelung* (1922) and *Die logischen Grundlagen der Mathematik* (1923) Hilbert outlines a new analysis of the concept of axiom. These two works can be regarded as belonging to a transition period in between the first and the second, not only chronologically but also from a conceptual point of view. In this period, axioms are not fundamental proposition able to catch the deep conceptual roots of a theory, but they have just a deductive supremacy, in comparison of other propositions. The work that clearly marks that a change has happened is *Über das Unendliche* (1925).

> Certain of the formulas correspond to mathematical axioms. The rules whereby the formula are derived from one another correspond to material deduction. Material deduction is thus replaced by a formal procedure governed by rules. The rigorous transition

[2] [Hilbert 1899], p. 1.

[3] For a detailed description of this period see [Sieg 1999].

> from a naïve to a formal treatment is effected, therefore, both for the axioms (which, though originally viewed naïvely as basic truth, have been long treated in modern axiomatics as mere relations between concepts) and for the logical calculus (which originally was supposed to be merely a different language)[4].

During the Twenties Hilbert's proof theory was born. Consequently the axiomatic method becomes a tool to discover the principles of the whole mathematics in its formalized representation. In this new perspective Hilbert defines a new kind of axiom.

> This program already affects the choice of axioms for our proof theory[5].

These new axioms are of a radical different kind. They are the axioms on which the mathematical building rests: logical and arithmetical in character and, moreover, they are axioms that deserve to be called true. Following the terminology of Feferman[6], we could call Hilbert's logical-arithmetical axioms *foundational* (while the axioms of a non foundational theory can be called *structural*). The importance of these axioms and their role in the foundation of mathematics is well expressed by Hilbert:

> The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds.

But now, since Hilbert's program was to reduce any piece of mathematics to finitary mathematics, logical-arithmetical axioms cannot express and gather every "fundamental fact of our intuition" of doing mathematics. Then their justification have to rest on something different from the intuition of the first period. Interesting enough, even in this case, the ultimate source of justification of mathematical knowledge is a form of intuition.

An important point that motivate also my work is that this analysis undermine a formalist conception of mathematics in Hilbert's thought.

Hibert and intuition

Intuition is what marks the difference between the two periods and the corresponding ideas of axioms. Moreover the intuitive character of every

[4] [Hilbert 1925], p. 381 in [Van Heijenoort 1967].
[5] [Hilbert 1923], p. 1138 in [Ewald 1996].
[6] [Feferman 1999]

axiom, in the first period, and only of the axioms for the proof theory, in the second period, is what distinguishes axioms from other true propositions[7].

Fist of all we need to stress the difference between "intuitive" and "evident", since the confusion between these two concepts has always been source of ambiguity. By "evident sentence" we mean a sentence that does not need to be analysed to exhibit its truth. By "intuitive sentence" we mean a sentence whose truth, in a given context, is immediately perceived, so that it is possible to skip some step of reasoning that, in other cases, would be necessary. An evident sentence need not to be justified and its validity is immediately given, while an intuitive sentence is justified thanks to a certain level of knowledge, but for which validity does not follow for free. Indeed, in this latter case, validity must be ascertained in a different way, for example with a proof. Unlike evidence, which is innate within our mind, intuition can be educated thanks to use and mathematical practice. There is an important *caveat* though. The intuition that we describe here, that we could call a *contextual* intuition, is not an intuition that depends on a specific faculty of the mind, different from intellect. In other words it is not a Kantian-style intuition, i.e. a faculty whose structure depends on pure forms, that are given once and for all, like space and time, and that governs sensible knowledge. On the contrary, we are proposing a kind of intuition that can be refined by the same knowledge that it helps to create.

We will see how the interplay between intuition and evidence helps in understanding the difference between two different conceptions of axioms.

Intuition in the first period

As described by Hilbert, the process of axiomatization starts from an intuition concerning a domain of facts (*Tatsachen*), then, while formalizing it, it tries to clear the logical relationships within the concepts of the theory. The process, as Hilbert describes it, leads from the subject matter of an informal theory to a conceptual level[8].

Axioms, in this period, are used to link intuition to mathematical practice. Indeed, they give meaning to signs, expressing some "fundamental fact of our intuition". However, mathematical practice changes and shapes mathematical intuition.

> [O]ne should always be guided by intuition when laying things

[7] Also in [Kitcher 1967] there is an interesting analysis of the role that intuition plays in Hilbert's works of the second period. On the same subject see [Legris 2005] and [Parsons 1998]. What it lacks is a parallel analysis for the first period and a reflection on their differences.

[8] Recall that at the beginning of [Hilbert 1899] Hilbert quotes Kants' *Critique of pure reason* and writes "All human knowledge begins with intuitions, thence passes to concepts and ends with ideas". This quotation, though not Kantian in spirit, explains how Hilbert wanted to use the axiomatic method in the first period.

> down axiomatically, and one always has intuition before oneself as a goal [*Zielpunkt*]. Therefore, it is no defect if the names always recall, and even make easier to recall, the content of the axioms, the more so as one can avoid very easily any involvement of intuition in the logical investigations, at least with some care and practice[9].

When working on the foundation of geometry Hlibert explains his goal in the following way: "we can outline our task as constituting *a logical analysis of our intuition [Anschauungsvermögens]*" ([Hilbert *1899], p. 2), i.e. an analysis of the most fundamental principles of geometry, conducted with formal means. Among these principles there are of course also our spatial intuitions, but "the question of whether spatial intuition has an *a priori* or empirical character is not hereby elucidated" ([Hilbert *1899], p. 2).

Hilbert's intuition is *modus operandi* acquired by habit developed in parallel with the demonstrative techniques[10]. It is not an innate intuition, but it is sufficiently reliable to be used as an heuristic criterion and that can be formalized. Indeed Hilbert is well aware of the distance between our mathematical intuition and spatial intuition.

> *A general remark on the character of our axioms I-V* might be pertinent here. The axioms I-III [incidence, order, congruence] state very simple, one could even say, original facts; their validity in nature can easily be demonstrated through experiment. Against this, however, the validity of IV and V [parallels and continuity in the form of the Archimedean Axiom] is not so immediately clear[11].

All these remarks show that at the beginning of Hilbert's foundational studies there is no coincidence between the notion of intuition and the notion of evidence. But the problem of adequacy between formalization and subject matter of an axiomatic system or, in other words, between spatial and contextual intuition, is left open.

Hilbert's solution is mathematical and in both cases of geometry and analysis is a Completeness Axiom; a bridge between intuition and formalization that, almost like a stipulation, fixes the reference of a theory and the tools that characterize it: continuity principles that, in the case of analysis, allows to identify the filed structure of the reals and our intuitions of the real line. I propose to interpret this axiom in the same vain of Church-Turing thesis and the second order Induction Axiom.

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[9] [Hilbert *1905], pp. 87-88. Translation in [Hallet 2008].

[10] It is similar to Klein's conception of intuition: *Mechanical experiences, such as we have in the manipulation of solid bodies, contribute to forming our ordinary metric intuition, while optical experiences with light-rays and shadows are responsible for the developement of a 'projective' intuition*, in [Klain 1897], p. 593.

[11] [Hilbert *1898-1899], p. 380 in [Hallet and Majer 2004].

The second period

In the Twenties, when engaged in the foundations of mathematics, Hilbert's new conception of axioms mirrors a deeper enquiry about the concept of intuition, in the direction of a Kantian-style notion.

> Kant taught [...] that mathematics treats a subject matter which is given independently of logic. Mathematics therefore can never be grounded solely on logic. [...] As a further precondition for using logical deduction and carrying out logical operations, something must be give in conception, viz. certain extralogical concrete objects which are intuited as directly experienced prior to all thinking[12].

Hilbert's purposes are clearly Kantian. It remains to see how much Hilbert's ideas towards the realization of those purposes are really Kantian. The affinity of the two thinkers looks in fact merely verbal and, maybe, for Hilbert, functional to philosophers' approval. Indeed at that time the forms of neo-Kantism were quite spread and often quite far from Kant's original ideas. Indeed we can describe Hilbert's work as a *critical deduction of mathematical knowledge*: one of the tasks of the *Neue Fries'sche Schule*[13] founded by the neo-kantian philosopher Leonard Nelson, who was a colleague of Hilbert in Göttingen, during the Twenties.

This new form of intuition is sensible, as also for Kant, but there is an important difference. Indeed for Hilbert intuition is evident as far as it is a kind of knowledge: "also [...] mathematical knowledge in the end rests on a kind of intuitive insight [*anshaulicher Einsicht*]"[14]. On the contrary intuition, for Kant, is not a kind of knowledge, since in the intuitive process it lacks the presence of the concepts under which the objects, given in the intuition, fall.

Hilbert tries to find a justification for mathematical knowledge, analyzing the main possibility of formal knowledge: the tension between syntax and semantics is solved in terms of an intuition that recognizes the parallelism between our mathematical thoughts and our use of mathematical signs. The tension between formalization and intuition is solved identifying the two at the level of arithmetic and logic.

Hilbert then goes a step further: he claims that our intuitions of symbols have not only an *a priori* character, but they also manifest typical features of evidence.

> The subject matter of mathematics is [...] the concrete symbols themselves whose structure is immediately clear recognizable[15].

[12] [Hilbert 1925], p. 376 in [Van Heijenoort 1967].
[13] See [Peckhaus 1990] on this subject.
[14] [Hilbert 1930], p. 1161 in [Ewald 1996].
[15] [Hilbert 1925], p. 376 in [Van Heijenoort 1967].

This is the main difference between the first and the second period, and also the main difference between Kant's and Hilbert's intuition. In the first period intuition and evidence are kept apart, in the second one they coincide, thanks to a Kantian-style (for Hilbert) notion of intuition, that is more similar to Descartes' or Leibniz's intuitive knowledge.

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