ALGEBRAIZATION OF PARACONSISTENT LOGIC $P^1$.

by

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INTRODUCTION.

In the last twenty five years or so, paraconsistent logics have been extensively developed. The best known of these logics are probably Da Costa’s $C_n$ systems. Several other logics have been studied such as Sette’s $P^1$, $[S]$, da Costa and D’Ottaviano’s $J3$, $[C-D’O]$ and many others. For references see $[A]$ and $[C-M]$.

In $[M]$, Mortensen proves that logic $C_1$ (and thus $C_n, n \geq 1$) is not algebraizable by showing that in (the formula algebra of) $C_1$ there is no congruence other than the identity compatible with the set of theorems of $C_1$. Mortensen gives no formal definition of algebraizability of a deductive system though.

Blok and Pigozzi in $[B-P, 1]$, introduce such a notion of algebraizability which is a natural generalization of the classical Lindenbaum-Tarski method. In this context, Mortensen’s result is given a precise mathematical interpretation. The reader is refered to $[B-P, 1]$ specially Chapter 5, Theorem 5.1 for details.

In this paper we prove that logic $P^1$ is algebraizable in this sense. Other paraconsistent logics such as $J3$ above mentioned have also been proven algebraizable.

§0. PRELIMINARIES.

By a propositional language we will understand some set $\mathcal{L}$ of propositional connectives. The $\mathcal{L}$-formulas are built in the usual way

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from propositional variables $P_0, P_1, P_2, \ldots$ using the connectives of $\mathcal{L}$. The set of all $\mathcal{L}$-formulas is denoted by $\text{Fm}_\mathcal{L}$.

By an inference rule over $\mathcal{L}$ we mean any pair $< \Gamma, \phi >$ where $\Gamma$ is a finite set of formulas and $\phi$ is a single formula of $\text{Fm}_\mathcal{L}$. A deductive system $S$ (over $\mathcal{L}$) is defined by a (possibly infinite) set of inference rules and axioms, it consists of the pair $S = < \mathcal{L}, \vdash_S >$ where $\vdash_S$ is the relation between sets of formulas and individual formulas defined by the following condition: $\Delta \vdash_S \phi$ iff $\phi$ is contained in the smallest set of formulas that includes $\Delta$ together with all substitution instances of the axioms of $S$, and is closed under direct derivability by the inference rules of $S$. Let $\mathcal{L}$ be a propositional language and $K$ any class of $\mathcal{L}$-algebras. Let $\models_K$ be the relation that holds between a set $\Gamma$ and a single equation $\phi \approx \psi$, in symbols, $\Gamma \models_K \phi \approx \psi$, if every interpretation of $\phi \approx \psi$ in a member of $K$ holds provided each equation in $\Gamma$ holds under the same interpretation. Thus, $\Gamma \models_K \phi \approx \psi$ iff for all $A \in K$ and every interpretation $a$ of the variables of $\Gamma \cup \{ \phi \approx \psi \}$ as elements of $A$, for every $\xi \approx \eta \in \Gamma$, $\xi^A(a) = \eta^A(a) \Rightarrow \phi^A(a) = \psi^A(a)$. In this case we say that $\phi \approx \psi$ is a $K$-consequence of $\Gamma$. The relation $\models_K$ is called the (semantical) equational consequence relation determined by $K$.

**DEFINITION 0.1**

Let $S = < \mathcal{L}, \vdash_S >$ be a deductive system and $K$ a class of algebras $K$ is called an algebraic semantics for $S$ if $\vdash_S$ can be interpreted in $\models_K$ in the following sense: there exists a finite system $\delta_i(p) \approx \varepsilon_i(p)$, for $i < n$, of equations with a single variable $p$ such that, for all $\Gamma \cup \{ \phi \} \subseteq \text{Fm}_\mathcal{L}$ and each $j < n$,

$$\Gamma \vdash_S \phi \iff \{ \delta_i[\psi/p] \approx \varepsilon_i[\psi/p] : i < n, \psi \in \Gamma \} \models_K \delta_j[\phi/p] \approx \varepsilon_j[\psi/p].$$

The $\delta_i(p) \approx \varepsilon_i(p)$, for $i < n$ are called the defining equations for $S$ and $K$.

In order to simplify notation we shall use $\delta \approx \varepsilon$ as an abbreviation for a system $\delta_i(p) \approx \varepsilon_i(p), i < n$.

**DEFINITION 0.2**

Let $S$ be a deductive system and $K$ an algebraic semantics for $S$
with defining equations \( \delta_i \approx \varepsilon_i \), for \( i < n \), i.e.,

(i) \( \Gamma \vdash \phi \iff \{ \delta(\psi) \approx \varepsilon(\psi) : \psi \in \Gamma \} \models K \delta(\phi) \approx \varepsilon(\phi) \).

K is said to be equivalent to S if there exists a finite system \( \Delta_j(p, q) \), for \( j < m \), of composite binary connectives (i.e. formulas with two variables) such that, for every equation \( \phi \approx \psi \),

(ii) \( \phi \approx \psi \vdash \models K \delta(\phi \Delta \psi) \approx \varepsilon(\phi \Delta \psi) \).

The system \( \Delta_j, j < m \), of composite binary connectives satisfying (ii) is called a system of equivalence formulas for S and K.

**COROLLARY 0.3**

Let K be an algebraic semantics for S with defining equations \( \delta \approx \varepsilon \). If K has equivalence formulas \( \Delta \), then for each set \( \Gamma \) of equations and each equation \( \phi \approx \psi \),

(i) \( \Gamma \models K \phi \approx \psi \iff \{ \xi \Delta \eta : \xi \approx \eta \in \Gamma \} \not\models \phi \Delta \psi \),

and for each \( \nu \in Fm \),

(ii) \( \nu \vdash \models \delta(\nu) \Delta \varepsilon(\nu) \).

Conversely, if there exists a system \( \Delta \) of formulas satisfying conditions (i) and (ii), then K is equivalent to S with equivalence formulas \( \Delta \). If K is an equivalent algebraic semantics for S, definition 0.2(i) guarantees that \( \vdash \models \) can be interpreted in \( \models K \), corollary 0.3(i) that \( \models K \) can be interpreted in \( \vdash \models \), and 0.2(ii), 0.3(ii) guarantee that these interpretations are, essentially, inverse to one another. \( \square \)

**DEFINITION 0.4**

A deductive system S is said to be algebraizable if it has an equivalent algebraic semantics.

**THEOREM 0.5**

Let S be a deductive system given by a set of axioms Ax and a set of inference rules Ir. Assume S is algebraizable with equivalence formulas \( \Delta \) and defining equations \( \delta \approx \varepsilon \). Then the unique equivalent quasivariety semantics for S is axiomatized by the identities

(i) \( \wedge(\phi) \approx \varepsilon(\phi) \),

for each \( \phi \in Ax \),

(ii) \( \wedge(p \Delta p) \approx \varepsilon(p \Delta p) \),

together with the following quasi-identities

(iii) \( \psi_0 \approx \varepsilon(\psi_0) \wedge \ldots \wedge \delta(\psi_{n-1}) \approx \varepsilon(\psi_{n-1}) \)
\[ \Rightarrow \delta(\phi) \approx \varepsilon(\phi), \]
for each \(<\{\psi_0, \ldots, \psi_{n-1}\}, \phi >\in \text{Ir}, \]
(iv) \(\delta(p \Delta q) \approx \varepsilon(p \Delta q) \Rightarrow p \approx q. \square \)

A more useful characterization of algebraizable deductive systems is the following:

**THEOREM 0.6**

A deductive system \(S\) is algebraizable iff there exists a system \(\Delta\) of formulas in two variables and a system \(\delta \approx \varepsilon\) of equations in a single variable such that the following conditions (i)-(v) hold for all \(\phi, \psi, \nu \in \text{Fm}\):

(i) \(\vdash_S \phi \Delta \phi\);
(ii) \(\phi \Delta \psi \vdash_S \psi \Delta \phi\);
(iii) \(\phi \Delta \psi, \psi \Delta \nu \vdash_S \phi \Delta \nu\);

For every primitive connective \(\omega\) and all \(\phi_0, \ldots, \phi_{n-1}, \psi_0, \ldots, \psi_{n-1} \in \text{Fm}\) where \(n\) is the rank of \(\omega\),

(iv) \(\phi_0 \Delta \psi_0, \ldots, \phi_{n-1} \Delta \psi_{n-1} \vdash_S \omega \phi_0 \ldots \phi_{n-1} \Delta \omega \psi_0 \ldots \psi_{n-1}\).

Finally, for all \(\nu \in \text{Fm}\),

(v) \(\vdash_S \delta(\nu) \Delta \varepsilon(\nu)\).

In this event \(\Delta\) and \(\delta \approx \varepsilon\) are systems of equivalence formulas and defining equations for \(S. \ \square\)

In [B-P,2], a semantical criterium to decide if a deductive system is algebraizable is given.

**DEFINITION 0.7**

An \(\mathcal{L}\)-matrix is a pair \(M =< A, D >\) where \(A\) is an \(\mathcal{L}\)-algebra and \(D\) is a subset of the universe \(A\) of \(A\). \(A\) is the algebra of \(M\) and \(D\) is the set of designated values. A class \(\mathcal{M}\) of \(\mathcal{L}\)-matrices determines a deductive system \(S_{\mathcal{M}} =< \mathcal{L}, \vdash_{\mathcal{M}} >\) in the following way:

\(\Gamma \vdash_{\mathcal{M}} \phi\) iff for each \(M =< A, D >\in \mathcal{M}\) and for every interpretation \(a\) of the variables of \(\Gamma \cup \{\phi\}\) as elements of \(A\), if \(\psi^A(a) \in D\) for each \(\psi \in \Gamma\), then \(\phi^A(a) \in D\).

**THEOREM 0.8 [B-P, 2]**

Let \(\mathcal{M}\) be a class of \(\mathcal{L}\)-matrices and \(S\) the deductive system semantically defined by \(\mathcal{M}\). \(S\) is algebraizable iff there exists unary formulas \(\delta_i, \varepsilon_i, i < n\), and binary \(\Delta_j, j < m\), such that, for each
\[ M = < A, D > \in M, R(M, \Delta) = \{ < a, b > \in A \times A : \Delta_j^A(a, b) \in D \text{ for all } j < m \} \text{ verifies:} \]

(i) \( R(M, \Delta) \) is a congruence on \( A \);

(ii) For each \( a \in A, a \in D \) iff \( \delta_i^A(a) = \varepsilon_i^A(a)(R(M, \Delta)) \) for all \( i < n \). \( \Box \)

§1 THE DEDUCTIVE SYSTEM \( P^1 = < \mathcal{L}, \vdash_{P^1} > \)

The propositional language \( \mathcal{L} \).

There are only two connectives: one unary \( \sim \), and one binary \( \rightarrow \).

AXIOMS: If \( X, Y \) and \( Z \) are w.f.f.s then the following are axioms.

\begin{align*}
\text{Ax.1} & \quad X \rightarrow (Y \rightarrow X) \\
\text{Ax.2} & \quad (X \rightarrow (Y \rightarrow Z)) \rightarrow ((X \rightarrow Y) \rightarrow (X \rightarrow Z)) \\
\text{Ax.3} & \quad (\sim X \rightarrow \sim Y) \rightarrow ((\sim X \rightarrow \sim Y) \rightarrow X) \\
\text{Ax.4} & \quad (X \rightarrow \sim \sim X) \rightarrow X \\
\text{Ax.5} & \quad (X \rightarrow Y) \sim \sim (X \rightarrow Y)
\end{align*}

Rule of inference.

Modus Ponens (M.P.) i.e. \( < \{ X, X \rightarrow Y \}, Y > \) is the only rule of inference.

Binary connectives \( \lor, \land, \leftrightarrow \) are defined as follows:

\begin{align*}
X \land Y \text{ abbreviates } & \sim \sim ((X \rightarrow X) \rightarrow (Y \rightarrow Y)) \rightarrow \sim \sim (X \rightarrow \sim Y)) \\
X \lor Y \text{ abbreviates } & \sim \sim (X \rightarrow \sim \sim X) \rightarrow (\sim X \rightarrow Y) \\
X \leftrightarrow Y \text{ abbreviates } & (X \rightarrow Y) \land (Y \rightarrow X).
\end{align*}

The deductive system \( P^1 \) is a Paraconsistent Logic system i.e. \( X \land \sim X \) is not in general a contradiction. It is maximal in the sense that, if \( \phi \) is a classical tautology which is not provable in \( P^1 \), then the propositional calculus obtained by adding \( \phi \) as a new axiom to \( P^1 \) is the Classical Propositional Calculus.

We say that a formula \( \phi \) is regular if \( \vdash_{P^1} (\phi \sim \sim \phi) \). All non-
atomic formulas are regular and the class of all regular formulas behave as in the Classical Propositional Calculus. For a proof of this as well as for

THEOREM 1.1 - 1.5, see [S].

PROPOSITION 1.1

For every formula \( \phi, \vdash_{P^1} \phi \rightarrow \phi \). \( \square \)

PROPOSITION 1.2

If \( \Gamma \) is a set of formulas, \( \phi \) and \( \psi \) two formulas such that \( \Gamma, \phi \vdash P^1 \psi \), then \( \Gamma \vdash_{P^1} \phi \rightarrow \psi \). \( \square \)

COROLLARY 1.3

(i) \( \phi \rightarrow \psi, \psi \rightarrow \xi \vdash_{P^1} \phi \rightarrow \xi \).

(ii) \( \phi \rightarrow (\psi \rightarrow \xi), \psi \vdash_{P^1} \phi \rightarrow \xi \). \( \square \)

PROPOSITION 1.4

The Following formulas are theorems of \( P^1 \).

a) \( (\phi \rightarrow \sim \sim \phi) \rightarrow (\sim \phi \rightarrow (\phi \rightarrow \psi)) \)

b) \( (\phi \rightarrow \sim \sim \phi) \rightarrow ((\sim \psi \rightarrow \sim \phi) \rightarrow (\phi \rightarrow \psi)) \)

c) \( (\psi \rightarrow \sim \sim \psi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\sim \psi \rightarrow \sim \phi)) \)

d) \( (\psi \rightarrow \sim \sim \psi) \rightarrow (\phi \rightarrow (\sim \psi \rightarrow \sim (\phi \rightarrow \psi))) \)

e) \( (\psi \rightarrow \sim \sim \psi) \rightarrow ((\phi \rightarrow \psi) \rightarrow ((\sim \phi \rightarrow \psi) \rightarrow \psi)) \). \( \square \)

Let \( M = \langle \{ T_0, T_1, F \}, \{ T_0, T_1 \} \rangle \) be the matrix where \( \{ T_0, T_1 \} \) are the designated values and \( \rightarrow, \sim \) are defined in the following way:

\[
\begin{matrix}
\rightarrow & T_0 & T_1 & F & \sim \\
T_0 & T_0 & T_0 & F & T_0 & F \\
T_1 & T_0 & T_0 & F & T_1 & T_0 \\
F & T_0 & T_0 & T_0 & F & T_0 \\
\end{matrix}
\]
PROPOSITION 1.5 (Completeness Theorem, see [S], Prop. 9.)

$M$ is a characteristic matrix of $P^1$ (i.e. For each $\phi \in \mathsf{Fm}$ and each interpretation $\nu$ of the variables of $\phi$ in $\{T_0, T_1, F\}, \nu(\phi) \in \{T_0, T_1\}$ if $\vdash_{P^1} \phi$.)

Let $M = \langle \{T_0, T_1, F\}, \{T_0, T_1\} \rangle$ be defined as before. We say that a set $\Gamma$ of formulas is satisfiable in $M$ if there is a valuation $\nu$ such that for every $\alpha \in \Gamma, \nu(\alpha) \in \{T_0, T_1\}$. In this case we say $\nu$ satisfies $\Gamma$. If every finite subset of $\Gamma$ is satisfiable, we say that $\Gamma$ is finitely satisfiable. We say that a set $\Gamma$ of formulas satisfies a formula $\phi$ if every valuation $\nu$ that satisfies $\Gamma$ satisfies $\phi$.

THEOREM 1.6

$\Gamma$ is satisfiable iff $\Gamma$ is finitely satisfiable.

PROOF.

This is a well known fact, considering that the language is finitary and that the matrix $M$ is finite. An easy proof can be obtained using König’s lemma.

COROLLARY 1.7

$\Gamma$ satisfies $\phi$ iff there exists a finite subset $\Gamma_0$ of $\Gamma$ that satisfies $\phi$.

PROOF.

From right to left the result is clear.

Suppose then that $\Gamma$ satisfies $\phi$. If $\phi$ is non-atomic, suppose that for every finite subset $\Gamma_0$ of $\Gamma$ there exists a valuation $\nu$ that satisfies $\Gamma_0$ but $\nu(\phi) = F$. Then $\nu(\neg \phi) \in \{T_0, T_1\}$, therefore $\Gamma \cup \{\neg \phi\}$ is finitely satisfiable, thus satisfiable. There exists a valuation $\nu$ that satisfies $\Gamma \cup \{\neg \phi\}$, and since $\neg \phi$ is non-atomic $\nu(\neg \phi) = T_0$; but $\phi$ is also non-atomic, so $\nu(\phi) = F$. But $\Gamma$ satisfies $\phi$, and we have a contradiction.

Assume $\phi$ is atomic. Since $\Gamma$ satisfies $\phi$, $\Gamma$ satisfies $(\phi \rightarrow \phi) \rightarrow \phi$ which is non-atomic, therefore there exists a finite subset $\Gamma_0$ of $\Gamma$ such that $\Gamma_0$ satisfies $(\phi \rightarrow \phi) \rightarrow \phi$; but if $\Gamma_0$ satisfies $(\phi \rightarrow \phi) \rightarrow \phi$, then $\Gamma_0$ satisfies $\phi$. □
THEOREM 1.8

If $\Gamma$ satisfies $\phi$, then $\Gamma \vdash_{P^1} \phi$. (i.e. The matrix $M$ is strongly characteristic.)

PROOF.

This follows from Corollary 1.7 and the fact that $M$ is characteristic for $P^1$. $\square$

§2 ALGEBRAIZATION OF $P^1$.

THEOREM 2.1

$P^1$ is algebraizable.

A SYNTACTICAL PROOF.

For $\phi, \psi$ formulas, let $\delta(\phi) = (\phi \rightarrow \phi) \rightarrow \phi$, $\varepsilon(\phi) = \phi \rightarrow \phi$, $\phi \Delta_1 \psi = \phi \rightarrow \psi$, $\phi \Delta_2 \psi = \psi \rightarrow \phi$, $\phi \Delta_3 \psi = \sim \phi \rightarrow \sim \psi$ and $\phi \Delta_4 \psi = \sim \psi \rightarrow \sim \phi$. We must show that $\delta$, $\varepsilon$ and $\Delta$ satisfy conditions (i) - (v) of Theorem 0.6.

(i): Proposition 1.1 implies $\vdash_{P^1} \phi \Delta \phi$.

(ii): $\phi \Delta \psi \vdash_{P^1} \psi \Delta \phi$ is obtained from the definition of $\Delta$.

(iii): $\phi \Delta \psi, \psi \Delta \psi \vdash_{P^1} \phi \Delta \phi$ is obtained from Corollary 1.3.

(iv): We must prove:

a) $\phi \Delta \psi \vdash_{P^1} \sim \phi \Delta \sim \psi$ and

b) $\psi_1 \Delta \psi_1, \phi_2 \Delta \psi_2 \vdash_{P^1} (\phi_1 \rightarrow \phi_2) \Delta (\psi_1 \rightarrow \psi_2)$.

These are straightforward using the theorems in §1 and the fact that $\sim \phi, \sim \psi, \phi_1 \rightarrow \phi_2$ and $\psi_1 \rightarrow \psi_2$ are regular.

(v): We must prove $\nu \vdash_{P^1} \delta(\nu) \Delta \varepsilon(\nu)$, i.e.

$$\nu \vdash_{P^1} (\nu \rightarrow \nu) \rightarrow \nu \Delta \nu \rightarrow \nu$$

To prove

$$\nu \vdash_{P^1} (\nu \rightarrow \nu) \rightarrow \nu \Delta \nu \rightarrow \nu$$

use the theorems in §1 and that both $(\nu \rightarrow \nu) \rightarrow \nu$ and $\nu \rightarrow \nu$ are regular. The other half is obvious.
A SEMANTICAL PROOF.

Let \( M, \delta, \varepsilon \) and \( \Delta \) be as before. We must prove the hypothesis of Theorem 0.8. Since \( M \) is strongly characteristic for system \( P^1 \), it is semantically defined by \( M = \{ M \} \).

We only have to prove that

(i) \( R(M, \Delta) \) is a congruence in \( \{ T_0, T_1, F \} \), and
(ii) For each \( X \in \{ T_0, T_1, F \} \), \( X \in \{ T_0, T_1 \} \) iff \( (\delta(X), \varepsilon(X)) \in R(M, \Delta) \).

(i) \((T_0, T_1) \notin R(M, \Delta) \) since \( \Delta_4(T_0, T_1) = F \);
\((T_1, T_0) \notin R(M, \Delta) \) since \( \Delta_3(T_1, T_0) = F \);
\((T_0, F) \notin R(M, \Delta) \) since \( \Delta_1(T_0, F) = F \);
\((F, T_0) \notin R(M, \Delta) \) since \( \Delta_2(F, T_0) = F \);
\((F, T_1) \notin R(M, \Delta) \) since \( \Delta_2(T_1, F) = F \);
\((T_1, F) \notin R(M, \Delta) \) since \( \Delta_1(T_1, F) = F \).

Thus \( R(M, \Delta) \) is the identity in \( \{ T_0, T_1, F \} \) so it is a congruence.

(ii) \( \delta(T_0) = (T_0 \rightarrow T_0) \rightarrow T_0 = T_0 \rightarrow T_0 = \varepsilon(T_0) = T_0 \);
\( \delta(T_1) = (T_1 \rightarrow T_1) \rightarrow T_1 = T_0 \rightarrow T_1 = T_0 = \varepsilon(T_1) \);
\( \delta(F) = (F \rightarrow F) \rightarrow F = T_0 \rightarrow F = F \). But \( (F, \varepsilon(F)) \notin R(M, \Delta) \).

So \( X \in \{ T_0, T_1 \} \) iff \( (\delta(X), \varepsilon(X)) \in R(M, \Delta) \). \( \square \)

COROLLARY 2.2

The algebraic semantics associated with \( P^1 \) is the quasivariety generated by the three element algebra \( \{ T_0, T_1, F \} \) after proposition 1.4. \( \square \)

References


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