THE CONSTRUCTION OF THE CALCULI $C_n$
OF DA COSTA

by

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The propositional calculi $C_n$, $1 \leq n \leq w$, of da Costa, constitute a special category of paraconsistent logic which was constructed by observing the following conditions:

1. The principle of contradiction, $\neg(A \land \neg A)$, should not in general be valid;

2. From two contradictory formulas, $A$ and $\neg A$, it should not be possible in general to deduce an arbitrary formula;

3. The $C_n$, $1 \leq n \leq w$, must contain the most part of the schemata and rules of the classical propositional calculus which do not interfere with the first conditions (see [2]).

However, some criticisms were pointed against $C_n$ systems. It was argued that they have serious drawbacks, both philosophical and technical, and that their construction is somehow artificial (see, for instance, [4], [5] and [6]).

In this paper, we will show that, contrary to this position, there exists a very natural construction for the calculi $C_n$ which is "dual"
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to the construction of Heyting's intuitionistic calculus.

Da Costa constructed a hierarchy of calculi $C_1, C_2, \ldots, C_\omega$ such that every calculus of the hierarchy is strictly stronger than those which follow it (see [1]). Here, we will concern ourselves only with the calculus $C_1$, observing that the construction of the remaining calculi is similar.

Let us define a calculus as a triple $C = <FOR, A, R>$, where $FOR$ is the set of well-formed expressions, $A$ is a subset of $FOR$, $A \neq \emptyset$ and $R$, $R \neq \emptyset$, is a set of relations on $FOR$, called inference rules. Given a calculus it is possible to define when a formula $A$ is a consequence of a set $\Gamma$ of formulas (in symbols, $\Gamma \vdash C A$). We will call $Cn(\Gamma)$ the set of consequences of $\Gamma$, e.i., $Cn(\Gamma) = \{ A : \Gamma \vdash C A \}$.

Following Tarski (see [7]), we can axiomatize the notion of consequence as follows:

\begin{enumerate}
  \item[(C)] $\Gamma \subseteq Cn(\Gamma)$ \hspace{1cm} Cumulativeness
  \item[(M)] $\Gamma \subseteq \Delta \Rightarrow Cn(\Gamma) \subseteq Cn(\Delta)$ \hspace{1cm} Monotonicity
  \item[(I)] $Cn(Cn(\Gamma)) \subseteq Cn(\Gamma)$ \hspace{1cm} Idempotence
\end{enumerate}

We can also axiomatize the classical propositional consequence operator by adding the conditions characterizing the usual functors:

\begin{enumerate}
  \item[(-)] $Cn(\{ \Gamma, \neg A \}) = FOR$ \hspace{1cm} iff \hspace{1cm} $A \in Cn(\Gamma)$
  \item[(\wedge)] $Cn(\{ \Gamma, A \wedge B \}) = Cn(\{ \Gamma, A, B \})$
  \item[(\vee)] $Cn(\{ \Gamma, A \vee B \}) = Cn(\{ \Gamma, A \}) \cap Cn(\{ \Gamma, B \})$
\end{enumerate}
\[ (\to) \quad B \in \mathcal{C}(\{\Gamma, A\}) \iff A \to B \in \mathcal{C}(\Gamma) \]

\[ (F) \quad A \in \mathcal{C}(\Gamma) \iff \text{for some } \Delta\text{-finite, } \Delta \subseteq \Gamma, \ A \in \mathcal{C}(\Delta). \]

Considerations about those conditions can be found in [3]. Classical logic is defined as the proof-closure of any set of formulas under the following conditions:

**RD** – Detachment rule: \( A, A \to B \vdash B \)

\[ (A1) \quad A \to (B \to A) \]

\[ (A2) \quad (A \to (B \to D)) \to ((A \to B) \to (A \to D)) \]

\[ (A3) \quad (A \to (B \to (A \land B))) \]

\[ (A4) \quad (A \land B) \to A \]

\[ (A5) \quad (A \land B) \to B \]

\[ (A6) \quad A \to (A \lor B) \]

\[ (A7) \quad B \to (A \lor B) \]

\[ (A8) \quad (A \to D) \to ((B \to D) \to ((A \lor B) \to D)) \]

\[ (A9) \quad (A \to B) \to ((A \to \neg B) \to \neg A) \]
(A10) \( A \rightarrow (\neg A \rightarrow B) \)

(A11) \( \neg \neg A \rightarrow A \)

Now, we concentrate our investigation on Tarski's condition about negation. The basic idea concerning negation is that logical opposition means inconsistency. Classically, in inconsistent theories, any formula is derivable and such theories are trivial; nevertheless, we can distinguish inconsistency from triviality. A logic is paraconsistent if it can be used as the underlying logic for inconsistent but nontrivial theories.

We can obtain consistent sublogics of classical logic by weakening Tarski's condition on negation (\( \neg \)).

The first natural modification of (\( \neg \)) is the version of "denied conclusion":

\[
(\neg^-) \quad \text{Cl}(\{\Gamma, A\}) = \text{FOR} \quad \text{iff} \quad \neg A \in \text{Cl}(\Gamma),
\]

which gives us the intuitionistic calculus (IL). This calculus is defined by RD, (A1)-(A10) above.

We can weaken the axiomatic bases of IL by deleting the negative axioms, defining the positive calculi as those satisfying Tarski's positive axioms, and the minimal calculi as positive calculi satisfying (A9).

The axiomatic minimal calculus called Johansson's minimal logic (MJL), has the following conditions on Tarski's style:

\[
(1J^-) \quad \neg B \in \text{Cl}(\Gamma) \quad \text{iff} \quad \neg A \in \text{Cl}(\{\Gamma, A\})
\]
(2J¬) \( \neg A \in Cn(\{B \land \neg B\}) \)

We can also consider two basic negative theses:

(MC) Minimal contraposition:

\( \neg B \rightarrow \neg A \in Cn(\{\Gamma, A \rightarrow B\}) \) and

(MR) Minimal reduction:

\( \neg A \in Cn(\{\Gamma, A \rightarrow \neg A\}). \)

Let PL (Positive logic) be defined as RD, (A1)-(A8). We can define a proper sublogic of Johansson’s minimal logic as follows:

\[ MCL = PL + (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A). \]

Then, we have the following result:

\[ MCL + (A \rightarrow \neg A) \rightarrow \neg A = MJL \]

\[ MJL + A \rightarrow (\neg A \rightarrow B) = IL \]

This can be visualized by the schema:
As it is well known, the inconsistency sign of classical logic, \( A \land \neg A \), is also a inconsistency sign in the intuitionistic logic, but it is not in Johánsson's minimal logic. In this case, \( A \land \neg A \) trivializes only the negative part of the calculus.

Now, in analogous way, we can find a route for the construction of the paraconsistent calculus \( C_1 \), by substituting the inconsistent sign, \( A \land \neg A \), by \( A \land \neg A \land A^0 \) or \( A \land \neg^* A \), now called triviality sign (we define \( \neg(B \land \neg B) \) by \( B^0 \) and \( \neg B \land B^0 \) by \( \neg^* B \)).

Then, Tarski's condition for negation is as follows:

\[
(P\neg) \quad \text{CN}(\{\Gamma, \neg A\}) = \text{FOR} \quad \text{iff} \quad A \land A^0 \in \text{CN}(\Gamma)
\]

or

\[
\text{CN}(\{\Gamma, \neg A\}) = \text{FOR} \quad \text{iff} \quad A \in \text{CN}(\{\Gamma, A^0\})
\]

With this modification we can obtain from Tarski's conditions, the following axioms:
(RD) \[ A, \quad A \rightarrow B/B \]

(A1)-(A8) as in classical calculus

(A9) \[ B^0 \rightarrow ((A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)) \]

(A10) \[ A^0 \rightarrow (A \rightarrow (\neg A \rightarrow B)) \]

(A11) \[ A \lor \neg A \]

If we add to these axioms the following schemata:

(A12) \[ A^0 \land B^0 \rightarrow (A \land B)^0 \]

(A13) \[ A^0 \land B^0 \rightarrow (A \lor B)^0 \]

(A14) \[ A^0 \land B^0 \rightarrow (A \rightarrow B)^0, \]

we obtain the calculus \( C_1 \) of da Costa. Observe that (A10) is equivalent in \( C_1 \) to \( \neg \neg A \rightarrow A \), which was an axiom on the original formulation (see [2]).

Now, we are going to construct \( C_1 \) from the positive calculus, taking \( A \land \neg A \land A^0 \) as a sign of triviality, that is, by assumption of \( (P \neg\neg) \). We can formulate:

\[ (MC)^0 \quad \text{Paraconsistent minimal contraposition:} \]

\[ \neg B \rightarrow \neg A \in \mathcal{Cn}(\{\Gamma, A \rightarrow B, B^0\}) \quad \text{or} \]
\[ \neg A \in \mathcal{Cn}(\{\Gamma, A \rightarrow B, B^0, \neg B\}). \]

We obtain the following logic:

\[ (\text{MCL})^0 = PL + B^0 \rightarrow ((A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)). \]

If we join minimal reduction (MR) to (MCL)$^0$, we obtain Paraconsistent Reductio ad Absurdum ((A9) of $C_1$) and we call this system (RAL1)$^0$.

The following step is to strengthen this logic by regularizing the composition of "well-behaved" formulas, that is, the formulas which obey the principle of contradiction. To this end, we add the following formulas:

\[ A^0 \land B^0 \rightarrow (A \land B)^0 \]

\[ A^0 \land B^0 \rightarrow (A \lor B)^0 \]

\[ A^0 \land B^0 \rightarrow (A \rightarrow B)^0 \]

\[ A^0 \rightarrow (\neg A)^0. \]

We then obtain a new system called (RAL2)$^0$.

We now have a choice: the principle of contradiction \((\neg(A \land \neg A)\) must be a valid schema; its inclusion in (RAL2)$^0$ gives us the Johansson minimal logic (MJL). If to this we add Duns Scotus Law we get Intuitionistic Logic (IL). Of course, if we add excluded
middle to IL, we have classical logic. But we can strengthen the calculus \((RAL2)^0\) in another direction, by assuming excluded middle instead of its "dual", the principle of contradiction. We then obtain what we call minimal paraconsistent logic \((MPL)\). If we add to the system \((MPL)\) the formula \(A^0 \rightarrow (A \rightarrow (\neg A \rightarrow B))\) (or, alternatively, \(\neg \neg A \rightarrow A\)), we obtain the calculus \(C_1\). As it is well known, \(C_1\) plus \(\neg (A \land \neg A)\) give us classical logic \((CL)\).

We then have the following picture:

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                                 PL
                                 
             (MCL)^0
             
             (MCL)^0
             (A \rightarrow \neg A) \rightarrow \neg A

             (RAL1)^0
             A^0 \land B^0 \rightarrow (\neg A)^0 \land (A \land B)^0
             \land (A \lor B)^0 \land (A \rightarrow B)^0

             (RAL2)^0
             (A \land \neg A) \land (A \land \neg A)

             MJL
             A \rightarrow (\neg A \rightarrow B)

             IL
             A \land \neg A

             CL
             (A \land \neg A)

             MPL
             A^0 \rightarrow (A \rightarrow (\neg A \rightarrow B))
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As we observed earlier, in Johansson’s minimal logic, we don’t have inconsistency sign. As a matter of fact, a contradiction in MJL trivializes only the negative part of the calculus, that is

\[ \vdash_{MJL} A \rightarrow (\neg A \rightarrow \neg B), \quad \text{but} \]

\[ \forall_{MJL} A \rightarrow (\neg A \rightarrow B). \]

In an analogous way, in the minimal paraconsistent logic (MPL), we have:

\[ \vdash_{MPL} A^0 \rightarrow (A \rightarrow (\neg A \rightarrow \neg B)), \quad \text{but} \]

\[ \forall_{MPL} A^0 \rightarrow (A \rightarrow (\neg A \rightarrow B)). \]

This means that, in MPL, we don’t have a sign of triviality; here, a contradiction, \( A \wedge \neg A \wedge A^0 \), trivializes only the negative part of the calculus.

We now present some properties of the calculi just described.

\[ \forall_{MPL} A \wedge \neg A \wedge A^0 \rightarrow B \]

\[ \forall_{MPL} \neg\neg A \rightarrow A \]

\[ \forall_{MPL} ((A \rightarrow B) \rightarrow A) \rightarrow A \]

\[ \vdash_{(RAL1)^0} B^0 \rightarrow ((A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)) \]

\[ \vdash_{C_1} \neg\neg A \rightarrow A \]
\[ \vdash_{C_1} ((A \rightarrow B) \rightarrow A) \rightarrow A \]

As it was observed, condition (P−) characterizes a calculus \( C \), which is weaker than \( C_1 \). If from this calculus we delete (A9) and substitute (A10) for \( \neg A \rightarrow A \), we obtain the calculus \( C_w \) of da Costa; then, the calculus \( C \) is situated between \( C_1 \) and \( C_w \).

We think that these considerations show that the calculi \( C_n \) of da Costa have a very "natural" construction from Positive Logic, in a way that we can consider them "dual" to the construction of the Intuitionistic Calculus. We think that it would be interesting to investigate the subcalculi here considered, from syntactic and semantic points of view. We plan to do this in future works.

References


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