ON THE DEVELOPMENT OF PARACONSISTENT LOGIC AND DA COSTA'S WORK

by

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Introduction

A theory T is said to be inconsistent if it has as theorems a formula and its negation; and it is said to be trivial if every formula of its language is a theorem.

A logic is paraconsistent if it can be used as the underlying logic for inconsistent but nontrivial theories, which are called paraconsistent theories.

In paraconsistent logics the scope of the principle of (non-)contradiction is, in a certain sense, restricted. We may even say as da Costa and Marconi 1989, that if the strength of this principle is restricted in a system of logic, then the system belongs to the class of paraconsistent logics.

In fact, in paraconsistent logics the principle of (non-)contradiction is not necessarily invalid, but in every paraconsistent logic from one formula and its negation it is not possible, in general, to deduce every formula.

When we name the principle or law of (non-)contradiction (L.N.-C.), we usually mean any of its several non-equivalent formulations, or any of its connected theses: one of which is the negation of the other, one is false; it is not the case of a statement and its negation: a predicate cannot simultaneously belong and not belong to the same

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Among the paraconsistent logics, we may identify the weak paraconsistent logics, which can be used not only as underlying logics of paraconsistent theories, but also of consistent ones, as for example the logics $C_n$, $1 \leq n \leq w$, of da Costa; and the strong paraconsistent logics, which can be used as underlying logics of paraconsistent theories, but not of consistent ones, as the systems DM and DL of Routley and Meyer 1976.

Arruda 1980b presents an interesting discussion about paradoxes, antinomies and Hegel’s thesis.

According to the meaning given for the terms paradox, the derivation in $T$ of a formula and its negation, and antinomy, a metalogical proof that $T$ is trivial, she claims that an antinomy from the point of view of classical logic may be a veridical paradox from the standpoint of paraconsistent logic.

Paraconsistent logic is closely related to other kinds of nonclassical logics, especially with dialectical and relevant logic.

Many present days authors use the term dialectical logic to designate those logics that are intentionally constructed to formalize aspects of the dialectical discourse of Hegel, Marx and their followers. Routley and Meyer 1976 give a necessary condition for a propositional logic being dialectical: to be closed under modus ponens, simply inconsistent (i.e. to include at least one formula and its negation as theses) and nontrivial.

Da Costa 1974b says that “dialectical logic is intimately connected with the theory of inconsistent systems. There are several conflicting conceptions of dialectic logic, and for most specialists it is neither formal nor even in principle formalizable. Nonetheless, employing techniques used in the theory of inconsistent systems, it is apparently possible to formalize some of the proposed dialectic logics the formalizations we are talking about intend only to make explicit certain ‘regularities’ of the ‘dialectical movement’”

Marconi 1979 also suggests that “probably any future account to think in logical terms about the ‘Hegelian discourses’ will have an obligatory passage in the theory of inconsistent formal systems”.

Relevant logic originated with the work Ackermann 1956 and was especially developed by Anderson and Belnap 1976. The basic idea of Ackermann was to obtain an implication, entailment
or relevant implication, which was free from the so-called paradoxes of material implication.

As argued in Arruda 1989, it is difficult to establish a general criterium of comparison between paraconsistent logic and dialectical logic or relevant logic, on account of the fact of the concepts of deduction and theorem in these logics being not always similarly defined.

The field of paraconsistent logic and the fields of dialectical logic (da Costa and Wolf 1980) and of relevant logic (Arruda 1989) have intersections, but they are, surely, distinct fields.

Paraconsistent logics are also related with other kinds of logics, like, for instance, paracomplete logic, fuzzy logic, the general theory of vagueness, intuitionistic and many-valued logics, Meinong’s theory of objects, as well as with the logical theses of the later Wittgenstein.

Da Costa 1982 claims that the existence of non-classical logics is reasonable from the philosophical point of view and gives an exposition of the most important effects of the impact of paraconsistent logic on the philosophical field.

Da Costa and Marconi 1989 distinguish between two kinds of philosophical issues that have been raised in connection with paraconsistent logic: those concerning to the logical peculiarities of paraconsistent systems, which main topics are the several ways in which a paraconsistent logic can be constructed and the meaning of the logical constants in paraconsistent systems; and those concerning to the philosophical motivations for paraconsistent logics, beginning by the purely logical motivations.

In fact, a deep and complete analysis of the philosophical import and philosophical consequences of paraconsistent logic seems not to have been made yet. But Marconi 1979 and 1984, da Costa 1974 and 1982, and da Costa and Marconi 1989 are good papers on the subject.

The study of paraconsistent logics, besides allowing the construction of paraconsistent theories, makes possible the direct study of logical and semantical paradoxes without trying to avoid them; the study of certain principles in their full strength as, for example, the principle of comprehension in set theory; and perhaps it permits us a better understanding of the concept of negation.
The aim of this paper is to present some aspects of the development of paraconsistent logic, from the work of Jaśkowski and da Costa, its founders, until recent results on Model Theory and Computer Science.

We do not analyse here, the philosophical aspects concerning to paraconsistent logics.

We begin by discussing the historical origins of paraconsistent approaches, presenting Łukasiewicz and Vasil'ev as the forerunners of paraconsistent logics and of non-classical logics in general, and introducing Jaśkowski and da Costa as the first logicians to construct systems of paraconsistent logics.

In §2, we study Jaśkowski’s discussive propositional calculus $D_2$, its axiomatics, its semantics and the algebraic structure associated to it.

In §3, we present da Costa’s hierarchies of paraconsistent propositional logics, predicate calculi and calculi of descriptions. The most important results about these calculi are studied, as well as their different kinds of semantics. Other paraconsistent logics are also introduced, and the algebrization of some paraconsistent systems and paraconsistent set theories are presented.

In §4 we discuss relevant and dialectical paraconsistent systems.

Some other systems and general results about paraconsistent logics are studied in §5.

Finally, in §6, we present several recent applications of paraconsistent logics, emphasizing those related to Computer Science, specifically to logical programming and knowledge representation.

As this work is a general overview on the development of paraconsistent logic, we neither analyse deeply the results nor give technical details and proofs.

The References do not intend to be exhaustive about the subject. We only mention the works named in the paragraphs.

In this paper we largely use the works Arruda 1980b and 1989, which present the historical evolution of paraconsistent logic until 1980. We also use da Costa and Marconi 1989, which gives an overview on the subject in the eighties.

§1. Historical Origins of Paraconsistent Logic
In spite of Eastern philosophy having generally been more tolerant of inconsistency than Western, paraconsistent approaches were not so exceptional in classical antiquity and were assumed by several philosophical schools, as for instance by the Sophists, by the Megarians and the Stoics.

But we may say that paraconsistent thinking begins in the West with Heraclitus of Ephesus.

Several elements of paraconsistent thinking can be found in Eastern philosophy, showing that in Chinese thought, as in Indian, contradictions were tolerated and used to illustrate certain points.

Orthodox philosophers in Christendom were strongly anti-paraconsistent, but there were isolated nonconsistent positions during the middle-age. Priest and Routley 1989a argue that the Neo-Platonism contained significant paraconsistent elements.

Since Heraclitus, meanwhile several philosophers, including Hegel, Marx, Engels and the contemporary dialectical materialists have proposed the thesis that contradictions are fundamental for the understanding of reality.

According to Priest and Routley 1989a, “in the modern period, beginning with the Renaissance and Enlightenment and running through until the beginning of the present century, two further major philosophical positions emerged which were congenial to paraconsistent approaches and took paraconsistent shape in some of their elaborations, namely idealism, especially as elaborated by Hegel. and, very differently, the philosophy of common sense. especially as presented by Reid”.

At the beginning of this century, paraconsistency was definitely discovered by several studious, all of them working independently of the others.

In 1910, we have the publication of a paper by Lukasiewicz. *On the principle of contradiction in Aristotle*: the first paper by Vasil’ev on non-classical logic; and the second revised edition of Meinong’s *Über Annahmen*, the basic text on Meinong’s theory of objects, with new points, in order to deal with Russell’s objections that the theory did not respect the principle of (non-)contradiction (L.N.C.).

Meinong’s theory of objects includes inconsistent objects and defective objects (like the Russell class) such as the logical and semantical paradoxes supplay. Under the theory there are contradictory
objects which, in virtue of their nature, have contradictory features and are amenable to logic treatment.

Meinong’s theory seems to have influenced Lukasiewicz’s work, unless in its initial stage.

Parsons 1980 and Plumwood and Routley 1982 give recent presentations and elaborations of the theory of objects.

Meinong worked within traditional logic, but none of them, neither Meinong nor Łukasiewicz and Vasil’ev used modern symbolic logic.

Ayda 1980b says that “strangely enough, even philosophers (non-logicians) who accepted Hegel’s thesis have not established any formal system of paraconsistent logic”.

The two real forerunners of paraconsistent logics, and even of non-classical logic in general, are Jan Łukasiewicz and Nicolaj Vasil’ev. Both, working independently of each other, claimed, in 1910 and 1911, inspired by the development of non-Euclidean Geometry, that a revision of the basic laws of Aristotelian logic would yield new non-Aristotelian systems of logic, in particular, systems admitting violation of the L.N.C.

But the two first logicians to construct systems of paraconsistent logics are the Polish Stanislaw Jaśkowski and the Brazilian Newton Carneiro Affonso da Costa.

1.1 Jan Łukasiewicz (1876-1956)

Łukasiewicz taught Philosophy at the University of Warsaw, Poland.


In these works Łukasiewicz shows that the arguments built in order to justify the principle of (non-)contradiction, L.N.C., all derived from Aristotle, are feeble. He conjectures that, like for the non-Euclidean geometries, “a fundamental revision of the basic laws of Aristotle’s logic might perhaps lead to new non-Aristotelian systems of logic”. And he did not exclude the possibility of non-trivial contradictory theories being true.
In the paper he presents a “historically critical exposition” of Aristotle’s formulations of L.N.C.: the ontological formulation (“It is impossible that the same thing belong and not belong to the same thing at the same time and in the same respect”), the logical formulation (“The most certain of all basic principles is that contradictory properties are not true together”) and the psychological one (“No one can believe that the same thing can be and not be”). He also criticises conclusively Aristotle’s attempts to establish them, saying that “none of the three formulations of the principle of contradiction is identical in meaning with the other...”, in spite of Aristotle having equated the logical and ontological formulations, and having failed in proving the psychological form from the logical one.

Łukasiewicz shows that the named proof “is incomplete because Aristotle did not demonstrate that acts of believing which correspond to contradictory properties are incompatible” and “inconclusive”, for “there are sufficient examples in the history of philosophy where contradictions have been asserted at the same time and with full awareness”.

He also says that “although Aristotle proclaims the nondemonstrability of the principle of contradiction... he strives in spite of that to give demonstrations for the principle” and he “proves not that the mere denial of the principle of contradiction would lead to absurd consequences, rather he attempts to establish the impossibility of the assumption that every thing is contradictory”. Therefore, he claims that “however, he who denies the principle of contradictions or who demands a proof of it, surely does not need to accept that everything is contradictory...”.

Łukasiewicz, then, states that Aristotle, like Meinong, “limits the range of validity of L.N.C. to actual existents only”, the “sensibly perceptible world, conceived as becoming and passing away, could contain contradictions” at least potentially, but beyond this ephemeral world is “another, eternal and non-ephemeral world of substantial essences, which remains intact and shielded from every contradiction”.

So, according to Łukasiewicz, Aristotle did conceive the possibility of the L.N.C. being not valid in general. But he did not openly reveal his “true position” on account of this diplomacy in trying to enforce the L.N.C. and “in holding high the value of scientific
research”. But Łukasiewicz seems to agree that “the principle is
the sole weapon against error and falsehood”, and that to deny it
“would have opened door and gate to every falsity and nipped the
young blossoming science in the bud”.

Łukasiewicz finally rejects the Aristotle’s view that L.N.C. is the
most final and highest logical principle.

He tries to show that there is no logical basis for the adoption
of L.N.C.. He writes:

“(A) The principle of contradiction cannot be proved by proclai-
ming it directly evident. For:

(a’) evidence does not appear to be a permissible criterium of
truth: it turns out that false propositions as well are held to be
evident...

(b’) the principle of contradiction does nor appear to be evident
to everyone:

(B) The principle of contradiction cannot be proved by setting
it up as a natural law determined by the psychical organization of
man. For:

(a’) it is possible to determine false propositions by our psychical
organization...

(b’) it is questionable whether the principle of contradiction can
be validated as a law determined by the psychical organization of
man.

(C) The principle of contradiction cannot be proven on the basis
of the definition of statements or negations ... For:

(a’) If one accepts that the negation “A is not B” means the
falsity of the affirmation “A is B”, then the principle of contradiction
is not to be deduced therefrom.

(b’) of course, if one prefers rather to avoid designating one
and the same proposition ad true and false, another definition of
falsity can be set up ... The basic notion of falsity is, ... that
false propositions correspond to nothing objective ... The principle
of contradiction can in no way be derived from this definition of
falsity” (Łukasiewicz 1971, pp. 505-6).

Łukasiewicz opened the way for non-classical logic but, in spite
of having faced the challenge of discussing the general validity of the
Aristotelian principle of (non-)contradiction, he didn’t create para-
consistent logic, having not constructed any paraconsistent logical system.

In fact, some years later, in Lukasiewicz and Tarski 1930, he introduced the first system of many-valued logic, as an attempt to investigate the modal propositions and the notions of possibility and necessity closely related to such propositions.

But in 1948, it was one of his students, Stanisław Jaśkowski, who constructed the first propositional formal system of paraconsistent logic.

1.2 Nikolaj A. Vasil’ev (1880-1940)

On account of his imaginary logics Vasil’ev has been considered a forerunner of many-valued and paraconsistent logic. He wrote: “we witness the logic of a new time... it is necessary to enlarge the limits of logic, believing in the possibilities of various logical systems”.

Vasil’ev was a physician, professor of Philosophy at the University of Kazan, Russia. It seems that he had no knowledge about Lukasiewicz’ works of 1910, but their arguments about the possibility of constructing non-Aristotelian logics are very similar. It is interesting that he also argues that the excluded middle law appeared in “Aristotle’s mind in order to refute his adversaries, and not for logical reasons”.

In Vasil’ev 1910, his first paper, he shows that it is possible to derogate the principle of the excluded middle in its formulation “two contradictory propositions cannot be both true and cannot be both false”, maintaining the principle of (non-)contradiction. He begins with a new classification of judgements in judgements about facts, which express a fact happening at a fixed instant of time, and judgements about concepts, which express non-temporal laws, concluding that the logic of judgements about concepts is a kind of non-Aristotelian logic.

Vasil’ev argues that there are only three different types of judgements about concepts: “Every S is P”, “No S is P” and “Only some (not all) S are P, and the remaining are not P”. As any two of these judgements cannot be simultaneously true but they can both be false, he introduces a kind of law of excluded fourth: for each concept A and predicate P, only one of these judgements must be
true, and a fourth judgement cannot be formulated.

In his papers of 1911, 1912 and 1913, Vasil'ev discusses the derogation of the L.N.C. and the consequent possibility of the construction of a new logic, non-Aristotelian, in which this law is not valid in general.

In the Proceedings of the Fifth Philosophical International Congress, Vasil'ev 1925 presents a three pages summary of his ideas.

Inspired by the methods of construction used by Lobatchewski in his non-Euclidean geometry, initially called imaginary geometry, Vasil'ev extends his views on his non-Aristotelian logic of concepts to a logic which he calls imaginary logic.

Vasil'ev claims that every logical system consists of two parts: the metalogic, containing the laws of thought which cannot be eliminated because all of them are necessary for every thinking and for logic maintaining its logical character; and the ontological basis of logic, containing the laws which depend on the properties of the objects we are considering.

According to Vasil'ev, metalogic is the "first logic", prior to every logic, the same for all systems and one logic of "dimension one", i.e., it is a logic of only one quality of judgements—affirmative judgements. For him, starting from metalogic, we can construct the Aristotelian logic, a logic of "dimension two", by adding judgements of a new quality, the negative judgements. From Aristotelian Logic we can build a logic of "dimension three", by adding judgements of a third quality, the "indifferent judgements" (S is P and not P).

Metalogic is a kind of universal logic, being a part of every logical system. But he does not clarify its laws, not even the laws of his imaginary logic: he only discusses the law of contradiction and non-self-contradiction. Nonetheless it becomes clear that for him there was not only one imaginary logic, but one logic for every imaginary world.

As Vasil'ev believed that contradictions do not exist in the real world but that they exist in a possible world imagined by our mind, he says that Aristotelian logic refers to real world while his imaginary logic refers to imaginary worlds created by imagination.

He supposes the existence of imaginary worlds of any finite "logical dimension" n, which have a logic of dimension n to describe them, with judgements of n different qualities and with an ontolo-
sical law of excluded \((n + 1)th\).

In Vasil’év imaginary logic of dimension three the ontological law of contradiction, taken in the Kantian form “no object can have a predicate which contradicts it” is not valid, for it depends on the properties of the objects. But the metalogical law of non-self-contradiction “one and the same judgement cannot be simultaneously true and false” is valid.

According to Arruda 1977, Vasil’év tried to show that this logic of dimension three, with its ontological law of excluded fourth, has a classical interpretation, as it happens with Lobachevski’s Geometry.

Vasil’év’s works seem not to have had influence during his time, remaining almost unknown up to 1962. His first work was discussed by Hessen 1910 and criticized in Smirnov 1911. In Vasil’év 1912 there is a note refuting Smirnov.

Several years later, Vasil’év’s papers were included by Church 1936 in his Bibliography of Symbolic Logic and mentioned by Korcik 1965. His ideas were well presented and discussed, for the first time, in Smirnov 1962 and in the review written by Comey 1965.

Kline 1965 considers Vasil’év a forerunner of many-valued logics. Rescher 1964 and Jammer 1974 agree with this view.

Arruda 1977 presents three different formal approaches to the intuitive ideas of Vasil’év for his imaginary logic of dimension three, obtaining in each case a paraconsistent logic. Her first and third systems are decidable by three-valued matrices, and the second one by a two-valued matrix, but it was not given any interpretation for the truth-values considered. Arruda believes that every possible formalization for Vasil’év’s imaginary logic would necessarily yield to a paraconsistent logic and not necessarily to a many-valued logic. Arruda argues against Kline’s interpretation, claiming that Vasil’év’s notion of dimension has nothing to do with the number of different truth-values involved but with “the number of... different qualities of a judgement”.

Priest and Routley 1989a present a deep discussion about both interpretations and they also suggest that Vasil’év could be placed as one of the founders, along with McColl and Lewis, of intensional logics.

Bochvar 1939 introduces a three-valued calculus and studies
its applications to the analysis of contradictions.

We cannot state that Bochvar proposed a paraconsistent treatment of the contradictions. It would depend on how the third truth-value would be interpreted and whether it would be designated, and Bochvar seems to suggest various interpretations.

Routley and Priest present Vasil’ev and Bochvar as the Russian forerunners of paraconsistent logic and say that Bochvar was perhaps the first logician to introduce a logic of paradox or calculus of antinomies.

1.3 Stanislaw Jaśkowski (1906-1965)

Under Łukasiewicz’ influence, one of his students, Jaśkowski, motivated by several problems related to contradictions, especially those concerning to “convincing reasoning which nevertheless yield two contradictory conclusions”, constructed the first system of paraconsistent logic.

His two papers were published originally in Polish by Studia Societatis Scientiarum Torunensis in 1948 and 1949. Later, only in 1969, a translation to English of Jaśkowski 1948 appeared in Studia Logica.

Initially, he does not explicitly propose simple acceptance of the paradoxes as truth. However, subsequently he does consider representation of the Liar antinomy in the paraconsistent system he constructs, he indicates that other paradoxes such as Russell’s can be similarly treated and remarks that ordinary procedures leading from inconsistency to triviality fail.

Jaśkowski opens, probably for the first time for inconsistent theories, the problem of non-triviality.

According to Arruda 1989, Jaśkowski’s main motivations to construct his system are the following: the problem of the systematization of theories which contain contradictions as it occurs in dialectics; the study of theories where there are contradictions caused by vagueness; and the direct study of some empirical theories whose postulates or basic assumptions are contradictory.

Jaśkowski 1969 says that “The principle that two contradictory statements are not both true and false is the most certain of all”. This is how Aristotle... formulates his opinion known as the
logical principle of contradiction. Examples of convincing reasonings which nevertheless yield contradictory conclusions were the reason why others sometimes disagreed with the Stagirite's firm stand. That was why Aristotle's opinion was not in the least universally shared in antiquity... The contemporary formal approach to logic increases the precision of research in many fields, but it would not be correct to formulate Aristotle's principle of contradiction as: 'Two contradictory sentences are not both true'. We have namely to add: 'in the same language' or 'if the words occurring in those sentences have the same meaning'. This restriction is not always observed in every day usage, and in science too we often use terms that are more or less vague... Any vagueness of the terms a can result in a contradiction of sentences, because with reference to the same object $X$ we may say that '$X$ is $a'$ and also '$X$ is not $a'$, according to the meaning of the term $a$ adopted for the moment'.

The paper also claims that "...it is known that the evolution of empirical disciplines is marked by periods in which the theorists are unable to explain the results of experiments by a homogeneous and consistent theory, but use different hypotheses, which are not always consistent with one another, to explain the various groups of phenomena... we have to take into account the fact that in some cases we have to do with a system of hypotheses which, if subjected to a too precise analysis, would show a contradiction among them or with a certain accepted law, but which we use in a way that is restricted so as not yield a self-evident falsehood".

Jaśkowski pointed clearly the difference between contradictory systems, which includes two theses such that one contradicts the other, and over-complete systems, in which all formulas are theses. He considers that classical logic is not adequate to the study of contradictory but not over-complete systems, because of the law which asserts that from one formula and its negation it is possible to deduce any formula, which he calls implicational law of over-completeness.

Based on these ideas, Jaśkowski proposed the problem of constructing a propositional calculus with the following properties: "1) when applied to contradictory systems would not always entail their over-completeness; 2) would be rich enough to enable practical inferences; 3) would have an intuitive justification".
He mentions some already known systems in which the mentioned law is not valid in general as, for instance, Kolmogorov's systems; but he claims that none of these systems provides a good solution for the problem.

He is concerned with paraconsistent logics as a whole. He does not, however, get far with their classification. Many-valued logics, for instance, are set aside as not providing solution, what in fact happens only with certain particular functionally complete ones.

Jaśkowski presents his own solution, only at the propositional level, obtained by translation of the modal system S5, known as discursive (or discursive) logic and denoted by $\mathbf{D}_2$, and states:

"Obviously, these conditions do not univocally determine a solution, since they may be justified in varying degrees, the satisfaction of condition 3 being rather difficult to appraise objectively".

Discursive logic is intended as a formalization of a logic of discourse. The term comes from considering that the theses advanced by different participants in a discussion, containing inclusive terms whose meaning is vague, are combined like assertions in a deductive system; their consequences do not reflect an uniform opinion and ought to be intuitively interpreted as if they were preceded by the symbol of possibility Pos, that is the sense "it is possible that". A formula is regarded as true if, and only if, it is true according to some source - the participant in the discussion.

Discursive logic, apart from being paraconsistent, is also non-adjectiv. For: "... from the fact that a thesis $A$ and a thesis $B$ have been advanced in a discourse it does not follow that the thesis $A \& B$ has been advanced, because it may happen that $A$ and $B$ have been advanced by different persons. And from the formal point of view, from the fact of $A$ is possible and $B$ is possible it does not follow that $A$ and $B$ are possible simultaneously".

He states: "Let such a system which cannot be said to include theses that express opinions in agreement with one another, be termed a discursive system".

The ideas underlying the construction of $\mathbf{D}_2$ are very interesting and link discursive logics to other classes of logics recently studied, as for instance the non-monotonic logics of interest to computer science, and the doxastic logics.

The system $\mathbf{D}_2$ will be introduced in §2.
1.4 Newton Carneiro Affonso da Costa

In spite of Jaśkowski having constructed a paraconsistent propositional calculus we may say, according to Arruda 1980b, that N.C.A. da Costa “is actually the founder of paraconsistent logic”.

In the fifties, without knowing Jaśkowski’s works on the system $D_2$, da Costa began to develop his ideas about the importance of the study of contradictory theories.

For him, contradictory theories cannot be a priori excluded, considering that consistent and inconsistent theories have the same status. He claims that the only particularity of the inconsistent ones is that they must be based on logical systems different from the classical one.

Priest and Routley 1989a say that “with da Costa’s work we arrive at something strikingly different from what had gone before, deliberately fashioned paraconsistent logical systems – not overtly matrix logics or translations of modal logics...”

In 1958 and 1959, da Costa published in Portuguese his first papers *A note on the concept of contradiction and Observations on the concept of existence in Mathematics.*

Da Costa 1958, after some considerations about the Hilbertian concept of existence in Mathematics, proposes the following *Principle of Tolerance in Mathematics:* “From the syntactical and semantical points of view, every theory is permissible, since it is not trivial”.

Da Costa’s ideas were completely worked out in 1963, when he began to publish a series of papers containing his hierarchy of first-order logics for the study of inconsistent but non-trivial theories. The first work *da Costa 1963a*, which introduces his systems, begins with the following words:

“Loosely speaking, the central idea of this paper is the following: a formalized system based on classical logic (or intuitionistic logic, or some many-valued logics...) if inconsistent is trivial in the sense that all its propositions are provable; then, from this point of view, it does not have any special mathematical interest. However, for many reasons as, for example, the comparative analysis with consistent systems, and for an adequate metamathematical analysis of the principle under consideration, it is convenient to study ‘directly’ the inconsistent systems. But for such study it is necessary to
construct new types of elementary logic appropriate to handle such systems”.

The papers which came after, published by the Comptes Rendus of Paris Academy of Sciences in 1963 and 1964, present a summary of the main results of the previous work. Further developments about the systems were also published by the Comptes Rendus between 1965 and 1974, and a general survey of the first results appeared in da Costa 1974b.

Da Costa firstly constructed a hierarchy of propositional calculi $C_n$, $1 \leq n \leq w$, satisfying the following conditions:

a) The principle of contradiction, in the form $\neg(A \& \neg A)$, should not be valid in general.

b) From two contradictory premises $A$ and $\neg A$, we should not deduce any formula whatever $B$.

c) They should contain the most important schemes and rules of classical logic compatible with the first two conditions.

Subsequently he extended the $C_n$ to a hierarchy of first-order predicate calculi $C_n^*$, $1 \leq n \leq w$, and to a hierarchy of first-order predicate calculi with equality $C_n^\equiv$, $1 \leq n \leq w$.

Then he extended them to a hierarchy of calculi of descriptions $D_n$, $1 \leq n \leq w$, and applied all of them to the construction of the hierarchy of set theories $NF_n$, $1 \leq n \leq w$, inconsistent but apparently non-trivial.

Da Costa’s hierarchies are introduced in §3.

Ayda Iznez Arruda was a graduate student of da Costa when he was working out his systems. In her paper of 1980b she says that she “would like to notice that he know about the work of Jaśkowski only when he was finishing the writing of da Costa 1963a, this is the reason for only a short note about Jaśkowski’s work in its Introduction”.

Da Costa, his disciples and collaborators, especially Arruda (between 1964 and 1983), have researched several paraconsistent systems, having inclusive obtained results related to the algebraic structures associated to such systems, to model theory and some applications to computer science.

Since 1964 da Costa’s logics have been largely studied by other non-Brazilian logicians, and many authors have contributed to the development of these logics and of paraconsistent logics in general.
1.5 Final Observations

Among the pioneer papers which introduce paraconsistent systems, we must yet mention Nelson 1959 and Asenjo 1966.

Nelson studied a systems of paraconsistent logic and applied it to arithmetic.

The Argentinian logician Asenjo presented a calculus of antinomies in his thesis in 1954, but he only published his system in 1966.

Asenjo’s system will be presented in §3.5.

Before finishing this historical sketch we observe that it would have been interesting an analysis of Wittgenstein’s work on inconsistency, preceeding the discussion about Jaśkowski’s contribution. But we prefered not to do it here, and to indicate the reading of Wrigley 1980 and Marconi 1984.

We may distinguish three different periods in Wittgenstein position about negation, contradiction and paradoxes: early, transitional and late.

Only during his late period, Wittgenstein had much in common with paraconsistent approach.

Wittgenstein 1964 states that a contradiction does not automatically destroy a calculus.

In spite of having not built any satisfactory paraconsistent logic as a calculus, Wittgenstein considered some of the basic requirements for paraconsistent theories and his philosophical work was meaningful for the “investigation of calculi containing contradictions” (Wittgenstein 1975).

Nowadays, paraconsistent logic constitutes an important subject among non-classical logics, being especially studied in Australia, Brazil, Bulgaria, Italy, Poland, the Soviet Union and the U.S.A.

In this paragraph, besides Arruda 1980b and 1989 we used Priest and Routley 1989a. This paper presents a historical study of inconsistency or contradictions, from antiquity until the modern revival and the contemporary paraconsistent and dialectical approaches.

The first elementary introduction to paraconsistent logic is the book Grana 1983. We also suggest the books Grana 1990a and 1990b.

Studia Logica, vol. 43, 1/2, 1984 was entirely devoted to paraconsistent logic and the IX Brazilian Logic Symposium, held in São Paulo in 1988, was dedicated to the twenty-five years of da Costa's paraconsistent logic.

Routley, Priest and Norman 1989 is a recent book on paraconsistent logic, edited by Philosophia Verlag.

It is interesting to mention that the term paraconsistent logic was introduced by the Peruvian philosopher Miró Quesada, in his lecture during the Third Latin – American Symposium on Mathematical Logic, held at the University of Campinas, Brazil, in 1976.

To finish this historical introduction, we remember Wittgensteins' 1964 words:

"Indeed, even at this stage a predict a time when there will be mathematical investigation of calculi containing contradictions and people will actually be proud of having emancipated themselves from contradiction."

§2. Paraconsistent Logics Based on Jaśkowski Discursive Logic

In this paragraph we introduce Jaśkowski's discursive propositional calculus $D_2$, presenting its axiomatics, given by da Costa, Dubikajtis and Kotas, and several of its properties; the semantics constructed for the system; the algebraic structure associated to it; and we present some other kinds of discursive logics.

2.1 Discursive Logic $D_2$

Jaśkowski's discursive propositional calculus was introduced in Polish notation, which is not used in this paper.

Let $L$ be the language of the propositional modal calculus $S5$ of Lewis.

We have the following definition for discursive conjunction, discursive implication and discursive equivalence, respectively:

$$p \&_d q =_d p \& \diamond q$$
\[ p \rightarrow_d q = \text{if } \Diamond p \rightarrow q \]

and

\[ p \leftarrow_d q = \text{if } (\Diamond p \rightarrow q) \& (\Diamond q \rightarrow p). \]

The symbols \&_d, \rightarrow_d and \leftarrow_d are considered as functors in the system S5 and the modal connective \( \Diamond \) can be read as “someone maintains that”.

The theorems of D2 are characterized by the clause: The formula A of L is a theorem of D2 if, and only if, \( \Diamond A \) is a theorem of S5.

So, A is discursively valid if, and only if, in any S5 - Kripke structure there exists a possible world in which A is satisfied.

We observe that in D2 the rule of modus ponens is not valid for material implication \( \rightarrow \), but, of course, it is valid for \( \rightarrow_d \).

D2, with the connectives \( \lor, \&_d, \rightarrow_d \) and \( \leftarrow_d \), has all the properties of classical positive logic, being a strong logic.

Da Costa and Dubikajtis 1968 present an axiomatization for Jaśkowski's propositional calculus and initiate a semantical study of it.

In this paper they introduce the system J, whose primitive connectives are \( \neg, \lor, \supset \) and \( \Box \) (necessity) and whose axioms and rules are the following:

\[
\begin{align*}
\text{J}_1 & : \Box((A \supset B) \supset ((B \supset C) \supset (A \supset C))) \\
\text{J}_2 & : ((\neg A \supset A) \supset A) \\
\text{J}_3 & : \Box(B \supset (A \lor B)) \\
\text{J}_4 & : \Box(\Box(A \supset B) \supset \Box(\Box A \supset \Box B)) \\
\text{J}_5 & : \Box((A \supset C) \supset ((B \supset C) \supset ((A \lor B) \supset C))) \\
\text{J}_6 & : \Box(A \supset (\neg A \supset B)) \\
\text{J}_7 & : \Box(A \supset (A \lor B)) \\
\text{J}_8 & : \Box(\Box A \supset A) \\
\text{J}_9 & : \Box(A \supset \Box \Diamond A) \\
\text{R}_1 & : \frac{A, \Box(A \supset B)}{B} \\
\text{R}_2 & : \frac{\Box A}{A}
\end{align*}
\]

Then, they show that J is equivalent to D2, after having defined discursive implication, left discursive conjunction, right discursive
conjunction and discussive equivalence, respectively, by:

\[ A \rightarrow B = _{df} \Diamond A \supset B \]

\[ A \wedge B = _{df} \Diamond A \cdot B \]

\[ A \& B = _{df} A \cdot \Diamond B \]

\[ A \leftarrow B = _{df} (A \rightarrow B) \wedge (B \rightarrow A) \]

We observe that \( A \cdot B \) is, as usually, \( \neg (\neg A \vee \neg B) \).

In another paper published in 1977, the same authors introduced two new equivalent axiomatics for \( D_2 \). The first one, denoted by \( \tilde{J} \), presents twenty-two axioms and the rule of modus ponens, having as primitive connectives the discussive implication \( \rightarrow \), the left discussive conjunction \( \wedge \), the disjunction \( \vee \) and the negation \( \neg \).

Later, Dubikajtis and his students (Achtelik et al 1979 and Dubikajtis et al 1980) reduced the number of axioms of \( \tilde{J} \) to fifteen and proved that fourteen of them were independent ones (see Arruda 1989).

In the same paper of 1977, da Costa and Dubikajtis constructed a higher order discussive logic based on \( S5_w \) and defined a semantics for it, which is an extension of Kripke’s semantics for modal logics.

Kotas 1975 investigates the algebrization of \( D_2 \). Based on a defined connective \( I \), he defines an equivalence relation \( \equiv \), which he proves to be a congruence in the algebra of formulas of the system. Then, he shows that \( D_2 \) is an implicative system with respect to \( I \), with minimal and maximal elements, and that not each of its thesis is its maximal element. He constructs the Lindenbaum algebra associated to \( D_2 \) and shows that a formula \( A \) is a thesis of the system if, and only if, the equality \( 1 \wedge a = 1 \), where \( a = |A| \), is true in the Lindenbaum algebra of \( D_2 \).

In this paper, Kotas also proves that \( D_2 \) is not decidable by finite matrices.

Furmanowski 1975 proves that the Deduction Theorem in its classical form holds in discussive sentencial calculus.
Kotas and da Costa 1977 show that \( D_2 \) fulfills all requisites for being a paraconsistent logic.

A large new variety of modal logics can be obtained by associating to every modal logic its corresponding Jaśkowski’s logic, which are studied in Kotas and da Costa 1977, being several of them interesting paraconsistent logics.


Da Costa and Kotas 1979 give another axiomatization \( SD_2 \) for discursive logic, with nineteen rules, construct the corresponding predicate calculus with equality and define a valuation semantics for it.

2.2 Other Results

Pinter 1980 introduces an interesting system, which he calls the logic of inherent ambiguity, stating that the need for this kind of logic “... arises naturally in social and human sciences, in organizational decision-making, and in the newest branch of computer sciences, as pattern recognition. All these require deduction and a small margin of imprecision”. An ambiguity is a statement of the form \( A \land \neg A \).

The primitive connectives of the system are \( \land, \lor, \neg \) (negation) and \( \neg \) (exclusion), observing that \( \land, \lor, \neg \) satisfy all the valid formulas of classical logic. Pinter defines:

\[
A \rightarrow B =_d \neg A \lor B \\
A \leftarrow B =_d (A \rightarrow B) \land (B \rightarrow A) \\
\star A =_d A \land \neg A \\
0A =_d \neg \neg A
\]

He also states the validity of:

\[
\star (A \land B) \leftarrow (\star A \land B) \lor (\star B \land A) \\
(\star B \land \neg A) \lor (\star A \land \neg B) \rightarrow \star (A \lor B)
\]
The axioms are the following:

\[ P_1 : 0 \phi, \text{ where } \phi \text{ is any classical valid formula in the connectives } \land, \lor \text{ and } \neg. \]
\[ P_2 : \neg(A \land \neg B) \leftrightarrow \neg A \lor \neg B \]
\[ P_3 : \neg(A \lor B) \leftrightarrow \neg A \land \neg B. \]

For Pinter, in the case of pattern recognition, the information received consists of units of information, which may be interpreted as vectors of information, which form a description space, which may be schematized in the following diagram:

```
 +------------------+
 |                  |
 | - P              |
 |                 |
 | * P              |
 |                 |
 | 0 P              |
```

Pinter’s system is a paraconsistent logic, being a slight modification of \( D_2 \).

According to da Costa and Marconi 1989, the weakly paraconsistent semantics proposed in Rescher and Brandom 1980 may also be seen as producing a version of discursive logic, by understanding that a formula is regarded as true if and only if it is true according to some “world”, being such a “world” identified with the “discursive source”. Hence, a contradiction cannot be true, but its components, one formula and its negation, can both be true. Marconi 1979a shows that Jaśkowski’s logic is a linguistic extension of a previous version of Rescher and Brandom’s system.

We can extend up \( D_2 \) to a first-order predicate calculus \( D_2^{+} \), using the language \( L^\ast \) of \( S5 \) with quantification and equality, denoted by \( S5Q^{=} \), and having that a formula \( A \) of \( L^\ast \) is a thesis of \( D^\ast \) if and only if \( \Diamond \hat{A} \) is a theorem of \( S5Q^{=} \), where \( \hat{A} \) is the universal closure of \( A \). Da Costa and Chuaqui 1999 prove that the logic
of a certain version of the concept of pragmatic truth is a form of D** (see Mikemberg et al. 1986).

The authors interpreted discourse logic as a species of logic of the acceptance of scientific hypotheses.

It is interesting to observe that, in Poland, research on paraconsistent logics is done especially in discourse logic. Dubikajtis and Kotas were Jaśkowski’s students and da Costa met Dubikajtis in Paris, in 1967. Since then, several Polish logicians have worked in collaboration with da Costa.

§3. Da Costa’s Paraconsistent logics

Da Costa created numerous kinds of paraconsistent logics, which have been studied and developed by him, his students and collaborators.

The hierarchy of propositional calculi $C_n$, $1 \leq n \leq w$, constructed in 1963 and satisfying the conditions expressed in §1.4, are the most widely known and studied.

In this paragraph we present the hierarchies $C_n$, $C_n^*$, $C_n^\equiv$ and $D_n$, $1 \leq n \leq w$, of da Costa’s propositional paraconsistent logics, predicate calculi and predicate calculi with equality and calculi of descriptions, respectively.

We give some definitions, discuss several of the most important results which characterize the systems, and introduce different kinds of semantics for da Costa’s calculi.

Then, we present other paraconsistent logics related to da Costa’s work.

After studying the algebraization of paraconsistent systems, we finally present several paraconsistent set theories.

3.1 The Hierarchy $C_n$, $1 \leq n \leq w$

The calculus $C_1$ has as primitive symbols propositional variables, the connectives $\neg$, $\lor$, $\land$, $\supset$ and parentheses.

The notions of formula, theorem, as well as the conventions and notations are the usual classical ones of Kleene 1952.

The operator $\circ$ is defined by:

$$A'' = \neg (A \land \neg A),$$
and $A''$ reads as $A$ is a well-behaved formula.

The axiom schemata and rule of $C_1$ are the following:

**AXIOM 1:** $A \supset (B \supset A)$

**AXIOM 2:** $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$

**AXIOM 3:** $A \supset (B \supset A \& B)$

**AXIOM 4:** $A \& B \supset A$

**AXIOM 5:** $A \& B \supset B$

**AXIOM 6:** $(A \supset C) \supset ((B \supset C) \supset (A \lor B) \supset C))$

**AXIOM 7:** $A \supset A \lor B$

**AXIOM 8:** $B \supset A \lor B$

**AXIOM 9:** $\neg \neg A \supset A$

**AXIOM 10:** $A \lor \neg A$

**AXIOM 11:** $B'' \supset ((A \supset B) \supset (A \supset \neg B) \supset \neg A))$

**AXIOM 12:** $A'' \& B'' \supset (A \supset B)'' \& (A \& B)''' \& (A \lor B)'''$

**R:** $\frac{A, A \supset B}{B}$

We observe that AXIOM 11 is just the usual *reductio ad absurdum*, with an additional hypothesis: that from $A \supset B$ and $A \supset \neg B$ we can obtain $\neg A$, only if $B$ is well-behaved; and AXIOM 12 asserts the sufficient conditions for propagation of well-behaviourism.

The above postulates have intuitive or informal justifications, as we can see in *da Costa and Carnielli 1986*.

New connectives are defined, for $0 < n < \omega$:

$$A^n = \underbrace{A'''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''
In every $C_n$, $1 \leq n < w$, the strong negation $\ast$ is defined by:

$$\neg \ast A =_{df} \neg A \& A^{(n)}$$

It is easy to see that, in every $C_n$ ($1 \leq n < w$), $\neg \ast$ has all the properties of classical negation. For that, it suffices to prove that

$$\vdash_{C_n} (A \supset B) \supset ((A \supset \neg \ast B) \supset \neg \ast A).$$

Let the classical propositional calculus be denoted by $C_w$.

Now, we state some of the most important results about the calculi $C_n$, $1 \leq n \leq w$, without proving them. Their proofs, when it is not indicated, are in the papers mentioned in §1.4.

**THEOREM 3.1.1:** All the rules and valid schemata of the classical positive propositional calculus are valid in $C_n$, $1 \leq n \leq w$.

**THEOREM 3.1.2:** If $A$ is a theorem in the intuitionistic positive propositional calculus, then $A$ is a theorem in $C_w$.

**THEOREM 3.1.3:** If $\Gamma$ is a set of formulas and $A_1, \ldots, A_m$ are the prime components of the formulas of $\Gamma \cup \{A\}$, then a necessary and sufficient condition for $\Gamma \vdash_{C_n} A$ is that $\Gamma, A_1^{(n)}, \ldots, A_m^{(n)} \vdash_{C_n} A$, for $1 \leq n \leq w$.

By this theorem, we have that all the theorems of $C_n$ are valid in $C_n$, $1 \leq n \leq w$, for “well-behaved” propositions.

The following theorem is the Deduction Theorem for the $C_n$, $1 \leq n \leq w$.

**THEOREM 3.1.4:** If $\Gamma$ is a set of formulas, $\Gamma, A \vdash_{C_n} B$ if, and only if, $\Gamma \vdash_{C_n} A \supset B$.

**DEFINITION 3.1.5:** A non-trivial system $S$ is finitely trivializable if there is a formula (not a schema) $F$ in $S$ such that the system obtained from $S$ by adding $F$ as a new postulate is trivial.

**THEOREM 3.1.6:** The systems $C_n$, $1 \leq n \leq w$, are non-trivial.
The $C_n$, $1 \leq n < w$, are finitely trivializable and $C_w$ is not.

**THEOREM 3.1.7:** All the axioms of $C_n$, $1 \leq n \leq w$, are independent. And every system in the hierarchy $C_n, C_1, \ldots, C_n, \ldots, C_w$ is strictly stronger than those which follow it.

The proof of this theorem is in Alves 1976, and according to it $C_w$ is the weakest calculus of the hierarchy. The next theorem is in Arruda 1975a.

**THEOREM 3.1.8:** The $C_n$, $1 \leq n \leq w$ are not decidable by finite matrices.

**THEOREM 3.1.9:** Let $A$ be a formula of $C_n$ and $A^*$ the formula obtained from $A$ by substituting $-^*$ for $\neg$. If $\vdash_{C_n} A$, then $\vdash_{C_n} A^*$, $1 \leq n \leq w$.

This theorem shows that $C_n$ can be obtained "inside" $C_n$, $1 \leq n \leq w$, because for $\neg^*$, $\vee$, $\&$, and $\supset$ all the theses of $C_n$ are valid in $C_n$.

We observe that the following schemata are provable in $C_n$, $1 \leq n \leq w$, the first one being the paraconsistent version of the rule of "reductio ad absurdum":

- If $\Gamma, B \vdash_{C_n} A^{(n)}$ and $\Gamma, B \vdash_{C_n} A$ and $\Gamma, B \vdash_{C_n} \neg A$, then $\Gamma \vdash_{C_n} \neg B$;
- $\Gamma, A, \neg A, A^{(n)} \vdash_{C_n} B$;
- If $\Gamma \vdash_{C_n} B^{(n)}$ and $\Gamma, A \vdash_{C_n} B$ and $\Gamma, A \vdash_{C_n} \neg B$, then $\Gamma \vdash_{C_n} \neg A$.

The following ones are not valid in $C_n$, $1 \leq n \leq w$:

- $\neg A \supset (A \supset B)$
- $\neg A \supset (A \supset \neg B)$
- $A \land \neg A \supset B$
- $A \supset \neg \neg A$
- $(A \supset B) \lor (\neg A \supset B)$

The schema $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$ is also not valid in $C_n$, $1 \leq n \leq w$. If we adjoin it to $C_1$, we obtain the classical propositional calculus $C_n$. 
Most of the usual results for $C_1$ also hold for $C_n$, $n \geq 2$, but some results, like for instance $(A \land \neg A)^n$, are theorems in $C_1$ but are not in $C_n$, $n \geq 2$.

Originally, $C_1$ was introduced with one thirteenth axiom, which was in fact a particular case of the following result.

**Theorem 3.1.10**: In $C_n$, $1 \leq n \leq w$, $A^{(n)} \supset (\neg A)^{(n)}$ is provable.

Between the systems $C_n$ and $C_{n+1}$, for $n \geq 0$, it is possible to construct a new hierarchy of calculi whose properties are similar to those of the $C_n$, $1 \leq n \leq w$. In these new calculi there is reduction of negations, what does not happen in $C_n$, $1 \leq n \leq w$.

Urbas 1989 attempts to remedy the fact of da Costa’s hierarchy $C_n$, $1 \leq n \leq w$, not to enjoy the property of intersubstitutivity of provable equivalents (S.E.).

The author argues that both in view of da Costa’s conditions for paraconsistent systems and for independent reasons, the named failure constitutes a deficiency of the systems. He claims that the property appears to be required for the systematic behavior of the connectives in a logic and that its absence from the $C$-systems is responsible for the difficult in obtaining natural algebraic and semantic perspectives on these systems. But we do not agree with this position.

After proving that the $C_n$ do not enjoy S.E., he introduces two extensions of the $C$-systems, one of them by addition of the rule $RC \left( \frac{(!C\rightarrow D)}{-D\rightarrow \bot} \right)$ and the other one by adding the rule $EC \left( \frac{(!C\equiv D)}{-D\rightarrow \bot} \right)$. Then, he shows that both extensions collapse all but the base system $C_w$ into classical logic and finally proves that there is no extension of the stronger $C$-systems which both enjoys S.E. and is weaker than classical logic.

The author suggests alternative methods to construct other related hierarchies, but we believe that a good approach would be by looking for an adequate definition for a “strong equivalence” compatible with the paraconsistent negation of the $C$-systems, which perhaps could yield to S.E.

We now observe that some results on the development of higher order logics corresponding to the systems $C_n$ may be found in Alves
and Moura 1978.

3.2 The Hierarchies $C^*_n$, and $C^=_n$, $1 \leq n \leq w$

Da Costa extended his hierarchy $C_n$, $1 \leq n \leq w$, to corresponding hierarchies of first-order predicate calculi $C^*_n$, $1 \leq n \leq w$, first-order predicate calculi with equality $C^=_n$, $1 \leq n \leq w$, and theories of description of description $D_n$, $1 \leq n \leq w$.

The axioms and rules of $C^*_n$, $1 \leq n < w$, are those of the corresponding $C_n$, plus the axioms and rules of classical first order predicate calculus related to the quantifiers, plus the following:

**AXIOM 1*: If $A$ and $B$ are congruent formulas (cf. Kleene 1952), or one is obtained from the other by the elimination of vacuous quantifiers, then $A = B$ is an axiom.

**AXIOM 2*: $\forall x (A(x))^{[n]} \supset (\forall x A(x))^{[n]}$

**AXIOM 3*: $\forall x (A(x))^{[n]} \supset (\exists x A(x))^{[n]}$.

The postulates of $C^*_w$ are those of $C_w$, plus the ones of classical first-order predicate calculus and AXIOM 1*.

The postulates of $C^=_n$, $1 \leq n \leq w$ are those of the corresponding $C^*_n$, plus the two usual axioms for equality $=.$

The previous theorems stated in 3.1 are easily extended to $C^*_n$ and $C^=_n$, $1 \leq n \leq w$.

The calculus $C^*_w$, obtained from $C^*_1$ by adding the schema $\neg (A \& \neg A)$ as a new postulate, is the classical first-order predicate calculus.

The main results about these predicate calculi are the following.

**THEOREM 3.2.1**: In $C^*_1$ the following are not valid:

- $\exists x \neg A(x) \equiv \forall x A(x)$
- $\forall x \neg A(x) \equiv \exists x A(x)$
- $\exists x A(x) \equiv \forall x \neg A(x)$
- $\exists x \neg A(x) \equiv \forall x A(x)$
THEOREM 3.2.2: The calculi $C_n^*$ and $C_n^=$, $1 \leq n \leq w$, are undecidable.

THEOREM 3.2.3: If $\Gamma \vdash C_n^* A$, then all the $k$-transforms of $A$ are deducible in $C_n$, $1 \leq n \leq w$ from the $k$-transforms of the formulas in $\Gamma$.

THEOREM 3.2.4: If $A$ denotes a formula of $C_n$, then $\vdash C_n^* A$ if, and only if, $\vdash C_n^= A$, $0 \leq n \leq w$. If the symbol $=$ does not occur in the formula $A$, then $\vdash C_n^= A$ if, and only if, $\vdash C_n^* A$, $1 \leq n \leq w$.

3.3 The Hierarchy $D_n$, $1 \leq n \leq w$

The calculi of descriptions $D_n$, $1 \leq n \leq w$, are obtained from $C_n$, $1 \leq n \leq w$, by introducing the description symbol $\nu$ and the following postulates, with the usual restrictions and with the symbols and conventions of Rosser 1953:

AXIOM 1$^D$: $\forall x F(x) \supset F(\nu yQ(y))$

AXIOM 2$^D$: $\forall x (P(x) \equiv Q(x)) \supset \nu x P'(x) \equiv \nu x Q(x)$

AXIOM 3$^D$: $\nu x F(x) = \nu y F(y)$

AXIOM 4$^D$: $P(\nu yQ(y)) \supset \exists x P(x)$

AXIOM 5$^D$: $\exists x P(x) \supset (\forall x ((\nu x P(x) = x) \equiv P(x)))$

Several results mentioned for the other hierarchies, hold for the $D_n$, $1 \leq n \leq w$.

THEOREM 3.3.1: $D_n$ is a conservative extension of $C_n^=$, for $1 \leq n \leq w$.

3.4 Decidability and Semantics for $C_n$, $1 \leq n \leq w$

Several logicians have introduced different kinds of semantics for
da Costa's calculi, proving their decidability.

Da Costa, Alves, Arruda and Loparic introduced an interesting semantics of valuations for the calculi $C_n$, $1 \leq n \leq w$, and the corresponding predicate calculi based on them, which generalize the classical valuation semantics and relative to which they proved that these logics are sound and complete.

The Henkin-style semantics for $C_n$, $1 \leq n \leq w$, presented in da Costa and Alves 1977 is based on the following definition of valuation.

**DEFINITION 3.4.1:** If $F$ is the set of formulas of $C_n$, $1 \leq n \leq w$, a *valuation* for $C_n$ is a function $v : F \rightarrow \{0, 1\}$ such that:

1. If $v(A) = 0$, then $v(\neg A) = 1$,
2. If $v(\neg \neg A) = 1$, then $v(A) = 1$,
3. If $v(B^{(n)}) = v(A \supset B) = v(A \supset \neg B) = 1$, then $v(A) = 0$,
4. $v(A \supset B) = 1$ iff either $v(A) = 0$ or $v(B) = 1$,
5. $v(A \& B) = 1$ iff $v(A) = v(B) = 1$,
6. $v(A \lor B) = 1$ iff either $v(A) = 1$ or $v(B) = 1$.
7. If $v(A^{(n)}) = v(B^{(n)}) = 1$, then $v((A \supset B)^{(n)}) = v((A \& B)^{(n)}) = v((A \lor B)^{(n)}) = 1$

**DEFINITION 3.4.2:** A valuation $v$ is a *model* of a set of formulas $\Gamma$ if, and only if, $v(A) = 1$ for every $A$ in $\Gamma$.

As usually, we denote that $v(A) = 1$ in every valuation $v$ which is a model of $\Gamma$, by $\Gamma \models A$.

**DEFINITION 3.4.2:** A set of formulas $\Gamma$ is said to be *trivial* if the set of consequences of $\Gamma$ is $F$; $\Gamma$ is said to be *inconsistent* if there is at least one formula $A$ such that $A$ and $\neg A$ are both consequences of $\Gamma$.

The properties of maximal consistent sets are easily extended to maximal non-trivial sets and, by substituting $\neg^*$ for $\neg$, the following
results are obtained.

THEOREM 3.4.4: Every non-trivial set of formulas, consistent or not, has a model.

THEOREM 3.4.5: \( \Gamma \vdash C_n A \) if, and only if, \( \Gamma \models A \), \( 1 \leq n \leq w \).

Using the concept of valuation and defining the quasi-matrices, Alves 1976 proves the decidability of da Costa's propositional systems.

THEOREMS 3.4.6: The calculi \( C_n \), \( 1 \leq n \leq w \), are decidable.

A two valued semantics and a decision method for \( C_w \) is given in Loparič 1977 and 1978. The main definitions and theorems, extremely technical, appear in Loparič and Alves 1980.

Arruda and da Costa 1977 extend the above semantics of \( C_n \) for \( C^r_n \) and \( D_n \), \( 1 \leq n \leq w \); but a valuation semantics for \( C^-_n \), \( C^-_w \) and \( D_w \) calls for a special treatment.

The method of semantical valuations can also be employed for the semantical study of higher-order paraconsistent logics.

Alves 1976 also introduces a similar semantics to those hierarchies of calculi constructed between \( C_n \) and \( C_{n+1} \), \( n \geq 0 \).

The Argentinian logician Fidel began to work on paraconsistent logic in Argentina, in 1969, with da Costa. In 1970, by algebraic methods, he had proved the decidability of \( C_n \), \( 1 \leq n \leq w \).

In Fidel 1977 a \( C_w \)-structure is defined as a system \( < X, \land, \lor, \top, 1, \{ N_x \}_{x \in X} > \), with \( < X, \land, \lor, \top, 1 > \) an implicative lattice, in the sense of Curry 1963, with a last element 1, and \( \{ N_x \}_{x \in X} \) a family of non-empty subsets of \( X \) satisfying, for every \( x \in X \), the conditions:

1. For every \( x' \in N_x \), \( x \lor x' = 1 \); and

2. For every \( x' \in N_x \), there exists \( x'' \in N_x \) such that \( x'' \leq x \).

DEFINITION 3.4.7: A \( w \)-valuation is a function \( \nu : F \rightarrow X \), such that:
1. $\nu(P) \in X$, with $P$ a propositional variable;

2. $\nu(A \& B) = \nu(A) \land \nu(B)$
   $\nu(A \lor B) = \nu(A) \lor \nu(B)$;

3. $\nu(\neg A) \in N_{\nu(A)}$, with $A$ a propositional variable or a formula
   of the form $B \& C$, $B \lor C$ or $B \supset C$;

4. $\nu(\neg \neg A) \in N_{\nu(\neg A)}$ and $\nu(\neg \neg A) \leq \nu(A)$.

**DEFINITION 3.4.8:** $A$ is $w$-valid in a $C_w$-structure $X$, what is denoted by $\models_w^X A$, if $\nu(A) = 1$ for any $w$-valuation $\nu$ into $X$; $A$ is valid, what is denoted by $\models_w A$, if for every $C_w$-structure it is true that $\models_w^X A$.

The next theorem gives the soundness and completeness of $C_w$.

**THEOREM 3.4.9:** For any formula $A$ of $C_w$, the following are equivalent:

1. $\vdash_{C_w} A$;

2. $\models_w A$; and

3. $\models_w^X A$, for any $N_{\Gamma}$-saturated $C_w$-structure with at most $2^2$ elements, $\Gamma$ being the number of subformulas of $A$.

From condition (3) above, we have that $C_w$ is decidable.

The method developed for $C_w$ can be extended to $C_n$, $1 \leq n \leq w$.

**Marconi 1980** presents another decision method for the $C_n$, $1 \leq n \leq w$, using semantical tableaux. He claims that the method gives a better understanding of the meaning of negation in the systems.

**Carnielli 1987**, using his systematization of finite many-valued logics through the method of tableaux, introduces a tableaux type approach to $C_1$, the system $TC_1$, also showing that $C_1$ is decidable.

**Raggio 1968**, when the decidability of the $C_n$, $1 \leq n \leq w$ was still an open problem, tries to solve the question through a Gentzen's sequence formalization of the calculi. The sequence calculi
equivalent to $C_n$, $1 \leq n \leq w$ could not be proved decidable, but he constructs the hierarchy of sequence calculi $WG_n$, $1 \leq n \leq w$, which are decidable and have similar properties to the calculi $C_n$, $1 \leq n \leq w$.

Semantics for logics having descriptions and the Hilbert $\varepsilon$-symbol are studied by Abar 1985 and Yamashita 1985. They also demonstrate that the general theory of variable binding term operators can be established on the basis of the predicate logics $C_n^\forall$, $1 \leq n \leq w$.

A natural question concerning the calculi $C_n$, $1 \leq n \leq w$ is if they may be naturally extendable to treat first-order theories. This is developed in Alves 1984 and 1989.

Alves changes the axiom $\neg\neg A \supset A$ by $\neg\neg A \equiv A$, in order to obtain a system $C_1^\forall$ really stronger than da Costa's $C_1^\forall$. He defines a pre-structure for the language of $C_1^\forall$ and constructs particular valuations, which differentiates the proof of the Completeness Theorem for $C_1^\forall$ from the classical one.

**THEOREM 3.4.10:** A $C_1^\forall$-theory is non-trivial if, and only if, it has a model.

Hence, he shows that a great part of the classical model theory can be extended to the paraconsistent first-order logics $C_n^\forall$, $1 \leq n \leq w$.

We observe that, in order to obtain a convenient Equivalence Theorem, it was necessary a special definition for a paraconsistent occurrence of a subformula $B$ in $A$.

A similar study had been done in D'Ottaviano 1982 for a kind of many-valued, modal and paraconsistent theories, named $J_3$-theories, which will be presented in a subsequent paragraph. In the $J_3$-Model Theory most of the classical model theorems were generalized.

We finally observe that the study of da Costa's systems provided the development of a general theory of valuations, in Loparić and da Costa 1984, which can be applied to any logic, whatsoever.

### 3.5 Other Systems of Paraconsistent Logics Related to da
Costa's Work

Working independently of da Costa, the Argentinian Asenjo defined a paraconsistent logic with interesting philosophical motivations, in a paper published in 1966.

Several other paraconsistent systems have been introduced by da Costa and his students and collaborators.

We present some of these systems in this paragraph.

It is interesting to observe that, as in the classical case, it is not difficult to extend several of the paraconsistent logics to modal and tense logic. And tense paraconsistent logic has been used to formalize dialectical logic, as in da Costa and Wolf 1980.

3.5.1. Asenjo's Calculus of Antinomies

Asenjo 1966 introduces a calculus of antinomies, a work begun in 1954 and written in 1964. The author defines his system using the three-valued matrix of Kleene 1952 (pp. 332-4), with the values 0, 1 and 2 for true, false and antinomic, respectively.

Da Costa's system $C_0$, is mentioned as an example of a calculus of antinomies.

Asenjo's system was modified by Tamburino 1973, being redefined by a three-valued matrix with two distinguished truth-values 0 and 2.

Tamburino presents an axiomatic $L$ for the system, using two different denumerable sets of propositional letters, those which take only the values 0 and 1, and those which take only the value 2. He extends $L$ to the first-order predicate calculus $K$, also with two different sets of predicate letters.

3.5.2. The Systems $P$, $P'$ and $J_n$, $1 \leq n \leq 5$

Looking for adequate systems for the construction of paraconsistent set theories, Arruda and da Costa 1966 and 1984 defined and studied the systems $P$ and $P'$, which are weak paraconsistent relevant logics and whose positive parts are a kind of minimal relevant logic.

Based on $P$ and $P'$, respectively, the authors obtained the predicate calculi $PQ$ and $PQ'$.

In these systems the law of modus ponens, $(A \& (A \rightarrow B) \rightarrow B)$,
the rule and the law of contraction of implications, \((A \rightarrow (A \rightarrow B))/A \rightarrow B\) and \((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)\), and the deduction theorem are not valid. They are not finitely trivializable, they are not decidable by finite matrices and can be extended to modal and tense logics.

The semantical study of the system \(P\) and some of its dialectical extensions are in Routley and Loparić 1979 and 1980.

Arruda and da Costa 1970, with similar objectives, introduced the systems \(J_n\), \(1 \leq n \leq 5\), in which the rule of modus ponens is also not valid. They also define the respective predicate calculi \(J_n^*\), \(1 \leq n \leq 5\).

Bunder 1983 studies the \(J_n\), \(1 \leq n \leq 5\).

3.5.3. The Logics of Vagueness, the Systems \(P^*\) and Two New Hierarchies

Arruda and Alves 1979a discuss the construction of logics adequate to the study of the problem of vagueness. They suppose that a logic of vagueness contains classical logic and characterize four kinds of vagueness related to negation, when the law of excluded middle or the law of contradiction is not valid.

Da Costa’s \(C_1\) is one of these four systems and the other ones are obtained similarly to it.

Arruda and Alves 1979b use the method of valuations in order to obtain a semantics for the logics of vagueness.

Sette 1973 introduces the three-valued paraconsistent logic \(P^*\), using the tables for implication and negation of da Costa 1968a. After axiomatizing the system, he shows that there is no calculus between \(P^*\) and the classical propositional calculus.

Bunder 1980 constructs a hierarchy of calculus \(B_0, B_1, B_2, \ldots\) having some properties similar to those of da Costa’s systems \(C_n\). \(B_0\) is the intuitionistic propositional calculus and its postulates are those of positive intuitionistic logic plus:

**AXIOM 1:** \(\vdash^2 A \supset B\)

**AXIOM 2:** \((A \supset^{n+1} B) \supset A\), where \(^{1}A = A, ^{1}A = A \& \neg A\) and \(^{n+1}A = ^{n}A \& \neg ^{n}A.\)
The Australian author also shows that intermediate or classical versions of $B_n$ are obtained by adjoining the axioms 10 or 11 of da Costa’s systems $C_n$.

Da Costa and Marconi 1986 study certain hierarchies $P_n$, $P_n^+$, $P_n^-$ ($1 \leq n \leq w$) of paracomplete logics (a proposition and its negation can be both false), duals in a certain sense of da Costa’s hierarchies. The systems are related to fuzzy logic and to paraconsistent logic.

Da Costa 199+ also introduce some logical systems which are simultaneously paraconsistent and paracomplete. These systems are related to the theory of vagueness and to Hegel’s logical conceptions.

3.5.4. $J_3$-Theories

Based on Jaśkowski’s ideas, D’Ottaviano and da Costa 1970 introduce a three-valued propositional system $J_3$, with two distinguished truth-values, which is paraconsistent and reflects some aspects of certain types of modal logics.

The calculus $J_3$ is given by the matrix $M = \{0, 1/2, 1\}$, $\vee, \nabla, \neg >$, where $\vee$, $\nabla$ and $\neg$ are defined by the following tables:

\[ A \lor B \]

<table>
<thead>
<tr>
<th>A \ B</th>
<th>0</th>
<th>1/2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>1</td>
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<td>1/2</td>
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<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ A \nabla A \]

<table>
<thead>
<tr>
<th>A</th>
<th>0</th>
<th>0</th>
<th>1/2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
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<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

D’Ottaviano 1982 and 1985a axiomatize $J_3$ and establish relations between this calculus and several known logical systems as, for example, intuitionism. She especially emphasizes the close analogy between $J_3$ and Łukasiewicz’ three-valued propositional calculus $L_3$.

The author also introduces the corresponding three-valued first-order $J_3$-Theories in whose languages may appear other equalities in addition to ordinary identity, and which extend the first-order predicate calculus $J_3^*$. She proves the Completeness Theorem and the Compactness Theorem for $J_3$-theories.

The model theory for $J_3$-theories reflects much of the classical
model theory. D’Ottaviano 1982, 1985b and 1987 obtain generalized versions of the following classical results: Keisler Model Extension Theorem, Loś-Tarski Theorem, Chang-Łoś Suszko Theorem, Tarski Cardinality Theorem, Löwenheim-Skolem Theorem, Joint-Consistency Theorem, Craig Interpolation Lemma, Beth-Padoa Definability Theorem, the Quantifier Elimination Theorem and many of the usual theorems on complete theories and categoricity.

In some cases, as for example the Extension Model Theorem and the Definability Theorem, she proves more than one generalization of the classical results, all of them compatible both with the many-valued aspects and the modal aspects of J₃-theories. The proofs reflect the existence of more than one distinguished truth-value in the matrix defining J₃-theories.

The system J₃ was the only paraconsistent system introduced by Epstein 1990, using D’Ottaviano and Epstein 1988.

3.6 The Algebrrization of Paraconsistent systems

In Cₙ, 1 ≤ n ≤ w, the equivalence relation given as usually by the basic (bi)conditional of the systems, is not a congruence relation. It seems that even if an adequate equivalence relation would be defined, in order to be a congruence, it would not reflect algebraically the logical properties of the Cₙ, on account of the characteristics of the basic negation of the calculi.

Arguing if it would be possible to obtain an algebrrization for the calculi Cₙ, 1 ≤ n ≤ w, in the line of Lindenbaum construction for the classical propositional calculus, Mortensen 1980 shows that, under certain assumptions, there exists no equivalence relation which can determine a non-trivial quotient algebra for Cₙ, 1 ≤ n ≤ w. This fact was expected since the construction of the Cₙ.

Three approaches to the algebrrization of the Cₙ, 1 ≤ n ≤ w, have already been considered.

The first algebrrization was presented in da Costa 1966 and studied by him and one of his students in da Costa and Sette 1969.

Da Costa 1966b defines an algebraic structure whose basic relation is an equivalence relation and having operators which are not compatible with this relation (non-monotone operators in the
sense of Curry 1963), but reflect the negation in the algebraic formulation.

According to da Costa, the algebra corresponding to $C_1$, denoted by $AC_1$, is an implicative lattice in the sense of Curry, because of the axioms 1-9 of $C_1$; and must have a first element 1, for $C_1$ is finitely trivializable.

In $AC_1$, he defines an operator $\prime$, for $p''$ being $(p \land p')'$, given by:

\[
\begin{align*}
 p'' & \leq (p \lor q) \lor ((p \lor q') \lor p') \\
 p'' \land q'' & \leq (p \lor q)'' \\
 p'' \land q'' & \leq (p \land q)'' \\
 p'' \land q'' & \leq (p \land q)'' \\
 p \lor p' & = 1 \\
 p' & \leq (p')'' \\
 p'' & \leq p
\end{align*}
\]

Sette 1971 defines the notion of a hyper-lattice structure which is a generalization of the notion of lattice, in order to obtain algebraic properties that are the counterparts of the logical properties of the calculus $C_w$.

The algebra $AC_w$ is an implicative lattice with first element 1 and a non-monotone operator $\prime$, satisfying:

\[
\begin{align*}
 p \lor p' & = 1 \\
 p'' & \leq p
\end{align*}
\]

As introduced in §3.4, Fidel 1977 presents a new kind of algebraic models which are, in a certain sense, the algebraic versions of the $C_w$, $1 \leq \tau \leq w$. This $C_w$-structure is essentially equal to Sette's algebra $AC_w$.

Carnielli and Alcântara 1984 propose the paraconsistent algebras of sets, which reflect in set theoretical terms the main features of $C_1$.

This work introduces the notion of da Costa algebra as a lattice theoretical structure which incorporates the properties of $C_1$, and then shows that any da Costa algebra is isomorphic to a paraconsistent algebra of sets.
Lewin et al 1990 proves that Sette's system $P^1$ is algebrasizable according to Block and Pigozzi 1989 concepts. They also study the kind of algebraic structures associated to the system.

It has also been proved that D'Ottaviano's calculus $J_3$ is Block-Pigozzi algebraizable, a $J_3$-algebra being a Łukasiewicz' algebra. But, for us, this algebraic characterization seems not to express adequately the logical characteristics of $J_3$.

3.7 Paraconsistent Set Theories

Arruda and da Costa introduced several paraconsistent set theories, using paraconsistent logics as their underlying logics and giving stronger formulations for the axiom schema of separation, without restrictions.

These theories try to construct inconsistent sets by avoiding antinomies, but not formal paradoxes.

According to Arruda 1980b, two problems have been investigated: the study of the properties of some sets which do not exist in the usual set theories, but which are supposed to exist in the paraconsistent ones; and the conjecture according to which weaker subjacent logics allow the construction of set theories existentially stronger than the usual ones, that is, set theories in which all the sets of the usual theories also exist but in which there exists at least one set not existant in the usual ones.

3.7.1 The Hierarchy $NF_n$, $1 \leq n \leq w$

Arruda 1964, 1970a, 1970b, 1975, 1980a and 1985 study the hierarchy of paraconsistent set theories of type $NF$, $NF_n$, $1 \leq n \leq w$, based on the systems $C_n$, $1 \leq n \leq w$, introduced in da Costa 1964d and Arruda and da Costa 1964.

Starting with $NF_\omega$, Arruda and da Costa construct the hierarchy of set theories $NF_n$, $1 \leq n \leq w$, having the corresponding calculi of descriptions $D_n$, $1 \leq n \leq w$, as their underlying logics.

Using a weak version of the system $NF$ of Rosser 1958, the theory $NF_\omega$ is defined. Its postulates are those of $D_\omega$, which is essentially equivalent to the calculus of descriptions given by Rosser, plus:
Extensionality Axiom: $\forall x \forall y \forall z \ (x \in y \equiv x \in z) \supset y = z$

Separation Axiom: $\exists y \forall x (x \in y \equiv F(x))$, with $x$ and $y$ being different variables, $y$ does not occurring free in $F(x)$ and $F(x)$ being stratified.

Arruda 1975b and 1980a show that many forms of the axiom schema of separation previously proposed by da Costa for the theories $\mathbf{NF}_n$, $1 \leq n \leq w$, lead to their trivialization. In fact, Arruda 1980a proves that da Costa's formulation of the schema of separation for $\mathbf{NF}_w$, although apparently not leading to the triviality of the system, leads to the "unpleasant result": $\forall x \forall y (x = y)$.

Arruda 1980b says that an apparently sure formulation of the schema of separation for $\mathbf{NF}_n$, $1 \leq n \leq w$, is the one proposed in her paper of 1980a mentioned above, in spite of it being not completely adequate for her purpose. She presents a weak form of $\mathbf{NF}_n$, $1 \leq n \leq w$, and a strong version of $\mathbf{NF}_n$.

The Russel set for $\mathbf{NF}_n$, is $R_n = \{x : x \not\in x\}$ and for $\mathbf{NF}_n$, $1 \leq n < w$, is $R_n = \{x : x \not\in x & (x \in x)^{(n)}\}$. Since $R_n$ trivializes $\mathbf{NF}_n$, it cannot exist in $\mathbf{NF}_n$, $1 \leq n < w$, but it can exist in $\mathbf{NF}_m$, $m > n$.

Arruda introduces the following specific postulates for $\mathbf{NF}_n$, $1 \leq n \leq w$.

**Schema of separation for $\mathbf{NF}_1$**: $\exists y \forall x (x \in y \equiv F(x))$, where $x$ and $y$ are different variables, $y$ does not occur free in $F(x)$, and $F(x)$ is stratified or, if not, $F(x)$ is $x \not\in x$.

**Schema of Separation for $\mathbf{NF}_n$, $2 \leq n < w$**: $\exists y \forall x (x \in y \equiv F(x))$, where $x$ and $y$ are different variables, $y$ does not occur free in $F(x)$, and $F(x)$ is stratified or, if not, $F(x)$ is any one among the following formulas $x \not\in y$, $((x \in x) & (x \in x)^{(n+1)})$, $1 \leq m < n$.

The specific postulates of $\mathbf{NF}_w$ are extensionality and the following, with the empty set $\phi$, the complement set $C$ and the universal set $\mathcal{U}$ being adequately defined:

**Schema of Separation for $\mathbf{NF}_w$**: $\exists y \forall x (x \in y \equiv F(x))$, where
$x$ and $y$ are different variables, $y$ does not occur free in $F(x)$, and $F(x)$ is stratified or, if not, the symbol $\supset$ does not occur in $F(x)$.

AXIOM $w^1 : \forall x \forall y(y = Cx \equiv x \cap y = \phi \& x \cup y = \mathcal{U})$

AXIOM $w^2 : \forall x \forall y(x \neq y \equiv \exists z(z \in x \& z \in Cy) \lor \exists z(z \in Cx \& z \in y)$.}

We observe that the sets $\phi$, $\mathcal{U}$ and $Cx$ play no special role in $\mathbf{NF}_w$, because its negation is very weak, but they are important, for Arruda shows that in $\mathbf{NF}_w$ the operations $\cup$, $\cap$ and $C$ and constants $\phi$ and $\mathcal{U}$ satisfy all the postulates of a Boolean algebra.

By introducing the notion of $x$ being a Quine individual if $x = \{x\}$, Arruda proves some strongly heterodox properties of Russell sets, giving the idea of some of the properties of inconsistent sets.

By supposing that the systems $\mathbf{NF}_n, 1 \leq n \leq w$ are non-trivial, she also shows that $\mathbf{NF}_{n+1}$ is existentially stronger than $\mathbf{NF}_n$, $0 \leq n < w$; and that $\mathbf{NF}_w$ is existentially stronger than each $\mathbf{NF}_n$, $0 \leq n \leq w$. But it is not true that $\mathbf{NF}_m$ is a proper subsystem of $\mathbf{NF}_n$ for any $m < n$.

Da Costa 1974b also proves that the usual incompleteness results can be extended to $\mathbf{NF}_n, 1 \leq n \leq w$.

Going on her studies on paraconsistent set theories, Arruda 1985 finally shows that in any da Costa paraconsistent set theory of type $\mathbf{NF}$, the axiom schema of separation must be formulated exactly as in $\mathbf{NF}$; for, in the contrary, some paradoxes are derivable that invalidate the theory.

She also proves that in any da Costa’s paraconsistent set theory with Russell’s set $R$, $\cup \cup R$ is the universal set. And that in any of these theories the existence of Russell’s set is incompatible with a general formulation of the axiom schema of separation and replacement.

In fact, Arruda and da Costa 1966 had early identified the problem. They observed that, as Moh Shaw-Kwei had shown, unrestricted separation principles were not compatible with systems which contain the rule of modus ponens and absorption principles of some order. Then, the authors introduced the systems $\mathbf{P}$ and $\mathbf{P}^*$, which are of much interest for other purposes, in order to deal with
the problem. They also defined the systems $J_n$, $1 \leq n \leq 5$, in 1968, under the same aims.

Based on the last Arruda's results, da Costa 1986 builds certain new versions of his set theories, which are in a precise sense stronger than the corresponding classical theories. After showing that if $\text{NF}_n$ is consistent then all the $\text{NF}_i$, $1 \leq n \leq w$ are non-trivial, he proves the following important result: if the classical systems of set theory correlated with da Costa's systems are consistent, then the latter are nontrivial.

3.7.2. Other Paraconsistent set Theories

Paraconsistent logics other than da Costa's $C_n$ systems are also adequate for the construction of set theories.

Arruda and da Costa 1970 introduce the set theories of type $ZF$ based on $J_n$, $1 \leq n \leq 5$, with the following axiom schema of separation.

$$\exists y \forall x (x \in y \equiv F(x)),$$

and proved, also by finitary methods, that they are not trivial; in 1974, they proved that, in the set theories based on $J_n^*$, $2 \leq n \leq 5$, every two sets are identical, what seems not to be true for the theory based on $J_1^*$.

Tamburino 1973 constructs an antinomic set theory based on the first-order predicate calculus $K$. For the only predicate letters $=$ and $\in$, we have that for any $x$ and $y$ it is not the case that $x = y$ and $x \neq y$, but there exists a pair $x$ and $y$ such that $x \in y$ and $x \notin y$.

Asenjo and Tamburino 1975 give another formulation of the axiom schema of separation of the system, such that the Russell set $R$ exists in their set theory and it is proved that $R \in R \& R \notin R$. It seems that the paradox of Curry is avoided in the theory, but Arruda 1989 says that, on account of the system $C_w$ being a subsystem of Asenjo's calculus of antinomies $L$, it is almost certain that it is possible to show that every two sets are equal.

Arruda and da Costa 1984 obtain some weak set theories based on $PQ$ and $PQ^*$ and prove by finitary methods that they are non-trivial. They also prove that some set theories, with no restriction in the axiom schema of separation and based on strong
logics without the rule of modus ponens, are non-trivial.

§4. Relevant and Dialectical Logics

Relevant and dialectical logics are studied mainly in Australia, New Zealand, Belgium, Italy and the United States.

In relevant logics schemas and rules like \( A \rightarrow (\neg A \rightarrow B) \) and \( A, \neg A \vdash B \) are not valid and they can be used as the underlying logics of inconsistent but nontrivial theories.

Paraconsistency was not the main motivation of relevant logics, in spite of most of them being paraconsistent.

Some of the earliest systems of relevant logic were paraconsistent, as for instance the logic of Lewis of 1912 and the weak implicational logic of Church 1951. But the important systems II' and II'' of Ackermann 1956 were not.

Anderson and Belnap 1976 reaxiomatize II', without any motivation related neither to paraconsistent systems nor to contradiction.

Even now, the American logicians who work on relevant logic, like Belnap 1977 and Dunn 1976, they do not agree with paraconsistent approaches.

On the contrary, most of the relevant Australian logicians have assumed strong paraconsistent positions.

4.1. The Australian Group

Some sporadic work of semantical aspect, the research on paradoxes from a statement-incapability and incompleteness aspect, the work on non-existent objects including impossible objects and quantificational theories that could include such objects in their domains, and the logical and especially semantical study of paradox-free implication and conditionality led the Australian relevant logic to the paraconsistent approach.

In Routley et al. 1983 there is a good study about relevant logic, and in Marconi 1979 we have a discussion about the relation between paraconsistent and relevant logic.

Routley, Meyer and their group of research have mainly worked on the relevant approach to dialectical logic, especially on the semantical aspects of it.
Routley and Meyer 1976, looking for the “minimal requirements on dialectical logic that can be extracted from the informal arguments standard texts employ”, construct the relevant systems \( \text{DM} \) and \( \text{DL} \), which are strong paraconsistent logics and are proposed to be a possible propositional bases for dialectical logic.

The primitive connectives of \( \text{DM} \) are \(-, \& \) and \( \lor \), \( A \lor B \) is defined as \( \neg(\neg A \& \neg B) \), and \( p_o \) is a sentential constant. The postulates and rules of \( \text{DM} \) are the following:

\[
\begin{align*}
\text{AXIOM D}_1 & : A \rightarrow A \\
\text{AXIOM D}_2 & : (A \rightarrow B) \& (B \rightarrow C) \rightarrow (A \rightarrow C) \\
\text{AXIOM D}_3 & : A \& B \rightarrow A \\
\text{AXIOM D}_4 & : A \& B \rightarrow B \\
\text{AXIOM D}_5 & : ((A \rightarrow B) \& (A \rightarrow C)) \rightarrow (A \rightarrow (B \& C)) \\
\text{AXIOM D}_6 & : (A \& (B \lor C)) \rightarrow ((A \& B) \lor (A \& C)) \\
\text{AXIOM D}_7 & : \neg\neg A \rightarrow A \\
\text{AXIOM D}_8 & : (A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A) \\
\text{AXIOM D}_9 & : (A \rightarrow B) \rightarrow (\neg(A \& \neg B)) \\
\text{AXIOM D}_{10} & : p_o, \& \neg p_o, \\
\text{RD}_1 & : A, A \rightarrow B \rightarrow B \\
\text{RD}_2 & : A, B \rightarrow A \& B \\
\text{RD}_3 & : A \rightarrow B, C \rightarrow D/(B \rightarrow C) \rightarrow (A \rightarrow D)
\end{align*}
\]

In \( \text{DL} \) the connective \( \lor \) is primitive and the postulates and rules are \( D_1 - D_7, D_{10} \) and \( RD_1 \). plus the following:

\[
\begin{align*}
\text{AXIOM D}_{11} & : A \rightarrow A \lor B \\
\text{AXIOM D}_{12} & : B \rightarrow A \lor B \\
\text{AXIOM D}_{13} & : ((A \rightarrow C) \& (B \rightarrow C)) \rightarrow (A \lor B) \rightarrow C \\
\text{AXIOM D}_{14} & : (\neg A \& \neg B) \rightarrow (A \lor B) \\
\text{AXIOM D}_{15} & : (\neg(A \& B)) \rightarrow (\neg A \lor \neg B) \\
\text{RD}_4 & : A \rightarrow B, \neg B \rightarrow \neg A.
\end{align*}
\]

The systems \( \text{DM} \) and \( \text{DL} \) are easily extended to first order predicate calculi with equality in Routley 1979.

In this paper, the author says that the relevant system \( \text{P} \) of Arruda and da Costa, mentioned in previous sections, together with a representative contradictory thesis \( p_o, \& \neg p_o \), is virtually a minimal dialectical logic.
Da Costa and Marconi 1989 write that "it is a rather debatable issue whether DM and DL do really constitute good or true approaches to extant dialectics". But they claim that the both systems have a powerful motivation and are among the best paraconsistent logics.

Brady and Routley 1989a formulate some dialectical set theories and prove their non triviality.

Several other Australian logicians have contributed to paraconsistent logic, as for example Priest, Plumwood, Bunder, Mortensen, Havas, Brady, Goddard, Slaney and Urbas.

4.2. Other Works

The Belgian logician Apostel, researching on the relation between dialectics and contemporary science, also worked on paraconsistent logic. Apostel 1979 discusses the relation between dialectics and logical systems, as well as between dialectical logic and paraconsistent logic.

The Spanish logician Peña introduced a new kind of paraconsistent logic, having some characteristics of fuzzy logics, and developed an alternative set theory. Peña 1979 is motivated by an explicit philosophical intuition.

Several other papers by Peña work out his ideas and Peña 1980 discusses the formalization of dialectics.

Da Costa and the American logician Wolf developed a new approach to dialectical logic.

Using the logic notions and techniques of paraconsistent logic, especially those of da Costa's systems, in the task of formalizing the notions of negation and contradiction in Hegel and Marx, da Costa and Wolf 1980 construct the propositional system DL of dialectical logic.

The classical positive logic is a part of the system; several important properties of classical negation are maintained and classical properties hold for well-behaved propositions.

The paper intends to formalize the fifth and sixth interpretations of the principle of the unity of opposites of Mc Gill and Parry 1948.

It is interesting also to see da Costa and Wolf 1985.
Von Wright 1986 introduces a system of paraconsistent logic, related to his work Von Wright 1968.

Smolenov 1982 suggests a new system of paraconsistent logic, whose intuition is adequate for Marxist philosophy.

Batens 1989 studies some aspects of dialectics.

§5. Some Other Systems and General Results About Paraconsistent Logics

In this paragraph we present some paraconsistent systems not strictly related to discursive logic, to da Costa’s hierarchies and relevant logic.

We mention general results which study the relations between paraconsistent logic and fuzzy logic; study the relations between dialectical logic, paraconsistent logic and relevant logic; characterize and classify paraconsistent logic; and systematize the field of a special class of paraconsistent logics.

Two papers by D’Ottaviano and López-Escobar, showing the possibility of studying paraconsistent logic with the conditional rather than negation, are also indicated.

Sette 1973 studies the relations between fuzzy logics and paraconsistent logics. He relates the degree of inconsistency of a system with the degree of fuzziness of it, by associating a fuzziness function f to a certain paraconsistent logic such that, intuitively speaking, the function measures how fuzzy the system is.

Bunder 1974 and 1989 proves that under certain conditions all the postulates of intuitionistic propositional and predicate calculi are obtained inside the combinatory logic of Curry and Feys 1958. These combinatory intuitionistic calculi are proved to be paraconsistent.

Priest introduced the Logic of Paradox LP, a system of paraconsistent logic to deal with logical paradoxes. The system is presented in Priest 1979, with an analysis of Gödel’s theorems on the consistency and completeness of arithmetic, and a semantical analysis of the propositional and predicate levels.

In LP some propositions may be true and false simultaneously.

The Italian logician Marconi since his Ph.D has worked in paraconsistent logic and its relations with dialectical logic and relevant
logic.

Marconi 1979 is an anthology on the formalization of dialectics. He presents a discussion on the relations between dialectical logic, paraconsistent logic and relevant logic.

After an interpretation of the so-called pseudo-Scotus law in three different ways, the paper presents a characterization of paraconsistent logics in three different kinds: syntactical weakly paraconsistent systems, syntactically strong paraconsistent systems and syntactically thesis paraconsistent systems. Then, Marconi classifies the systems of paraconsistent logics in seven different types.

The Belgian logician Batens has worked in paraconsistent logics and its philosophical import.

Batens 1980 is a very important paper, for it classifies the different kinds of paraconsistent logics, giving a systematization of the field of paraconsistent logics that are extensions of classical positive propositional calculus.

In his paper, Batens introduces a system PI, which is "close to classical propositional calculus but do not presuppose that the world is consistent" and is considered as a basic logic, that may be extended to richer systems. He characterizes PI semantically and axiomatizes it.

Batens also discusses questions related to the philosophy of paraconsistent logic, as for instance presenting special definitions for different kinds of paraconsistent logics and analysing the meaning of the negation in some paraconsistent extensions of PI.

D'Ottaviano and López-Escobar 1986 consider a valuation $V$ as an indexed set of unary functions, $V = \{v_i : i \in I\}$, each of the $v_i$ being a traditional two-valued assignment, such that $v_i(\alpha) = t$ if and only if, the formula $\alpha$ is true with a degree of error no greater than $i$. Certain uniform conditional error functions are defined, coherent with that interpretation for the $v_i, i \in I$.

Most of paraconsistent logics can be viewed as classical logic with a weakened negation. If a logic is considered as the embodiment of its theorems, then that approach to paraconsistent logic is very reasonable.

In another view of logic, one can be more interested in the inferences than in the theorems. Using the results mentioned above, D'Ottaviano and López-Escobar 1985 show the possibility of
developing paraconsistent logics with the conditional rather than negation.

§6. Applications of Paraconsistent Logic and Some Open Problems

There are several topics whose relations with paraconsistent logics have been studied such as inductive logic, many-valued logic, modal logic, fuzzy logic, ethics, the theory of belief, ontology, the logic of acceptance of scientific hypotheses, quantum mechanics, infinitesimal calculus, probability and computation.

The research about some of these relations is just starting.

6.1. Some Open Questions

There are several interesting open problems concerning paraconsistent logics.

Many questions related to paraconsistent set theory still remain to be studied and a good discursive set theory was not even formulated.

A complete study about the algebraic structures associated to paraconsistent calculi, a semantical analysis of several paraconsistent calculi, an adapted world semantics to the C_n and to other paraconsistent logics, a systematic study about the relations between relevant and paraconsistent logics, and the development of paraconsistent higher-order modal and tense logics are open problems.

In C_1 we define A'' as ¬(A&¬A). Is it possible to present another axiomatic treatment for the notion of "well-behaviorism", which does not make such a strong commitment to negation?

It can be proved that no C_n is a finite many-valued logic. One problem to be studied is if it is possible to characterize all the C_n, 1 ≤ n ≤ w, as infinite many-valued logics.

Another interesting question to be formulated is if it is possible to find a characterization which identifies all the paraconsistent logics that are amongst the finite many-valued logics.

6.2. General Applications

Some of the applications of paraconsistent logic in Philosophy, Natural and Social Sciences, Logic and Mathematics were conside-
and in the previous paragraphs of this paper, under other purposes.

Now, we shall discuss only some recent uses of paraconsistent logic.

Theories based on a *semantically closed language* are inconsistent, for they allow the construction of semantic paradoxes. *Priest 1979* and *Priest and Routley 1989b* show why these theories are related to paraconsistent logic.

*Quantum mechanics Theory*, in its Everett-Wheeler interpretation, according to which all possible alternative values of a quantum theoretic measurement are simultaneously actual, also suggests paraconsistent formalization in many of its parts. *Rescher and BBrandon 1980* and *Priest and Routley 1984* discuss the problem. *Priest and Routley 1989b*, in the paragraph on the subject, begin with the formalization of the Dirac delta function in terms of Hilbert spaces, in order to study the areas of special sensitivity as regards consistency.

The *Infinitesimal Calculus* under one interpretation of its original formulation, shows that infinitesimals are genuine inconsistent objects. Robinson redefined the infinitesimal calculus in terms of non-standard analysis, showing that the theory is not really inconsistent, but his theory seems not to be the original one. *Priest and Routley 1989b* say that the theory is highly suitable for a paraconsistent formalization, but the subject was not still adequately studied.

One of the recent applications of paraconsistent logic was its use in *ethics*, in the field of *deontic logic*. In order to deal with the so-called moral dilemmas (propositions of the form “α and not α are obligatory”) and other analogous problems, paraconsistent deontic logics seem to be appropriate. *Routley and Plumwood 1984* suggest it. *Puga 1985* and *da Costa and Carnielli 1986* deal with these questions, *da Costa and Carnielli* by introducing a system which is an extension of $C_\perp$. *Puga et al 199* continue their line of research and study some paraconsistent systems containing alethic and deontic modalities; this approach allows them, by using the symbols $\circ$ and $\lozenge$ for the deontic operators “it is obligatory that” and “it is permissible that” respectively, to treat the principles of Kant ($\circ A \rightarrow \lozenge A$) and of Hintikka ($\Box A \rightarrow \circ A$) from the classical and the paraconsistent points of view, and to propose systems in
which either, both or neither of the principles are valid.

Priest and Routley 1989b claim that paraconsistent deontic logics are particularly important, for they “rectify the gross distortions of the concepts of obligation that are obtained in classically based deontic logics”. As the sphere of obligation is wide, they observe that it must also be easy to produce inconsistent obligations by considering several other kinds of situations, as well as inconsistent orders, leading to paraconsistent *imperatival logics*.

Theories which admit *inconsistent beliefs* have also been regarded as paraconsistent. Routley and Routley 1975, Rescher and Brandom 1980, and Priest and Routley 1984 and 1989b discuss this kind of application of paraconsistent logic.

Elements of *doxastic logic* have been studied by direct analogy with weak modal logics, but on an apparently inadequate classical basis. Belief is not deductively closed, such that a person may believe A and not to believe B, though B is deducible from A. Thus some usual consistency postulates and theses must be rejected, and the more general worlds-theory of paraconsistent logics enables to reject these principles without problems.

Some cases of inconsistent believes are of special interest for Computer Science.

Da Costa and French have recently worked on belief and contradiction, and on a logic of self-deception.

In *da Costa and French 1988* they argue that we do have inconsistent beliefs and that the simplest, most natural and most intuitive way of formalising our belief-structures is to adopt a paraconsistent doxastic logic. In their paper 1989 they present and discuss three systems of doxastic logic and they conclude that the idea of using paraconsistent logic for modelling belief systems sheds new light on Moore's paradox and the problem of self-deception.

*Da Costa and French 1990* argue that, contrary to certain recent studies of the apparently paradoxical nature of self-deception, the most natural way to understand the problem is in terms of holding the contradictory beliefs using a non-classical doxastic logic.

Another very recent research is *da Silva 1990*. The work is about the interplay between rationality, relativism and logic. It is an approach about the implications of paraconsistent logics on the problem of relativism, showing that heterodox relativism is com-
compatible with rationality. It offers new versions of relativism with the help of some formalisms which can be interpreted as modal, paraconsistent systems. The result is the presentation of heterodox forms of relativism and rationality, which are no longer antinomic or self-refuted.

The standard approaches to probability theory are based on classical logic. But, according to Priest and Routley 1989b, if they are based on paraconsistent logic we may have several advantages. One of the easiest approaches to classical probability theory which can be adapted to paraconsistency is Carnap’s. If $C$ is a class of paraconsistent logic and $m$ is a normal measure function defined on $C$, the probability of a formula $A$, $Pr(A)$, may be defined as $m(\{x \in C / A \text{ holds at } x\})$.

We observe that paraconsistent probability theory diverges from the classical theory in the vicinity of negation. We can have sensible evaluations of the probabilities of statements relative to inconsistent data.

If we have the theory of rational acceptance, so that a claim should be rationally accepted just if it has a high enough probability, we may have some contradictions that are rationally acceptable.

### 6.3. Paraconsistent Logic and Computer Science

Reasoning in the presence of inconsistency is a field of growing importance in logic programming and knowledge representation.

The problem of inconsistency in data basis has been discussed and recognized as a threaten to real computer systems by several authors.

Bodies of data stored in a computer memory may naturally be inconsistent and the computer ought to be able to reason on inconsistent premisses without triviality.

Some authors agree that certain kinds of heterodox logic could be a possible solution to this problem.

Many-valued and relevant logic solutions have been proposed, but just as theoretical solutions. The problem is also often treated by using the so-called non-monotonic logics.

The pioneering work in this area is Belnap 1977, which uses a system of many-valued logic that is also paraconsistent. Beck
applies to the problem the system of von Wright 1986.

It is well known that the systems $C_n$, $1 \leq n \leq w$, do not coincide with any finite many-valued logics. Carnielli 1990a shows, however, that $C_1$ can be related to two distinct three-valued logics through a new concept of semantics of possible interpretation; and proves that the system, viewed in this way, offers a good foundation for the problem of representing plausible reasoning.

Carnielli and Marques 1990 discuss the problem of inconsistency in computer systems, comparing some views about the problem. The main objective of this work is to propose a mechanizable strategy for treating knowledge in the presence of contradictions, using an analytic tableau-type version of paraconsistent logic.

This approach permits to obtain a logic system such that conflicting situations in knowledge representation can be tolerated and reported, and new knowledges can be obtained from conflicting situations.

The authors also discuss the foundational potentiality of this new paradigm for the problem of knowledge representation.

Another problem related to Computer Science which is being studied by W. Carnielli, is the use of a paraconsistent approach to deal with the reasoning of distributed systems.

Concerning paraconsistent logic programming we have several researches worked out by Blair, Subrahmanian, da Costa, Lu and Vago.

H. Blair and V. Subrahmanian are involved in the design of programming languages for the development of very large expert systems, which may be considered to be just a set of formulas of some logic. For them, the design of this kind of systems is one of the main research thrusts of artificial intelligence, such systems being normally designed by a team of programmers who interact with a group of experts in the domain of interest. But different experts often have different opinions and its is therefore possible that an expert system ES thus constructed is inconsistent, in spite of a very large subset of ES may be consistent. So, they claim that a robust semantics is needed to define the declarative meaning of an expert system, believing that the only method known to make sense of inconsistent sets of formulas is through the use of paraconsistent semantics.
Programs written in the languages worked out by the both authors are considered to be formulas of a weak paraconsistent logic. The robust paraconsistent semantics of these languages is extremely useful in reasoning in the presence of inconsistent and/or erroneous information. So far, a theoretical framework has been developed for three languages: one of each for quantitative logic programming, in Subrahmanian 1987; an evidential logic programming, in Subrahmanian 199+, and Blair and Subrahmanian 1987a; and generally Horn logic programming in Blair and Subrahmanian 1987b and 1987c. Blair and Subrahmanian 1987b and 1988 extend a formalism called annotated logic, originally proposed by Subrahmanian 1987 which develops a four valued semantics for logic programs that may contain inconsistencies. These logics have now been used in logic programming, inheritance networks and object oriented databases. It seems that such logics have to be studied carefully.

Da Costa and Subrahmanian 1989 discuss various alternative schemes for paraconsistent reasoning and show that it applies to the design of very large knowledge bases and databases where inconsistent information may often be present.

Da Costa, Henschen et al 199+ develop a linear resolution style proof procedure for mechanical reasoning in the family of paraconsistent annotated logics proposed by Subrahmanian 1987. The experimental theorem prover obtained in the paper is the first implementation of a theorem prover for full fledged annotated logics.

In another paper, da Costa, Subrahmanian et al 199+ give a fullfledged study of the model theoretic and proof theoretic properties of a \( \tau \)-valued annotated logic introduced by Blair and Subrahmanian 1988 and based on a given complete lattice \( \tau \) of truth-values.

It seems that, nowadays, Computer Science is the most promising field for the application of paraconsistent logic.

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